Characters of Nonlinear Groups

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Nonlinear Groups

Non Nonlinear Groups

Atlas (lectures last week):

G = connected, complex, reductive, algebraic group
$G = G(\mathbb{R})$

$GL(n,\mathbb{R})$, $SO(p,q)$, $Sp(2n,\mathbb{R})$ not $\widetilde{Sp}(2n,\mathbb{R})$

Primary reason for this restriction: Vogan Duality

Atlas parameters for representations of real forms of G:

$$\mathcal{Z} \subset \prod_k K_i\backslash G/B \times \prod_j K_j^\vee\backslash G^\vee/B^\vee$$

Vogan duality: $\mathcal{Z} \ni (x,y) \rightarrow (y,x)$

Not known in general for nonlinear groups

Outline

Character/representation theory of:

(1) $\widetilde{GL}(2)$ (Flicker)

(2) $\widetilde{GL}(n,\mathbb{Q}_p)$ (Kazhdan/Patterson)

(3) $\widetilde{GL}(n,\mathbb{R})$ (A/Huang)

(4) $\widetilde{Sp}(2n,\mathbb{R})$ and $SO(2n+1)$

(5) $\widetilde{G(\mathbb{R})}$ (G simply laced)

Characters and Representations

π = virtual representation of $G(\mathbb{R})$

$\pi = \sum_{i=1}^{n} a_i \pi_i$ ($a_i \in \mathbb{Z}$, π_i irreducible)

$\theta_\pi = \sum_i \theta_{\pi_i}$ = virtual character

conjugation invariant function on $G(\mathbb{R})_0$ (regular semisimple elements)

Identify (virtual) characters and (virtual) representations

(4) $\widetilde{Sp}(2n, \mathbb{R})$ and $SO(2n+1)$

$\mathbb{F}$ local, characteristic 0

W symplectic/$\mathbb{F}$, $Sp(W) = Sp(2n, \mathbb{F})$

(V, Q): $SO(V, Q)$ = special orthgonal group of (V, Q)

Fix $\delta \in \mathbb{F}^\times/\mathbb{F}^{\times 2}$

Proposition [Howe + ϵ] There is a natural bijection

$$\{\text{regular semisimple conjugacy classes in } Sp(W)\}$$

and

$$\coprod_{(V,Q)} \{\text{ (strongly) regular ss conjugacy classes in } SO(V, Q)\}$$

union: $\dim(V) = 2n + 1$, $\det(Q) = \delta$

Proposition implies relation on characters/representations of $Sp(W)$, $SO(V,Q)$?

Naive guess: π representation of $SO(V,Q)$

Definition: $\mathrm{Lift}_{SO(V,Q)}^{Sp(W))}(\theta_\pi)(g) = \theta_\pi(g') \quad (g \leftrightarrow g')$

= conjugation invariant function on $Sp(W)_0$

Is this the character of a (virtual) representation π' of $Sp(W)$? If so:

$$\mathrm{Lift}_{SO(V,Q)}^{\widetilde{Sp}(2n,\mathbb{R})}(\theta_\pi) = \theta_{\pi'}$$

or

$$\mathrm{Lift}_{SO(V,Q)}^{\widetilde{Sp}(2n,\mathbb{R})}(\pi) = \pi'$$

Obviously not

Less naive guess:

$$\mathrm{Lift}_{SO(V,Q)}^{Sp(W)}(\theta_\pi)(g) = \frac{|\Delta_{SO}(g')|}{|\Delta_{Sp}(g)|}\theta_\pi(g')$$

$|\Delta_G(g)|$ = Weyl denominator (absolute value is well defined, independent of choice of positive roots)

Less obviously not

$p : \widetilde{Sp}(W) \to Sp(W)$ (metaplectic group)

$\omega^\psi = \omega_+^\psi \oplus \omega_-^\psi$ = oscillator representation
(choice additive character ψ, see Savin's lecture…

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(choice additive character ψ, see Savin's lecture…drop it from notation)

Definition: $\widetilde{g} \in \widetilde{Sp}(2n, \mathbb{R})_0$:

$$\Phi(\widetilde{g}) = \theta_{\omega_+}(\widetilde{g}) - \theta_{\omega_-}(\widetilde{g})$$

Lemma: $\widetilde{g} \in \widetilde{Sp}(W)_0, g = p(\widetilde{g}) \to g' \in SO(V, Q)$:

$$\begin{aligned} |\Phi(\widetilde{g})| &= \frac{|\Delta_{SO}(g')|}{|\Delta_{Sp}(g)|} \\ &= |\det(1+g)|^{-\frac{1}{2}} \end{aligned}$$

Digression: $G = Spin(2n)$, π = spin representation

$$|\theta_\pi(\widetilde{g})| = |\det(1+g)|^{\frac{1}{2}}$$

Stabilize:

Work only with $SO(n+1,n)$ (split)

$\pi \;\; SO(n+1,n), \theta_\pi$ is stable if $SO(2n+1,\mathbb{C})$ conjugation invariant

Definition: $Sp(2n,\mathbb{R}) \ni g \overset{st}{\longleftrightarrow} g' \in SO(n+1,n)$ if g, g' have the same nontrivial eigenvalues
(consistent with [is the stabilization of] earlier definition)

π = stable virtual character of $SO(n+1,n)$

Definition:

$$\boxed{\mathrm{Lift}_{SO(n+1,n)}^{\widetilde{Sp}(W)}(\theta_\pi)(\widetilde{g}) = \Phi(\widetilde{g})\theta_\pi(g') \quad (p(\widetilde{g}) \overset{st}{\longleftrightarrow} g')}$$

Theorem (A, 1998)

$\mathrm{Lift}_{SO(n+1,n)}^{\widetilde{Sp}(W)}$ is a bijection between

stable virtual representations of SO(n+1,n)

and

stable genuine virtual representations of $\widetilde{Sp}(2n,\mathbb{R})$

Write $\widetilde{\pi} = \mathrm{Lift}_{SO(n+1,n)}^{\widetilde{Sp}(2n,\mathbb{R})}(\pi)$

$\widetilde{Sp}(W)$: stable means $\theta(\widetilde{g}) = \theta(\widetilde{g}')$ if
(1) $p(\widetilde{g})$ is $Sp(2n,\mathbb{C})$ conjugate to $p(\widetilde{g}')$

(2) $\Phi(\widetilde{g}) = \Phi(\widetilde{g}')$.

proof: Hirai's matching conditions.

(necessary and sufficient conditions for a function to be the character of a representation)

Problem: Find an integral transform or other natural realization of this lifting.

Note: This result (in fact this entire talk) is consistent with, and partly motivated by, results of Savin (for example his lecture from this conference)

(1)-(3): Lifting from $GL(n,\mathbb{F})$ to $\widetilde{GL}(n,\mathbb{F})$

(Flicker, Kazhdan-Patterson, A-Huang)

$G = GL(n,\mathbb{F}) = GL(n)$ $\mathbb{F}$ is p-adic or real

$p : \widetilde{GL}(n) \to GL(n)$ non-trivial two-fold cover

Definition: $\phi(g) = s(g)^2$ ($s : GL(n) \to \widetilde{GL}(n)$ any section)

Definition: $h \in GL(n), \widetilde{g} \in \widetilde{GL}(n)$

$$\Delta(h,\widetilde{g}) = \frac{|\Delta(h)|}{|\Delta(\widetilde{g})|}\tau(h,\widetilde{g})$$

where $\tau(h,\widetilde{g})^2 = 1 \ldots$

(1)-(3): Lifting from $GL(n,\mathbb{F})$ to $\widetilde{GL}(n,\mathbb{F})$

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$$\Delta(h,\widetilde{g}) = \frac{|\Delta(h)|}{|\Delta(\widetilde{g})|}\tau(h,\widetilde{g})$$

where $\tau(h,\widetilde{g})^2 = 1 \ldots$(a little tricky to define)

Definition:

$$\boxed{\mathrm{Lift}_{GL(n)}^{\widetilde{GL}(n)}(\theta_\pi)(\widetilde{g}) = c \sum_{p(\phi(h))=p(\widetilde{g})} \Delta(h,\widetilde{g})\theta_\pi(h)}$$

next result:
Flicker: $n=2$, all $\mathbb{F}$
Kazhdan and Patterson: all n, $\mathbb{F}$ p-adic
A-Huang: all n, $\mathbb{F}=\mathbb{R}$

Theorem: π = virtual representation of $GL(n)$

(1) $\mathrm{Lift}_{GL(n)}^{\widetilde{GL}(n)}(\theta_\pi)$ is (the character of) a virtual representation or 0

(2) If π is irreducible and unitary then $\mathrm{Lift}_{GL(n)}^{\widetilde{GL}(n)}(\theta_\pi)$ is $\pm$ irreducible and unitary or 0

(3) $\mathrm{Lift}_{GL(n)}^{\widetilde{GL}(n)}(\mathbb{C})=\widetilde{\pi}_0$: a small, irreducible, unitary representations with infinitesimal character $\rho/2$ [Huang's thesis, Wallach's talk (n=3)]

Remark: Lift commutes with the Euler characteristic of cohomological induction (surprising)

Remark: Renard and Trapa have an example where π is irreducible (but not unitary) and $\mathrm{Lift}(\pi)$ is reducible.

(5) Lifting for simply laced real groups (joint with R. Herb)

G: complex, connected, reductive, simply laced
for this talk assume G_d simply connected (ρ exponentiates to G_d suffices)

$G(\mathbb{R})$ real form of G

$p : \widetilde{G(\mathbb{R})} \to G(\mathbb{R})$: admissible two-fold cover of $G(\mathbb{R})$
(admissible: nonlinear cover of each simple factor for which this exists)
Recall (Wallach's talk): nonlinear covers almost always exist

Definition:

$$\phi(g) = s(g)^2 \quad (g \in G(\mathbb{R}), s : G(\mathbb{R}) \to \widetilde{G(\mathbb{R})} \text{ any section})$$

Lemma:

(1) ϕ is well defined (independent of s)

(2) ϕ induces a map on conjugacy classes

(3) $g \in H(\mathbb{R}) = \text{Cartan} \Rightarrow \phi(g) \in Z(\widetilde{H(\mathbb{R})})$

proof:

(1) obvious

(2) obvious

(3) obvious

Lemma:

(1) ϕ is well defined (independent of s)

(2) ϕ induces a map on conjugacy classes

(3) $g \in H(\mathbb{R}) = \text{Cartan} \Rightarrow \phi(g) \in Z(\widetilde{H(\mathbb{R})})$

proof:

(1) obvious

(2) obvious

(3) obvious ($\phi(g) \in \widetilde{H(\mathbb{R})^0} \subset Z(\widetilde{H(\mathbb{R})})$)

[Suppressing for this talk: replace $G(\mathbb{R})$ by $\overline{G}(\mathbb{R})$ for an (allowed) quotient $\overline{G}$ of G - still true, less obvious, need stable in (2)]

$\widetilde{\pi}$ genuine representation of $\widetilde{G(\mathbb{R})}$,

$\widetilde{g} \in \widetilde{H(\mathbb{R})}$ = Cartan

Lemma (originally in Flicker)

$$\widetilde{g} \notin Z(\widetilde{H(\mathbb{R})}) \Rightarrow \theta_{\widetilde{\pi}}(\widetilde{g}) = 0$$

proof: $\widetilde{h} \notin Z(\widetilde{H(\mathbb{R})})$

$\widetilde{g}\widetilde{h}\widetilde{g}^{-1} \neq \widetilde{h} \quad (\widetilde{g} \in \widetilde{H(\mathbb{R})})$

projecting to $H(\mathbb{R})$ implies

$\widetilde{g}\widetilde{h}\widetilde{g}^{-1} = z\widetilde{h}$ (p(z)=1)

$\theta_\pi(\widetilde{h}) = \theta_\pi(\widetilde{g}\widetilde{h}\widetilde{g}^{-1}) = \theta_\pi(z\widetilde{h}) = -\theta_\pi(\widetilde{h})$

[Heisenberg group over $\mathbb{Z}/2\mathbb{Z}$]

Transfer Factors

Assume G is semisimple, simply connected ($\Rightarrow G(\mathbb{R})$ is connected)

$H(\mathbb{R})$ = Cartan, Φ^+ positive roots

$$\Delta(g,\Phi^+) = e^{\rho}(g)\prod_{\alpha\in\Phi^+}(1-e^{-\alpha}(g))$$

Definition: $h \in H(\mathbb{R})^0$, $\widetilde{g} \in \widetilde{H(\mathbb{R})}$, $p(\widetilde{g}) = h^2 \in H(\mathbb{R}) \cap G(\mathbb{R})_0$

$$\Delta(h,\widetilde{g}) = \frac{\Delta(h,\Phi^+)}{\Delta(g,\Phi^+)}\frac{\phi(h)}{\widetilde{g}}$$

$p(\phi(h)/\widetilde{g}) = h^2/p(\widetilde{g}) = 1$: $\quad \phi(h)/\widetilde{g} = \pm 1$, genuine function in $\widetilde{g}$

Obvious: $\Delta(h, \widetilde{g})$ is independent of choice of Φ^+ ($h \in H(\mathbb{R})^0$ here)

Punt: It is possible to extend the previous construction to general

$G(\mathbb{R})$, and to put conditions on $\Delta(h, \widetilde{g})$ so that the number of allowed extensions to $H(\mathbb{R}) \cap G(\mathbb{R})_0$ is acted on simply transitively by $G(\mathbb{R})/G(\mathbb{R})^0$.

(hard: reduction to the maximally split Cartan subgroup, Cayley transforms, need to make the Hirai conditions hold...)

So: fix transfer factors $\Delta(h,\widetilde{g})$

Definition: π = stable virtual representation of $G(\mathbb{R})$:

$$\boxed{\mathrm{Lift}_{G(\mathbb{R})}^{\widetilde{G(\mathbb{R})}}(\theta_\pi)(\widetilde{g}) = c \sum_{p(\phi(h))=p(\widetilde{g})} \Delta(h,\widetilde{g})\theta_\pi(h)}$$

Theorem: (joint with R. Herb)

(1) $\mathrm{Lift}_{G(\mathbb{R})}^{\widetilde{G(\mathbb{R})}}(\theta_\pi)$ is the character of a virtual genuine representation $\widetilde{\pi}$ of $\widetilde{G(\mathbb{R})}$ or 0

Theorem: (joint with R. Herb)

(1) $\mathrm{Lift}_{G(\mathbb{R})}^{\widetilde{G(\mathbb{R})}}(\theta_\pi)$ is the character of a virtual genuine representation $\widetilde{\pi}$ of $\widetilde{G(\mathbb{R})}$ or 0 - write $\mathrm{Lift}(\pi) = \widetilde{\pi}$

(2) Infinitesimal character: $\lambda \to \lambda/2$

(3) Every genuine virtual character of $\widetilde{G(\mathbb{R})}$ is a summand of some $\mathrm{Lift}_{G(\mathbb{R})}^{\widetilde{G(\mathbb{R})}}(\pi)$

(4) Lift takes (stable) standard modules to (sums of) standard modules

More on (4):

$I^{st}(\chi)$ = stabilized standard module defined by character χ of $H(\mathbb{R})$

$I^{st}(\chi) = \sum_w I(w\chi) \quad (w \in W(M)\backslash W_i)$

$$\boxed{\mathrm{Lift}(I^{st}(\chi)) = \sum_w I(\mathrm{Lift}_{H(\mathbb{R})}^{\widetilde{H}(\mathbb{R})}(w\chi))}$$

proof: Hirai's matching conditions

Very subtle point: need stability for the matching conditions to hold.

Remark:

(1) Some terms on the RHS are 0.

(2) The non-zero terms on the RHS have distinct central characters.

Remark: The notion of stability is probably not interesting for $\widetilde{G(\mathbb{R})}$ in the simply laced case; “L-packets” are (close to) singletons.

Question: Irreducibility of $\mathrm{Lift}(\pi)$?

Remark: The notion of stability is probably not interesting for $\widetilde{G(\mathbb{R})}$ in the simply laced case; "L-packets" are (close to) singletons.

Question: Irreducibility of $\text{Lift}(\pi)$? Unitarity?

Application to small representations

Related to lectures here by: Savin, Wallach, Kobayashi, Howe;

Application to small representations

Related to lectures here by: Savin, Wallach, Kobayashi, Howe;
Work by Friedberg, Loke, Sanchez, Trapa, Vogan, Weissman, Zhu, many others...

Corollary: $\widetilde{\pi}_0 = \mathrm{Lift}_{G(\mathbb{R})}^{\widetilde{G(\mathbb{R})}}(\mathbb{C})$ is a (non-zero) small virtual genuine character of $\widetilde{G(\mathbb{R})}$ of infinitesimal character $\rho/2$.

Usually (always?) $\widetilde{\pi}_0$ is irreducible or the sum of a very small number of irreducible representations, with distinct central characters

Small: If $\widetilde{\pi}_0$ is irreducible, it has Gelfand-Kirillov dimension $\le \frac{1}{2}(|\Delta| - |\Delta(\frac{\rho}{2})|)$

$\Delta(\frac{\rho}{2}) = \{\alpha \mid \langle \frac{\rho}{2}, \alpha^\vee \rangle \in \mathbb{Z}\}$ (integral roots defined by $\frac{\rho}{2}$)

Character formula I:

(Roughly):

$$\theta_{\widetilde{\pi}_0}(\widetilde{g}) = \frac{\sum_{w \in W(\Delta(\rho/2))} \operatorname{sgn}(w) e^{w\rho/2}(\widetilde{g})}{\Delta(\widetilde{g})}$$

(can be made precise)

Direct application of the lifting formula

Character formula II:

$$\widetilde{\pi}_0 = \sum_{(\widetilde{H(\mathbb{R})}, \widetilde{\chi})} \pm I(\widetilde{H(\mathbb{R})}, \chi)$$

where the sum runs over all $H(\mathbb{R})$ and most (all?) genuine irreducible representations χ of $\widetilde{H(\mathbb{R})}$ with $d\chi = \rho/2$

proof: Lift the Zuckerman character formula for $\mathbb{C}$

Other results

Better: replace $G(\mathbb{R})$ with $\overline{G}(\mathbb{R})$

Example: $\mathrm{Lift}_{PGL(2,\mathbb{R})}^{\widetilde{SL}(2,\mathbb{R})}$ is better than $\mathrm{Lift}_{SL(2,\mathbb{R})}^{\widetilde{SL}(2,\mathbb{R})}$:

$$\mathrm{Lift}_{SL(2,\mathbb{R})}^{\widetilde{SL}(2,\mathbb{R})}(\mathbb{C}) = \omega_+^{\psi} + \omega_+^{\overline{\psi}}$$

$$\mathrm{Lift}_{PGL(2,\mathbb{R})}^{\widetilde{SL}(2,\mathbb{R})}(\mathbb{C}) = \omega_+^{\psi}$$

Point: $\phi(\overline{H}(\mathbb{R}))$ is a bigger subset of $Z(\widetilde{H(\mathbb{R})})$

Hard work in A-Herb to allow (certain) $\overline{G}$

Two root length case (work in progress with R. Herb); Lifting will be from real form of $G^{\vee}(\mathbb{R})$ (generalizing $\widetilde{Sp}(2n,\mathbb{R})/SO(n+1,n)$ case)

Vogan Duality for nonlinear groups

Closely related to Lifting, and to the "L-group (?)" for nonlinear groups.

1) $\widetilde{Sp}(2n, \mathbb{R})$: D. Renard, P. Trapa

2) $\widetilde{G(\mathbb{R})}$ in type A: Renard, Trapa

3) $\widetilde{Spin}(2n+1)$: S. Crofts

4) $\widetilde{G(\mathbb{R})}$ for G simpy laced: A, Trapa

Long term goal:

Bring nonlinear groups into the Langlands program

or as a first step:

Bring nonlinear groups into the Atlas program