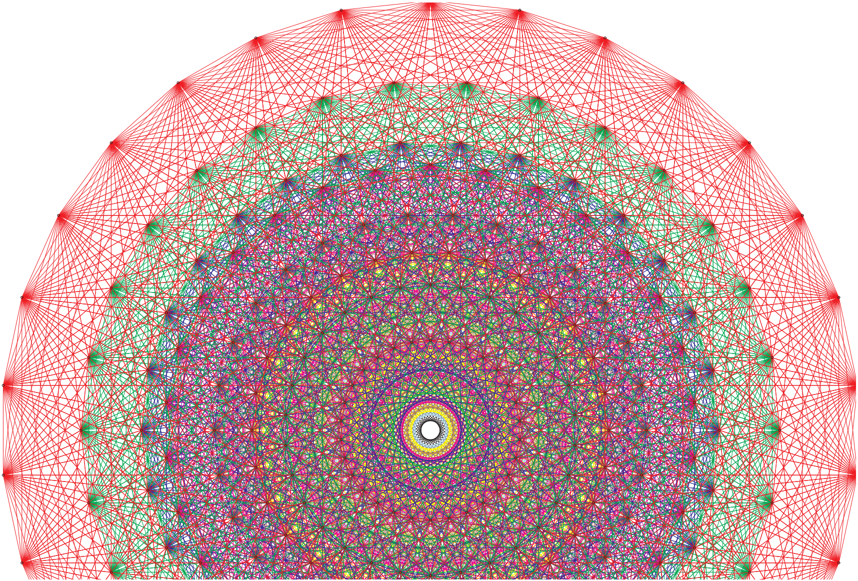
Atlas of Lie Groups and Representations

www.liegroups.org



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Atlas of Lie Groups and Representations:

Take this idea seriously

p-adic groups

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Question: Is there a finite algorithm to compute:

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(So far the answer seems to be no. . .)

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(2,157 of them = .41% are unitary)

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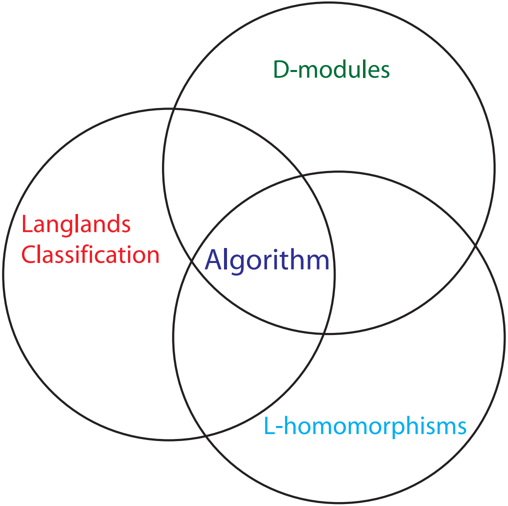
1) explicit: a computable combinatorial set
2) natural: make the Kazhdan-Lusztig-Vogan polynomials computable


D-modules
Langlands
Classification
Algorithm
L-homomorphisms

Three pictures

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Langlands classification: induced from discrete series, characters of Cartan subgroups

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$\mathcal{D}$-modules: local systems on $K(\mathbb{C})$ orbits on $G(\mathbb{C})/B(\mathbb{C})$

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For now assume G is simply connected, adjoint and $\mathrm{Out}(G) = 1$ (Examples: $G = G_2, F_4$ or E_8)

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Definition:

$$\mathcal{C}(G(\mathbb{R}),\rho)=\{(H(\mathbb{R}),\chi)\}/G(\mathbb{R})$$

$H(\mathbb{R})$=Cartan subgroup
χ = character of $H(\mathbb{R})$ with $d\chi=\rho$

The Langlands Classification

$(H(\mathbb{R}),\chi) \rightarrow I(H(\mathbb{R}),\chi)$=standard module (induced from discrete series of $M(\mathbb{R})$)
$\rightarrow \pi(H(\mathbb{R}),\chi)$ (unique irreducible quotient)

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Theorem: The map $(H(\mathbb{R}),\chi) \rightarrow \pi(H(\mathbb{R}),\chi)$ induces a canonical bijection:

$$\boxed{\Pi(G(\mathbb{R}),\rho) \overset{1-1}{\longleftrightarrow} \mathcal{C}(G,\rho)}$$

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In particular:

$$|\Pi(G(\mathbb{R}), \rho)| = \sum_i |W/W(G(\mathbb{R}), H(\mathbb{R})_i)||H(\mathbb{R})/H(\mathbb{R})_i|$$

$H(\mathbb{R})_1, \ldots, H(\mathbb{R})_n$ are representatives of the conjugacy classes of Cartan subgroups.

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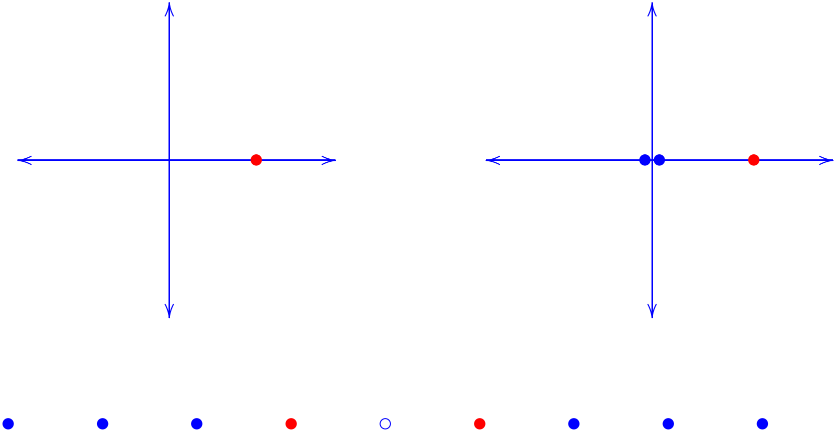
$SL(2,\mathbb{R})$ has 4 irreducible representations of infinitesimal character ρ

Example: $G = SL(2, \mathbb{R})$, infinitesimal character $=\rho$



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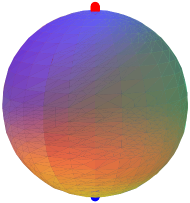
Theorem: (Vogan, Beilinson/Bernstein) There is a natural bijection

$$\boxed{\Pi(G(\mathbb{R}), \rho) \overset{1-1}{\longleftrightarrow} \mathcal{D}(G, K, \rho)}$$

Example: $G = SL(2, \mathbb{C})$, $G(\mathbb{R}) = SL(2, \mathbb{R})$

$\mathcal{B}$ is the sphere = $\mathbb{C} \cup \infty$



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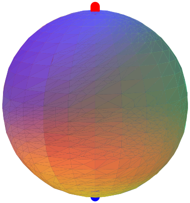

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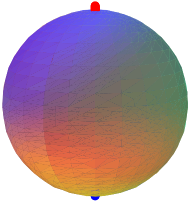

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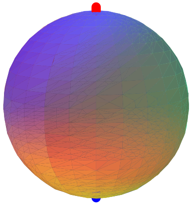

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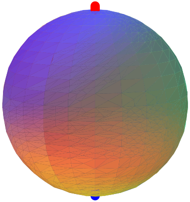

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Isotropy group: 1,1,$\mathbb{Z}/2\mathbb{Z} \to$ 4 representations

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Definition:

$$\mathcal{L}(G,\rho) = \{(\phi,\chi)\}/G^\vee$$

$\phi : W_{\mathbb{R}} \to G^\vee$, ($\phi(\mathbb{C}^\times)$ is semisimple, “infinitesimal character ρ”)
χ = local system on $\Omega^\vee = G^\vee \cdot \phi$
= character of $\mathrm{Stab}(\phi)/\mathrm{Stab}(\phi)^0$

Note: different real forms of G all have the same $G^\vee$ (no K here).
This version must take this into account (Vogan's super packets)

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Theorem: There is a natural bijection

$$\boxed{\coprod_i \Pi(G(\mathbb{R})_i, \rho) \overset{1-1}{\longleftrightarrow} \mathcal{L}(G, \rho)}$$

where $G_1(\mathbb{R}), \ldots, G_n(\mathbb{R})$ are the real forms of G.
(this version: book by A/Barbasch/Vogan)

Recapitulation

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(1) Character Data (orbits of $G(\mathbb{R})$ on Cartans):

$$\Pi(G(\mathbb{R}), \rho) \stackrel{1-1}{\longleftrightarrow} \mathcal{C}(G(\mathbb{R})) = \{(H(\mathbb{R}), \chi)\}/G(\mathbb{R})$$

(2) $\mathcal{D}$-modules (orbits $\mathcal{O}$ of K on G/B):

$$\Pi(G(\mathbb{R}), \rho) \stackrel{1-1}{\longleftrightarrow} \mathcal{D}(G, K, \rho) = \{(x, \chi)\}/K$$

(3) L-homomorphisms (orbits $\Omega^\vee$ of $G^\vee$ on L-homomorphisms):

$$\coprod_{i=1}^{n} \Pi(G(\mathbb{R})_i, \rho) \stackrel{1-1}{\longleftrightarrow} \mathcal{L}(G, \rho) = \{(\phi, \chi)\}/G^\vee$$

In each case there is some geometric data, and a character of a finite abelian group (two-group)

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We'd rather talk about orbits than characters of $(\mathbb{Z}/2\mathbb{Z})^n$
(Matching the three pictures: easy up to χ)

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Definition: Orbit $\Omega^\vee$ of $G^\vee$ on L-homomorphisms $\rightarrow$ L-packet

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Theorem (Vogan): The intersection of an L-packet and an R-packet is at most one element.

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Corollary: $\Pi(G(\mathbb{R}), \rho)$ is parametrized by a subset of pairs

$$(K \text{ orbit on } \mathcal{B}, G^\vee \text{ orbit on L-homomorphisms})$$

via

$$(\mathcal{O}, \Omega^\vee) \to \Pi_R(G(\mathbb{R}), \mathcal{O}) \cap \Pi_L(G(\mathbb{R}), \Omega^\vee)$$

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Which pairs?...

K-orbits on the dual side

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$K_1^\vee, \dots, K_n^\vee =$ complexified maximal compacts of real forms of $G^\vee$.

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$G^\vee$ orbits of L-homomorphisms are exactly the same thing as K orbits on G/B on the dual side.

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Corollary: $\Pi(G(\mathbb{R}), \rho)$ is parametrized by a subset of pairs

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This reduces the problem to:

$$\boxed{\text{Parametrize } K \text{ orbits on } \mathcal{B} = G/B}$$

(applied to G and $G^\vee$)

K orbits on G/B

Definition:

$$\mathcal{X} = \{x \in \mathrm{Norm}_G(H) \mid x^2 = 1\}/H$$

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(union over real forms, corresponding $K_1, \ldots, K_n$)

Sketch of Proof

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$$\mathcal{P} = \{(x, B)\}/G \ (x^2 = 1, B = \text{Borel})$$

$$\coprod_i K_i\backslash\mathcal{B} \xleftarrow[x]{1-1} \mathcal{P} \xrightarrow[B]{1-1} \mathcal{X}$$

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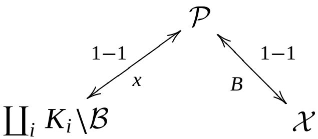

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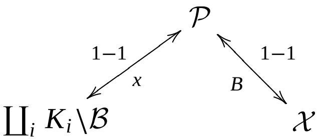

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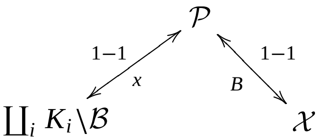

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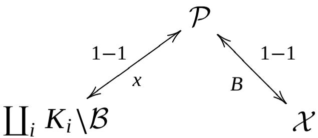

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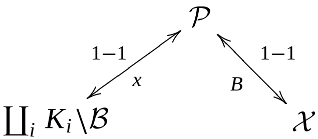

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$$(x, B) \sim_G (x', B_0) \to x' \in \mathcal{X} \qquad (\text{wlog } x' \in \text{Norm}(H))$$

$K\backslash G/B$ for $Sp(4,\mathbb{R})$ and $SO(3,2)$:

$Sp(4,\mathbb{R})$:

```
 0:     1   2     6   4  [nn]  0  0
 1:     0   3     6   5  [nn]  0  0
 2:     2   0     *   4  [cn]  0  0
 3:     3   1     *   5  [cn]  0  0
 4:     8   4     *   *  [Cr]  2  1  2
 5:     9   5     *   *  [Cr]  2  1  2
 6:     6   7     *   *  [rC]  1  1  1
 7:     7   6    10   *  [nC]  1  2  2,1,2
 8:     4   9     *  10  [Cn]  2  2  1,2,1
 9:     5   8     *  10  [Cn]  2  2  1,2,1
10:    10  10     *   *  [rr]  3  3  1,2,1,2
```

$SO(3,2)$:

```
0:    0  1    3  2  [nn]  0  0
1:    1  0    *  2  [cn]  0  0
2:    5  2    *  *  [Cr]  2  1  2
3:    3  4    *  *  [rC]  1  1  1
4:    4  3    6  *  [nC]  1  2  2,1,2
5:    2  5    *  6  [Cn]  2  2  1,2,1
6:    6  6    *  *  [rr]  3  3  1,2,1,2
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Proposition

1) Real forms of $G \overset{1-1}{\longleftrightarrow} \mathcal{X}_1/W$ ($\mathcal{X}_1$= fiber over $1 \in W$)
2) Cartan subgroups in all real forms: $\mathcal{X}/W$
3) $W(G(\mathbb{R}), H(\mathbb{R})) = \mathrm{Stab}_W(x)$

The Parameter Space $\mathcal{Z}$

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$$\mathcal{Z} \subset \coprod_i K_i \backslash \mathcal{B} \times \coprod_j K_j^\vee \backslash \mathcal{B}^\vee$$

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(Canonical up to characters of $G_{qs}(\mathbb{R})/G_{qs}(\mathbb{R})^0$, $G^\vee_{qs}(\mathbb{R})/G^\vee_{qs}(\mathbb{R})^0$)

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In general:

1. Fix an inner class of real forms
2. Need twists $G^\Gamma = G \rtimes \Gamma$, $G^\vee \rtimes \Gamma$ ($\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$)
3. Require $x^2 \in Z(G)$ (not $x^2 = 1$)
4. Need several infinitesimal characters
5. Need strong real forms

The General Algorithm

$$\mathcal{X} = \{x \in \mathrm{Norm}_{G^\Gamma\backslash G}(H) \,|\, x^2 \in Z(G)\}/H$$

$\mathcal{X}^\vee$ similarly, $\mathcal{Z} = \{(x,y)\,|\,\dots\} \subset \mathcal{X} \times \mathcal{X}^\vee$ as before.

Theorem: There is a natural bijection

$$\mathcal{Z} \overset{1-1}{\longleftrightarrow} \coprod_{i\in S} \Pi(G(\mathbb{R})_i, \Lambda)$$

Λ = certain set of infinitesimal characters
S is the set of "strong real forms"

Reference: Algorithms for Representation Theory of Real Reductive Groups, preprint (`www.liegroups.org`), Fokko du Cloux, A

Cayley Transforms and Cross Actions

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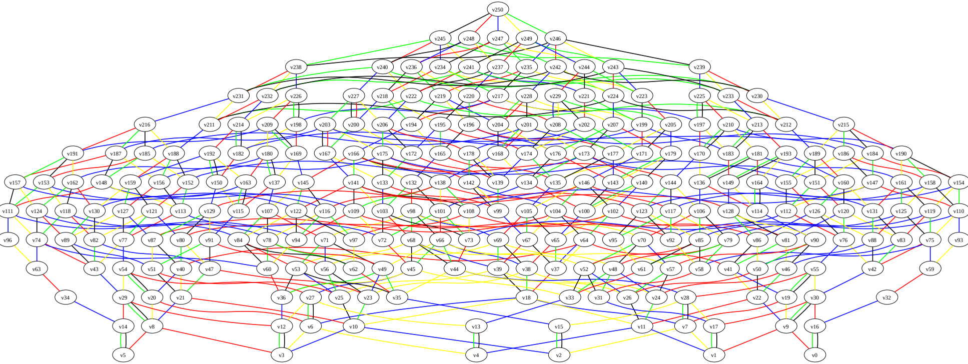
$$w \to w' = s_\alpha w \in W_2$$

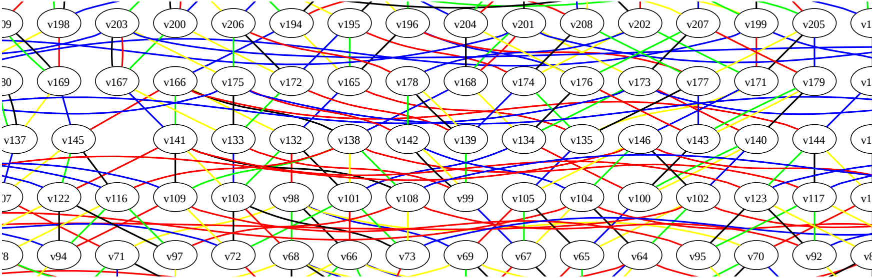
lifts to

$$x \to x' = \sigma_\alpha x$$

(Multivalued due to choice of σ_α: x' or $\{x_1', x_2'\}$)
This is the Cayley transform



$K\backslash G/B$ for $SO(5,5)$



Closeup of $SO(5,5)$ graph

Example: SL(2)/PGL(2)

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Example: SL(2)/PGL(2)

$\mathcal{O}$	x	x^2	K	$G_x(\mathbb{R})$	λ	rep	$\mathcal{O}^\vee$	y	y^2	$K^\vee$	$G_y^\vee(\mathbb{R})$	λ	rep
$\cdot$	I	I	G	$SU(2,0)$	ρ	$\mathbb{C}$	$\mathbb{C}^\times$	w	I	$O(2,\mathbb{C})$	$SO(2,1)$	2ρ	PS_+
$\cdot$	-I	I	G	$SU(0,2)$	ρ	$\mathbb{C}$	$\mathbb{C}^\times$	w	I	$O(2,\mathbb{C})$	$SO(2,1)$	2ρ	PS_-
$\{0\}$	t	-I	$\mathbb{C}^\times$	$SU(1,1)$	ρ	DS_+	$\mathbb{C}^\times$	w	I	$O(2,\mathbb{C})$	$SO(2,1)$	ρ	$\mathbb{C}$
$\{\infty\}$	-t	-I	$\mathbb{C}^\times$	$SU(1,1)$	ρ	DS_-	$\mathbb{C}^\times$	w	I	$O(2,\mathbb{C})$	$SO(2,1)$	ρ	sgn
$\mathbb{C}^\times$	w	-I	$\mathbb{C}^\times$	$SU(1,1)$	ρ	$\mathbb{C}$	$\{\infty\}$	t	I	$O(2,\mathbb{C})$	$SO(2,1)$	ρ	DS
$\mathbb{C}^\times$	w	I	$O(2,\mathbb{C})$	$SU(1,1)$	ρ	PS	$\cdot$	I	I	$G^\vee$	$SO(3)$	ρ	$\mathbb{C}$

$SL(2)/PGL(2)$ via atlas output

```
main: type
Lie type: A1 sc s
main: block
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
entering block construction ...
2
done
Name an output file (return for stdout, ? to abandon):
0(0,1):  1    (2,*)    [i1]  0
1(1,1):  0    (2,*)    [i1]  0
2(2,0):  2    (*,*)    [r1]  1  1
```

Example: $Sp(4, \mathbb{R})$

```
main: type
Lie type: C2 sc s
main: block
(weak) real forms are:
0: sp(2)
1: sp(1,1)
2: sp(4,R)
enter your choice: 2
possible (weak) dual real forms are:
0: so(5)
1: so(4,1)
2: so(2,3)
enter your choice: 2
entering block construction ...
10
done
Name an output file (return for stdout, ? to abandon):
 0( 0,6):   1   2    ( 6, *)  ( 4, *)    [i1,i1]  0
 1( 1,6):   0   3    ( 6, *)  ( 5, *)    [i1,i1]  0
 2( 2,6):   2   0    ( *, *)  ( 4, *)    [ic,i1]  0
 3( 3,6):   3   1    ( *, *)  ( 5, *)    [ic,i1]  0
 4( 4,4):   8   4    ( *, *)  ( *, *)    [C+,r1]  1  2
 5( 5,4):   9   5    ( *, *)  ( *, *)    [C+,r1]  1  2
 6( 6,5):   6   7    ( *, *)  ( *, *)    [r1,C+]  1  1
 7( 7,2):   7   6    (10,11)  ( *, *)    [i2,C-]  2  2,1,2
 8( 8,3):   4   9    ( *, *)  (10, *)    [C-,i1]  2  1,2,1
 9( 9,3):   5   8    ( *, *)  (10, *)    [C-,i1]  2  1,2,1
10(10,0):  11  10    ( *, *)  ( *, *)    [r2,r1]  3  1,2,1,2
11(10,1):  10  11    ( *, *)  ( *, *)    [r2,rn]  3  1,2,1,2
```

Example: E_8

```
real: type
Lie type: E8 sc s
main: blocksizes
              compact quaternionic split
compact       0       0            1
quaternionic  0       3,150        73,410
split         1       73,410       453,060
```

Unipotent Representations

Proposition: From the output of `atlas` one can list the special unipotent representations associated to a given nilpotent orbit.

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*) Fix a block $\mathcal{B}$ (`block`)

*) Fix nilpotent orbit $\mathcal{O}$ for $\mathfrak{g}^\vee$. Let $S = \{i_1, \ldots, i_r\}$ be the nodes of Dynkin diagram labelled 2. Let λ = corresponding infinitesimal character.

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(wgraph + calculation with character table of W)
4) For each such C list $\pi \in C$ with $\tau(\pi) = S$ (block)
5) Push these to λ
David Vogan has carried this out for E_8
(70 nilpotent orbits; 20 even ones; 143 unipotent representations with integral infinitesimal character for $E_8(split)$)
Conjecture (Arthur): These representations are unitary.

```
block dual to split group:
Rep        (x,y)      length Cartan  roots
133   (   133,320205)     0       0  [i1,i1,i1,i1,i1,i1,i1,i1]
140   (   140,320204)     1       1  [i1,i1,i1,i1,i1,i1,C+,r1]
42248 ( 40972,306175)     16      3  [C+,i1,C+,i1,C+,C+,C+,C-]
82083 ( 77494,287709)     21      6  [C-,C+,rn,C+,rn,C+,rn,C+]
124391(114466,263402)     24      2  [i1,C+,i1,i1,C+,C+,C-,i1]
124432(114507,263398)     24      2  [i1,i1,C+,C+,C-,i1,C+,i1]
132306(120375,257307)     25      3  [C+,C+,i1,i1,C+,C+,C-,C+]
191385(168884,220459)     29      1  [i1,i1,i1,i1,i1,i1,i1,C-]
198367(172894,213960)     30      4  [C-,C+,C+,C+,C+,C+,C+,C+]
205069(179284,210683)     30      2  [r1,i1,C+,i1,i1,i1,i1,C-]
225144(192668,195053)     32      5  [i1,rn,i1,C+,rn,C+,rn,C-]
233376(200324,190190)     32      2  [C-,i1,C+,C-,C+,C+,C-,C+]
233395(200343,190188)     32      2  [C-,C+,i1,i1,i1,C-,C+,C+]
237240(201594,186548)     33      6  [rn,C-,C+,C+,C+,C+,C+,C+]
243756(206740,180794)     33      3  [C+,i1,C+,C+,C-,C+,i1,C+]
244076(207060,180688)     33      3  [C+,C+,C+,C-,C+,C+,C+,C-]
252552(212118,174728)     34      4  [C+,C+,C+,C+,C-,C+,C+,C+]
258013(216823,170023)     34      3  [C+,C+,C+,i1,i1,i1,C-,C+]
258048(216858,170012)     34      3  [C+,i1,i2,C+,C-,C+,i1,C+]
288684(238673,147429)     36      2  [C+,i1,i1,i1,i1,C-,C+,i1]
309166(250360,129909)     38      4  [C+,C-,C+,C+,C+,C+,C+,C-]
320784(257336,120344)     39      4  [C+,C+,C+,i2,C-,C+,C+,C+]
...
453058(320205,   133)     64      9  [r2,r2,r2,r2,r2,r2,r2,r2]
...
block dual to compact group:
0    (0,    320205)       0       0  [ic,ic,ic,ic,ic,ic,ic,ic]
```

What next?

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