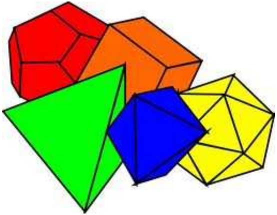
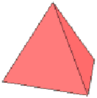
Regular Polyhedra in n Dimensions

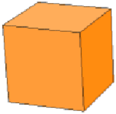


Jeffrey Adams
SUM Conference
Brown University
March 14, 2015
Slides at www.liegroups.org

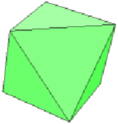
Platonic Solids



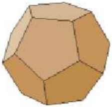
tetrahedron



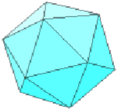
cube



octahedron



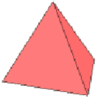
dodecahedron



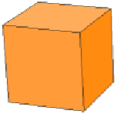
icosahedron

- What is a Platonic solid?
- Why are there exactly 5 of them?
- What about in other dimensions?

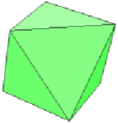
Platonic Solids



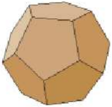
tetrahedron



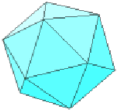
cube



octahedron



dodecahedron



icosahedron

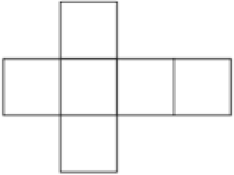
A Platonic Solid is a convex body such that:

- Each face is a regular polygon
- The faces are all identical
- The same number of faces meet at each vertex

Why are there 5 Platonic Solids?

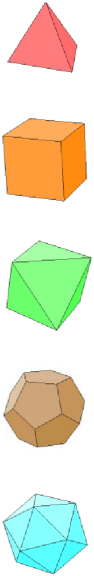
Many proofs...

Consider the graph of P:



Euler: $F - E + V = 2$

	F	E	V
tetrahedron	4	6	4
cube	6	12	8
octahedron	8	12	6
dodecahedron	12	30	20
icosahedron	20	30	12



Why there are 5 Platonic Solids

Suppose m_1 faces meet at each vertex ($m_1 = 3, 4, 5 \ldots$)

Each face is an m_2-gon ($m_2 = 3, 4, 5 \ldots$)

$m_1 V = 2E$, $m_2 F = 2E$

plug into $F - E + V = 2 \rightarrow$

$$\frac{2E}{m_2} - E + \frac{2E}{m_1} = 2$$

$$\boxed{\frac{1}{m_2} + \frac{1}{m_1} = \frac{1}{2} + \frac{1}{E}}$$

$3 \leq m_1, m_2 \ldots$ $m_1 = 6, m_2 = 3 \rightarrow \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ (contradiction)

Why there are 5 Platonic Solids

Only solutions:

$3 \leq m_1, m_2 \leq 5$

m_1	m_2	E	
3	3	6	tetrahedron
3	4	12	cube
4	3	12	octahedron
3	5	30	dodecahedron
5	3	30	icosahedron

$\{m_1, m_2\}$ is the Schläfli symbol of P

The faces and vertices being the same: the symmetry group of P acts transitively on faces, and on vertices.

The number of proper symmetries is $2E$.

G_0= proper symmetry group (rotations, not reflections)

	order of G_0	G_0
tetrahedron	12	A_4
cube/octahedron	24	S_4
dodecahedron/icosahedron	60	A_5

Regular Polyhedra in Higher Dimension

A polytope in $\mathbb{R}^n$ is a finite, convex body P bounded by a finite number of hyperplanes.

Each hyperplane ($n-1$ dimensional plane) intersects P in an $n-1$-dimensional polytope: a face F_{n-1}.

Repeat: $P \supset F_{n-1} \supset F_{n-2} \cdots \supset F_k \supset \cdots \supset F_0 = \text{vertex}$

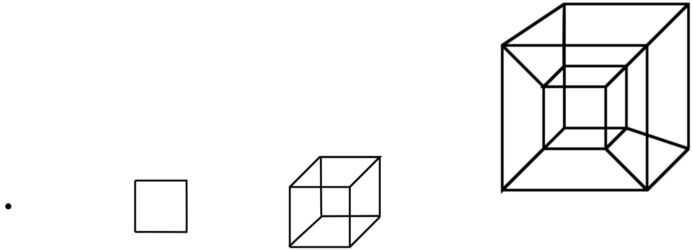
Definition: A Regular Polyhedron is a convex polytope in $\mathbb{R}^n$,such that the symmetry group acts transitively on the k-faces for all $0 \leq k \leq n$.

Each F_k is a k-dimensional regular polyhedron

Example: n-cube C_n or hypercube

Vertices $(\pm 1, \pm 1, \ldots, \pm 1)$

k-face: $(\overbrace{1, \ldots, 1}^{n-k}, \pm 1, \pm 1, \ldots, \pm 1) = C_k$



Symmetry group G:

S_n (permute the coordinates)

$\mathbb{Z}_2^n$ (2^n sign changes)

$G = S_n \ltimes \mathbb{Z}_2^n$ (wreath product)

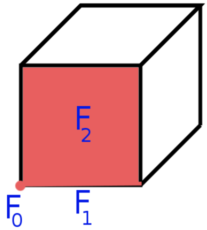
Regular Polyhedra in Higher Dimension

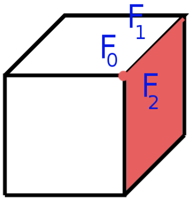
Equivalent definition:

A flag is a (maximal) nested sequence of faces:

$$\text{vertex} = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = P$$

Each face F_i is a lower-dimensional regular polyhedron.

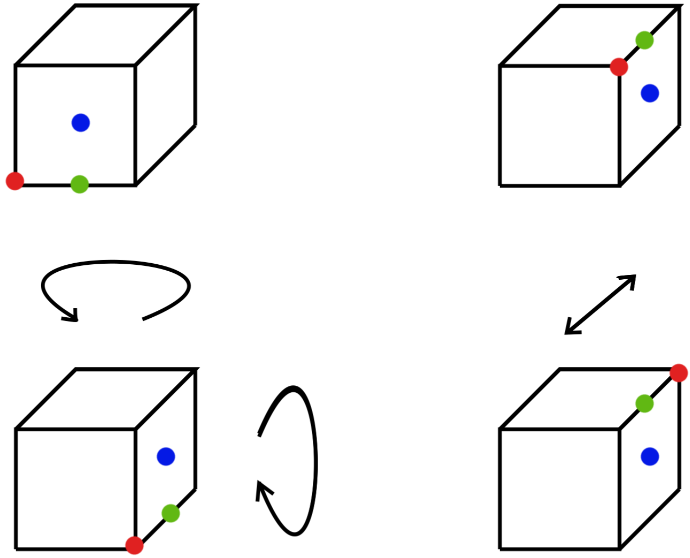




A Regular Polyhedron is a polytope whose symmetry group acts transitively on flags.

Any one flag can be taken to any other by a symmetry of P.

Symmetry Group and Flags



Symmetry Group and Flags

Theorem: The Symmetry group of G act simply transitively on Flags.

transitively: take any flag to any other

simply: Exactly one such symmetry (the only symmetry fixing a flag is the identity)

G= full symmetry group$\supset G_0$ of index 2

$$G \stackrel{1-1}{\leftrightarrow} \{\text{the set of flags}\}$$

	order of G
tetrahedron	$4 \times 3 \times 2 = 24$
cube/octahedron	$6 \times 4 \times 2 = 8 \times 3 \times 2 = 48$
dodecahedron/icosahedron	$12 \times 5 \times 2 = 20 \times 3 \times 2 = 120$

REFLECTIONS

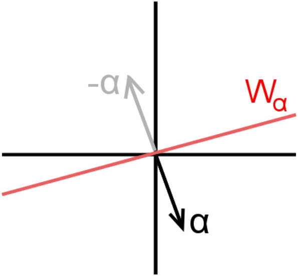
$\mathbb{R}^n$ with the usual inner product: $\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w}$.

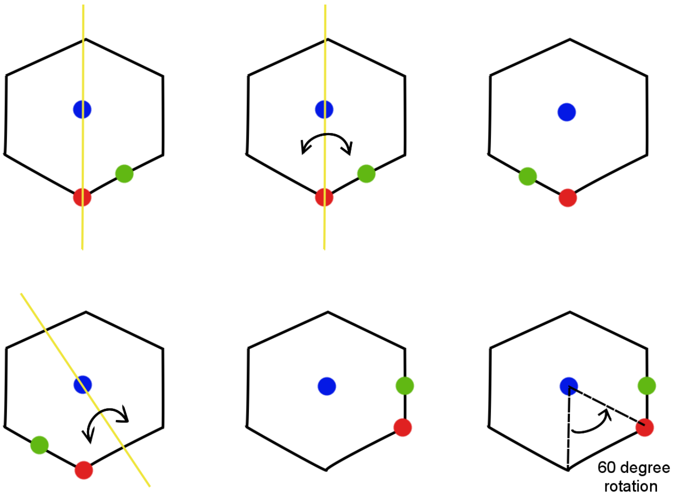
$\alpha \in V \rightarrow s_\alpha$:

$$s_\alpha(v) = v - \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

W_α=hyperplane orthogonal to α (dimension $n-1$)

$$s_\alpha(w) = w \quad (w \in W_\alpha)$$
$$s_\alpha(\alpha) = -\alpha$$




60 degree
rotation

Reflections in G

P is our regular polyhedron, with symmetry group G

fix a flag: $\mathcal{F} = F_0 \subset F_1 \cdots \subset F_n$

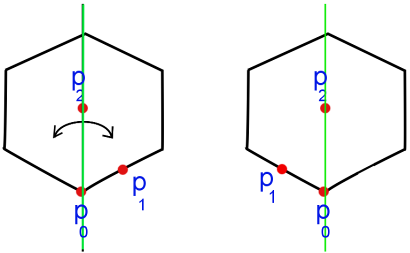
p_i : center of mass of F_i

Define a reflection s_i:

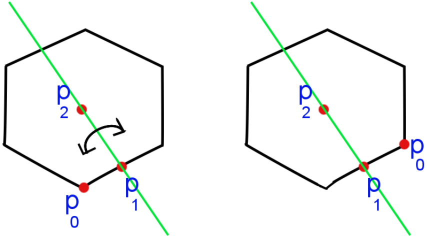
s_i is reflection fixing the hyperplane through $p_0, \ldots, p_{i-1}, \widehat{p_i}, p_{i+1}, \ldots, p_n$

Theorem: s_i is a symmetry of P, fixing all faces F_j except for F_i.

Reflections of Polyhedra



Fixes F_0, F_2 and moves F_1



Fixes F_1, F_2 and moves F_0

Theorem: s_i is a symmetry of P, fixing all faces F_j except for F_i.

Theorem: Every symmetry of P is a product of the reflections s_i

That is: you can move any flag $\mathcal{F}'$ to our fixed flag $\mathcal{F}$ by a series of these reflections, changing one face F_i at a time.

In other words: G is generated by $\{s_0, \ldots, s_{n-1}\}$

Coxeter Groups

A Coxeter group is generated by "reflections":

Abstractly: G has generators $s_1, \ldots, s_n$, and relations

$$s_i^2 = 1; \quad (s_i s_j)^{m_{ij}} = 1 \quad (m_{ij} = 2, 3, 4 \ldots, \infty)$$

Encode this information in a Coxeter graph:
Connect node i to node j with

Convention for common cases:

$m_{ij} = 2$:$(s_i s_j)^2 = 1$, $s_i s_j = s_j s_i$ (commute): no line

$m_{ij} = 3$:$(s_i s_j)^3 = 1$, unlabelled line

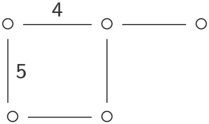
Coxeter Groups

Example:

$s_i^2 = 1, (s_i s_{i+1})^3 = 1,$ all other s_i, s_j commute

This is the symmetric group S_{n+1}, $\quad s_i = (i, i+1) = i \leftrightarrow i+1$

Very general construction (any graph with labels ≥ 2)

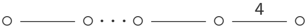
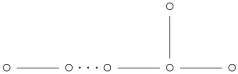
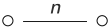
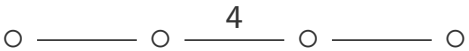
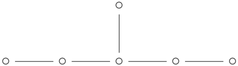
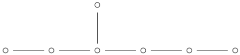


When is this abstract group finite?

Beautiful algebraic/geometric classification of these.

1. No loops
2. At most one branch point
3. $\circ\overset{a}{—}\circ\overset{b}{—}\circ \quad \Rightarrow a$ or $b \leq 3$
4. ...

This leads (quickly!) to the classification of finite Coxeter groups.

type	dimension	diagram	group
A_n	$n \geq 1$		S_{n+1}
B_n/C_n	$n \geq 2$		$S_n \ltimes \mathbb{Z}_2^n$
D_n	$n \geq 4$		$S_n \ltimes \mathbb{Z}_2^{n-1}$
$I_2(n)$	2		dihedral, order $2n$
H_3	3		120
H_4	4		14,400
F_4	4		11,52
E_6	6		51,840
E_7	7		2,903,040
E_8	8		696,729,600

Back to Regular Polyhedra

Question: Which of these groups can be the symmetry group of a regular polyhedron?

Recall s_i moves only the i-dimensional face F_i.

Key fact 1: If $j \neq i \pm 1$ then $s_i s_j = s_j s_i \Rightarrow$

Answer: The graph is a **line**

$$\circ \overset{m_{12}}{\text{——}} \circ \overset{m_{23}}{\text{——}} \circ \overset{m_{34}}{\text{——}} \circ \cdots \circ \overset{m_{(n-1)n}}{\text{——}} \circ$$

Key fact 2: $s_i s_{i+1}$ acts transitively on:

$$F_{i-1} \subset \{i - \text{faces}\} \subset F_{i+2}$$

The number of these i-faces is m_i: the Schläfli symbol of P

$s_i s_{i+1}$ has order m_i

Example: Cube

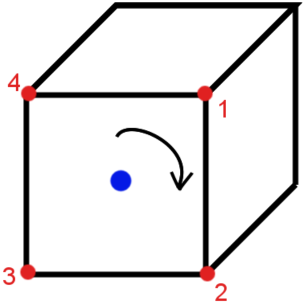
$s_i s_{i+1}$ acts transitively on

$F_{i-1} \subset \{i - \text{faces}\} \subset F_{i+2}$

$s_0 s_1$

$\emptyset \subset \{\text{vertices}\} \subset F_2$

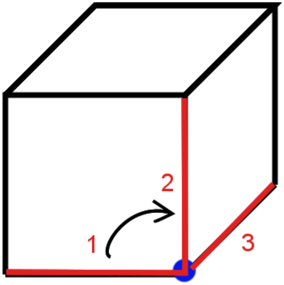
$m_0 = 4$



$s_1 s_2$

$F_0 \subset \{\text{edges}\} \subset F_3$

$m_1 = 3$



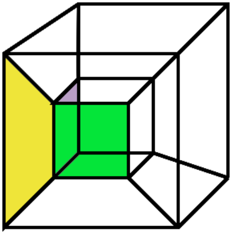
Example: 4-Cube

$$s_2 s_3$$

$$F_1 \subset \{\text{2-faces}\} \subset F_4$$

$$m_2 = 3$$

Schläfli symbol is$\{4, 3, 3\}$



Regular Polyhedra in n-dimensions

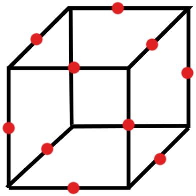
dimension	Polyhedron	diagram	G
2	n-gon	○ $\overset{n}{\text{———}}$ ○	dihedral
$n \geq 2$	n-simplex	○ ——— ○ · · · ○ ——— ○	S_{n+1}
$n \geq 3$	n-cube n-octahedron	○ ——— ○ · · · ○ ——— ○ $\overset{4}{\text{———}}$ ○	$S_n \ltimes \mathbb{Z}_2^n$
3	icosahedron dodecahedron	○ ——— ○ $\overset{5}{\text{———}}$ ○	120
4	600 cell 120 cell	○ ——— ○ ——— ○ $\overset{5}{\text{———}}$ ○	14,400
4	24 cell	○ ——— ○ $\overset{4}{\text{———}}$ ○ ——— ○	1,152

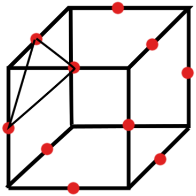
Other polyhedra with lots of symmetries?

See Coxeter's beautiful classics Regular Polytopes and Regular Complex Polytopes

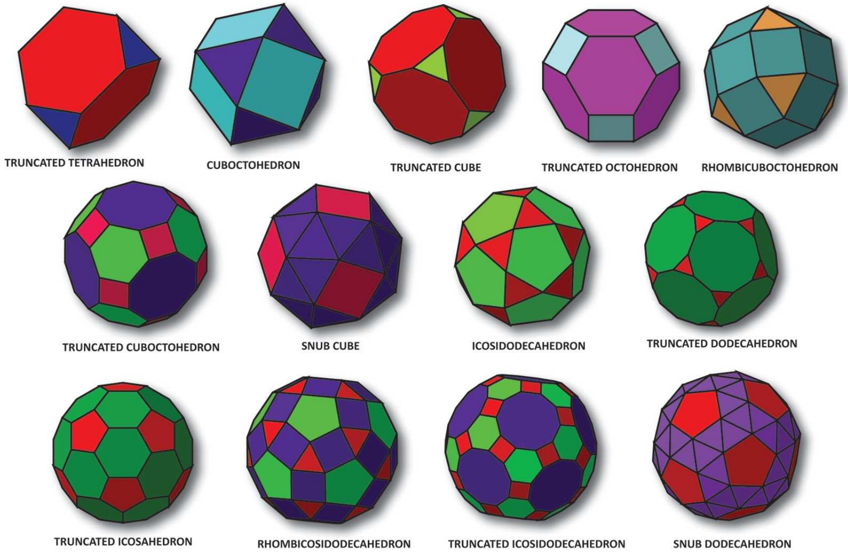
Example: root system of type A_3

Take the $G = S_4$-orbit of a midpoint of an edge of the cube:



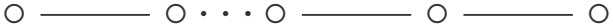
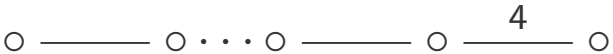
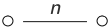
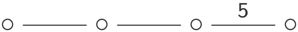
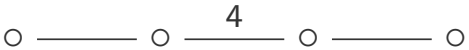
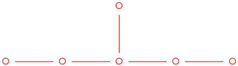
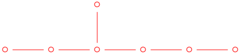


The Archimedean Solids



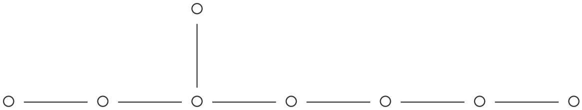
Exercise: Compute the symmetry group of each of these.

What about the other Coxeter groups?

type	dimension	diagram	group
A_n	$n \geq 1$		S_{n+1}
B_n/C_n	$n \geq 2$		$S_n \ltimes \mathbb{Z}_2^n$
D_n	$n \geq 4$		$S_n \ltimes \mathbb{Z}_2^{n-1}$
$I_2(n)$	2		dihedral, order $2n$
H_3	3		120
H_4	4		14,400
F_4	4		11,52
E_6	6		51,840
E_7	7		2,903,040
E_8	8		696,729,600

The root system E_8

E_8:



$G = W(E_8)$, $|G| = 696{,}729{,}600$

E_8 root system:

$\vec{v} = (a_1, a_2, \ldots, a_8) \subset \mathbb{R}^8$

- all $a_i \in \mathbb{Z}$ or all $a_i \in \mathbb{Z} + \frac{1}{2}$
- $\sum a_i \in 2\mathbb{Z}$
- $||\vec{v}||^2 = 2$

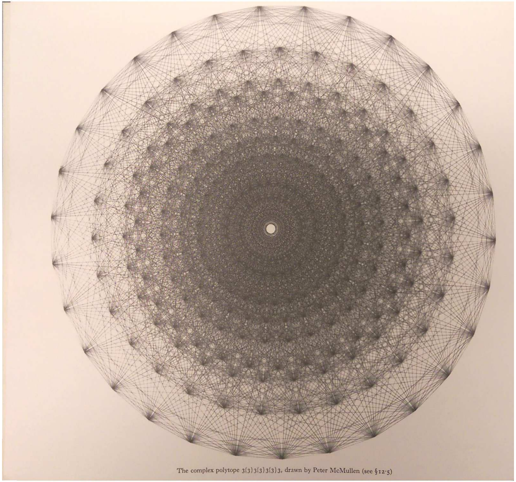
Exercise: This gives 240 vectors of length $\sqrt{2}$ in $\mathbb{R}^8$.

Question: What does this polyhedron look like?

Project it into $\mathbb{R}^2$

THE ROOT SYSTEM E_8

Frontspiece of *Regular Complex Polytopes* by H.S.M. Coxeter



The complex polytope 3{3}3{3}3{3}3 drawn by Peter McMullen

The root system E_8