Atlas of Lie Groups
and Representations

www.liegroups.org

The Unitary Dual

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RTNCG Conference

Institute Henri Poincaré
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- **David Vogan**
- Wai-Ling Yee
- Jiu-Kang Yu
- Gregg Zuckerman

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Atlas of Lie Groups and Representations (2002): Study this by computer

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2) Progress on Arthur's conjectures (giving a conceptual description of a large part of the unitary dual)

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Work in the setting of admissible $(\mathfrak{g}, K)$-modules (finite K-multiplicities)

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$\mathcal{M}_\gamma$ is finite dimensional, spanned by $\{$irreducible modules$\}$ *or* $\{$standard modules$\}$

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How do you compute the signature of a Hermitian form on an infinite dimensional vector space?

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So: compute the Hermitian form on $I(x, \lambda, \nu)$ by deforming to 0 (where it becomes tempered, i.e. unitary) and keeping track of the sign changes (inductive calculation)

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Conclusion: it is very difficult to formulate a precise algorithm

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So: write $I_c(\Gamma), J_c(\Gamma)$ for these representations equipped with their canonical c-forms.

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2022: Dougal Davis and Kari Vilonen proved a (slightly) weak version of this conjecture. Roughly speaking: the Hodge filtration, reduced $\pmod 2$ gives the c-form (later)

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2) π finite length $\Rightarrow \pi|_K = \sum_{i=1}^{n} a_i \pi_i|_K$ (π_i tempiric) (unique finite formula)

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Similarly: $\sum_{i=1}^{n}(a_i + b_i s)\pi_i$ represents the form on $(a_i + b_i)$ copies of $\pi_i|_K$, with a_i/b_i positive/negative forms.

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By induction: get an explicit formula

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Put (*) and (**) together:

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Then $J(\gamma)$ is unitary if and only if $z_i' \in \mathbb{Z}$ for all i (i.e. $z_i' = a_i + b_i s$ and $b_i = 0$).

Some Technical points

Strictly speaking $J_h(\gamma)$ isn't well defined. There are some choices along the way, but in the end the Hermitian form on $J(\gamma)$ (if it exists) is unique up to real scalar, and $J(\gamma)$ is unitary if and only if this form is positive or negative definite.

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To go from $I_c(\gamma)$ to $I_h(\gamma)$ is easy in the equal rank case. Otherwise it requires a long (painful!) digression in twisted KLV polynomials. See Lusztig-Vogan (2014) and Adams-Vogan (2015).

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$(a_0 + a_1 v + \dots a_n v^n) * \mu$ means: μ has multiplicity a_i in the i^{th} graded piece of the Hodge filtration

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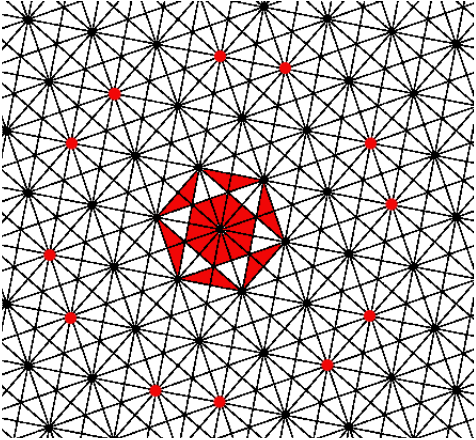

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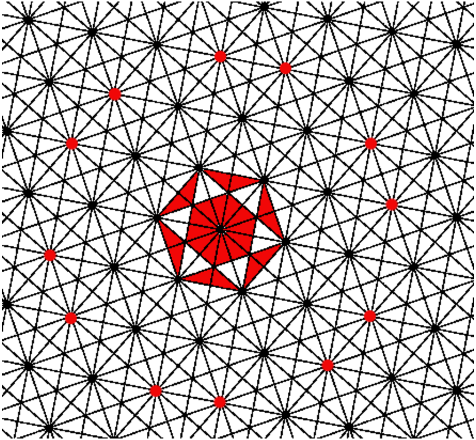

How does this generalize?

The FPP conjecture

Definition: The fundamental parallelepiped (FPP) is

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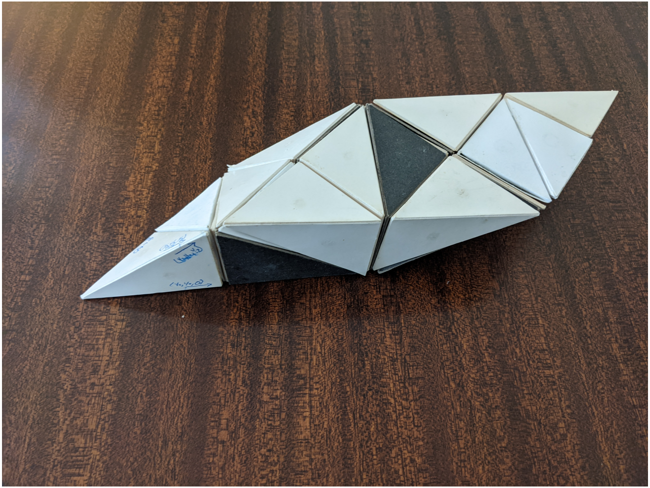

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Remark: The proof (by Davis/Mason-Brown) is an application of Schmid and Vilonen's Hodge theory conjectures (proved by Davis/Vilonen).

Description of the Unitary Dual

Remark: The set of pairs (x, λ) such that there exists ν such that $\gamma = \frac{(1+\theta_x)\lambda}{2} + \frac{(1-\theta_x)\nu}{2} \in \mathrm{FPP}$ is a finite set.

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So: computing the (x, λ, ν) in the FPP which are unitary is a finite calculation, and gives a complete description of the unitary dual.

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Theorem: The FPP has a finite facet decomposition; unitarity is constant on facets:there is a finite calculation to compute $\widehat{L(\mathbb{R})_{\mathrm{FPP}}}$ for each of (finitely many) $Q = LU$.

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2) General Arthur packets: In general $\Pi(\Psi)$ is known to be unitary in many cases (see the next slide)

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Davis/Mason-Brown: uniform proof: 1) for complex classical groups and many cases for real groups (using Hodge theory). Plus Adams/Mason-Brown/Ionov (unpublished): 1) in all cases, plus 2) in all "generic" casess.

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Ongoing work of Mason-Brown, Loseu, Davis, . . . : define a large class of representations, including those of Arthur type, so that the full unitary dual is obtained from this by compementary series deformations.

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Merci