Atlas of Lie Groups and Representations

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Cells for representations of real groups

or

Carrying coals to Newcastle

Carrying coals to Newcastle:

a) to take something to a place where its kind exists in great quantity.
b) to do something wholly unnecessary.

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Atlas Project

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Unipotent Representations

G: connected reductive group defined over a local field F of characteristic 0

For simplicity: assume $G(F)$ is (inner to) a split group

$G^\vee$: complex dual group of G

$\mathcal{O}^\vee$: unipotent orbit in $G^\vee$

Conjecture (Arthur): Associated to $\mathcal{O}^\vee$ is a finite set $\Pi(\mathcal{O}^\vee)$ of irreducible unitary representations of $G(F)$ satisfying certain conditions, including stability...

We call $\Pi(\mathcal{O}^\vee)$ a Weak unipotent Arthur packet

(Later: honest unipotent Arthur packets)

Unipotent Arthur packets

Arthur did not define $\Pi(\mathcal{O}^\vee)$, and there is no definition in general (that I am aware of). Even if one can give conditions to determine $\Pi(\mathcal{O}^\vee)$ uniquely, computing it might be difficult.

$F = \mathbb{R}$ or $\mathbb{C}$: Barbasch and Vogan gave a definition of $\Pi(\mathcal{O}^\vee)$. Computing $\Pi(\mathcal{O}^\vee)$ is another matter.

Today: Defining and computing $\Pi(\mathcal{O}^\vee)$ for real groups

Representations with fixed infinitesimal character

π: irreducible representation of $G(\mathbb{R})$, (a $(\mathfrak{g}, K)$-module), χ_π=infinitesimal character

Fix an infinitesimal character χ for G, that of a finite dimensional representation of G

$\mathcal{M}_\chi$: Grothendieck group of representations with infinitesimal character χ

$$\mathcal{M}_\chi = \mathbb{Z}\langle\{J(\gamma) \mid \gamma \in \mathcal{P}_\chi, J(\gamma) \text{ irreducible}, \chi_{J(\gamma)} = \chi\}\rangle$$

where $\gamma \in \mathcal{P}(\mathcal{M}_\chi) =$ a (finite) set of parameters.

Each $J(\gamma)$ is the unique irreducible quotient of a standard module $I(\gamma)$ (I is for "induced")

Fact:

$$\mathcal{M}_\chi = \mathbb{Z}\langle\{I(\gamma) \mid \gamma \in \mathcal{P}(\mathcal{M}_\chi)\}\rangle$$

EXAMPLE: $SL(2,\mathbb{R})$

Fix infinitesimal character χ of the trivial representation

There are 4 irreducible representations: $\mathbb{C}, DS_+, DS_-$ and PS_-. These are the trivial representation, two discrete series (one holomorphic, one anti-holomorphic) and PS_- is the irreducible, non-spherical principal series.

$$\mathcal{M}_\chi = \mathbb{Z}\langle \mathbb{C}, DS_+, DS_-, PS_- \rangle$$

Let PS_+ be the reducible principal series:

$$PS_+ = \mathbb{C} + DS_+ + DS_-$$

(in the Grothendieck group).

Standard modules:

$$\mathcal{M}_\chi = \mathbb{Z}\langle PS_+, DS_+, DS_-, PS_- \rangle$$

$$\mathbb{C} = PS_+ - DS_+ - DS_-$$

Kazhdan-Lusztig-Vogan polynomials

Change of basis matrix: Kazhdan-Lusztig-Vogan polynomials evaluated at $q = 1$ (up to some elementary signs)

$SL(2, \mathbb{R})$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Special case: ordinary Kazhdan-Lusztig modules for category O (Verma modules)

Coherent Continuation

Definition (Zuckerman): There is a natural representation of W on $\mathcal{M}_\chi$ (the coherent continuation representation)

Theorem: (Lusztig/Vogan) $(\mathcal{M}_\chi, \mathcal{P}(\mathcal{M}_\chi))$ has a natural structure of W-graph in the sense of [Kazhdan-Lusztig, 1979]).

As representations of W:

$$\mathcal{M}_\chi = \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$$

($\mathcal{B}_i$ is a block: $\sim$ generated by $\mathrm{Ext}(X, Y) \neq 0$)

Each block has the structure of a W-graph.

$SL(2, \mathbb{R})$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Cells

Definition: Given a block $\mathcal{B}$, a Harish-Chandra cell is a cell for the W-graph of $\mathcal{B}$ (as in [KL,1979])

A cell $\mathcal{C}$ carries a representation of W on

$$\mathbb{Z}\langle\{J(\gamma) \mid \gamma \in \mathcal{C}\}\rangle$$

Empirical fact (McGovern, Binegar): If $G(\mathbb{R})$ is a real form of $GL(n,\mathbb{C}), SO(n,\mathbb{C}), Sp(2n,\mathbb{C})$ or a simple exceptional group, then every Harish-Chandra cell is isomorphic to a left cell.

Special Representations

Theorem:

1) There is an integer $d(\mathcal{C})$ such that

$$\mathrm{Hom}_W(\pi_{\mathcal{C}}, \mathrm{Sym}^k(\mathrm{ref})) = \begin{cases} 0 & k < d(\mathcal{C}) \\ 1 & k = d(\mathcal{C}) \end{cases}$$

2) The cell contains a unique special representation $\sigma_{\mathcal{C}}$, which also occurs in $\mathrm{Sym}^{d(\mathcal{C})}(\mathrm{ref})$.

Vogan Duality

$G = G(\mathbb{C})$, $G(\mathbb{R})$, χ, $M_\chi \supset \mathcal{B}$

Theorem (Vogan)
There exists a real form $G^\vee(\mathbb{R})$ of $G^\vee(\mathbb{C})$, a block $\mathcal{B}^\vee$, and a bijection

$$\mathcal{P}(\mathcal{B}) \ni \gamma \to \gamma^\vee \in \mathcal{P}(\mathcal{B}^\vee)$$

with the following property: Define a perfect pairing $\mathcal{B} \times \mathcal{B}^\vee$ by:

$$\langle J(\gamma), J(\tau^\vee) \rangle = \delta_{\gamma,\tau}$$

Then:

$$\langle I(\gamma), I(\tau^\vee) \rangle = \delta_{\gamma,\tau}$$

Equivalently: the matrices of KLV polynomials for $G(\mathbb{R})$ and $G^\vee(\mathbb{R})$ are inverses.

Vogan Duality

Vogan duality:
(1) reverses inclusion of primitive ideals;

(2) takes small representations to large ones

(3) interchanges discrete series and (minimal) principal series (of a split group)

(4) takes cells to cells

(5) induces $\sigma \to \sigma^*$ on $\widehat{W} \simeq \widehat{W^\vee}$

If π is an irreducible representation of $G(\mathbb{R})$ we will write $\pi^\vee$ for the corresponding irreducible representation of some real form of $G^\vee(\mathbb{C})$.

Infinitesimal character

$\mathcal{O}^\vee$: nilpotent orbit for $G^\vee$

Jacobson-Morozov: $\mathcal{O}^\vee \mapsto \{H, E, F\}$

$$\mathcal{O}^\vee \mapsto \frac{1}{2}H \in \mathfrak{h}^\vee \simeq \mathfrak{h}^* \mapsto \chi(\mathcal{O}^\vee)$$

For simplicity: assume $\mathcal{O}^\vee$ is even ($\Leftrightarrow \chi(\mathcal{O}^\vee)$ is integral).

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{gr}(\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi))$$

$$\pi \mapsto \mathrm{gr}(I) \subset \mathrm{gr}(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g}) \mapsto \mathcal{V}(\mathrm{gr}(I)) \subset \mathfrak{g}^*$$

$\mathcal{V}(\mathrm{gr}(I))$ is $G(\mathbb{C})$-invariant and contained in the nilpotent cone.

Fact: (Borho/Brylinski/Joseph) $\mathcal{V}(\pi)$ is the closure of a single nilpotent orbit $\mathcal{O}$.

Definition (Vogan): $\overline{\mathcal{O}}$ (or simply $\mathcal{O}$) is the Associated Variety of the annihlator of π:

$$\mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}}$$

DEFINITION OF $\Pi(\mathcal{O}^\vee)$

Given $G = G(\mathbb{C}), G(\mathbb{R}), G^\vee(\mathbb{C})$

a nilpotent orbit $\mathcal{O}^\vee$ of $G^\vee(\mathbb{C})$.

Assume $\delta^\vee(\mathcal{O}^\vee) = \mathcal{O}^\vee$ where $\delta^\vee$ is the involution defining the L-group ${}^L G$. This is automatic if $G(\mathbb{R})$ is (inner to) a split group.

Definition: $\Pi(\mathcal{O}^\vee)$ is the set of irreducible representations π of $G(\mathbb{R})$ satisfying:

(1) $\chi_\pi = \chi(\mathcal{O}^\vee)$

(2) $\mathrm{AV}(\mathrm{Ann}(\pi^\vee)) = \overline{\mathcal{O}^\vee}$

Algorithm

Fix $G(\mathbb{C})$, $G(\mathbb{R})$. For simplicity assume $G(\mathbb{C})$ is simply connected.

0) Compute the conjugacy classes of $W = W(G(\mathbb{C}))$.

1) Explicitly compute $\mathcal{M}_\rho$, $\mathcal{P}(\mathcal{M}_\rho)$

2) Compute the blocks in $\mathcal{M}_\rho$, and for each block $\mathcal{B}$ the dual block $\mathcal{B}^\vee$

3) Run over the blocks $\mathcal{B}^\vee$. Compute the KLV polynomials for each $\mathcal{B}^\vee$.

4) Compute the cells $\mathcal{C}_1^\vee, \dots, \mathcal{C}_n^\vee$ in $\mathcal{B}^\vee$

5) For each cell $\mathcal{C}^\vee$ compute the representation $\pi_{\mathcal{C}^\vee}$ of $W^\vee$ on $\mathcal{C}^\vee$, and its character $\theta_{\mathcal{C}^\vee} = \text{trace}(\pi_{\mathcal{C}^\vee})$

6) Compute $d = \min\{k \in \mathbb{Z} \mid \langle \theta_{\mathcal{C}^\vee}, \theta_{S^k(\text{ref})} \rangle \neq 0\}$

ALGORITHM

7) Let $P_{\mathcal{C}^\vee} = \sum_{w \in W} \theta_{S^d}(\mathrm{ref})(w)\pi_{\mathcal{C}^\vee}(w) \in \mathrm{End}(\mathcal{C}^\vee)$ (this is a projection, up to scalar)

8) Compute the representation $\sigma_{\mathcal{C}^\vee}$ of W on the image of $P_{\mathcal{C}^\vee}$: this is the special representation in the cell $\mathcal{C}^\vee$.

Fix a complex even nilpotent orbit $\mathcal{O}^\vee$

9) Check if the nilpotent orbit attached to $\sigma_{\mathcal{C}^\vee}$ (by the Springer correspondence) is equal to $\mathcal{O}^\vee$. If so: translate (apply a Zuckerman translation functor to) the irreducible representations in $\mathcal{C}$ to infinitesimal character $\chi(\mathcal{O}^\vee)$

10) $\Pi(\mathcal{O}^\vee)$ is the set of (non-zero) irreducible representations obtained this way.

Primitive Ideals

11) The columns of $P_{\mathcal{C}^\vee}$ correspond to the irreducible representations in the cell. Two such representations have the same primitive ideal $\Leftrightarrow$ the corresponding columns are multiples of each other.

[Interlude: some examples]

Coda: Honest Arthur packets

An honest unipotent Arthur packet is attached to a homomorphism

$$\Psi : \mathbb{Z}_2 \times SL(2,\mathbb{C}) \rightarrow {}^L G$$

(Note: $W_{\mathbb{R}}/W_{\mathbb{R}}^0 \simeq \mathbb{Z}_2$)

So a weak unipotent Arthur packet is the union (not necessarily disjoint) of honest ones.

Recall $\Pi(\mathcal{O}^\vee)$ was defined in terms of AV(Ann($\pi^\vee$)), the closure of single complex nilpotent orbit.

There is a finer invariant AV(π) which is a union nilpotent $K(\mathbb{C})$ orbits on $(\mathfrak{g}/\mathfrak{k})^*$ (in bijection with: nilpotent $G(\mathbb{R})$-orbits on $\mathfrak{g}_0$). These "honest" Arthur packets are defined in [Adams/Barbasch/Vogan, 1992].

Coda: Honest Arthur packets

Another Algorithm: Vogan has given an outline of an explicit algorithm to compute the K-equivariant K-theory of the nilpotent cone. Assuming this can be made precise:

a) This will prove a version of the Lusztig-Vogan conjecture for real groups (complex case: Lusztig/Bezrukavnikov)

b) This gives an effective algorithm to compute $\mathrm{AV}(\pi)$, and therefore honest unipotent Arthur packets.