ADMISSIBLE W-GRAPHS AND COMMUTING CARTAN MATRICES

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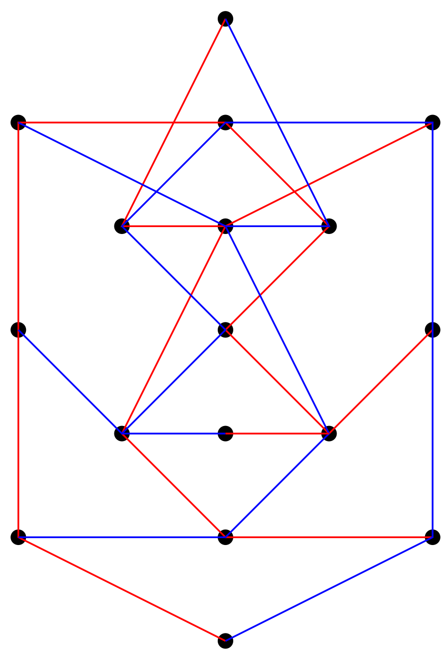


1.1 A Classification Problem

We want to classify commuting pairs of simply-laced Cartan matrices.[1]

What is a simply-laced Cartan matrix?

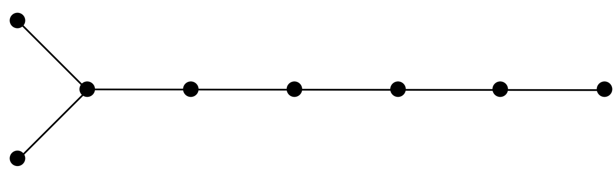
- 2's on the diagonal; $-1, 0$ off-diagonal;
- symmetric, positive definite.

Nicer to replace A with $2 - A$ (adjacency matrix of Dynkin diagram).

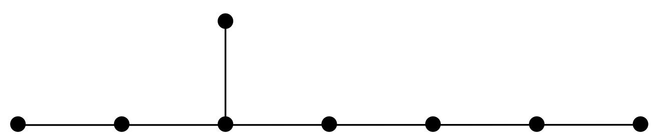
Recall the familiar A-D-E classification:

A_n

D_n $(n \geqslant 4)$

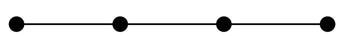


E_n $(n = 6, 7, 8)$



Note. We identify graphs with their adjacency matrices.

$A_4 \equiv$



$$\equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

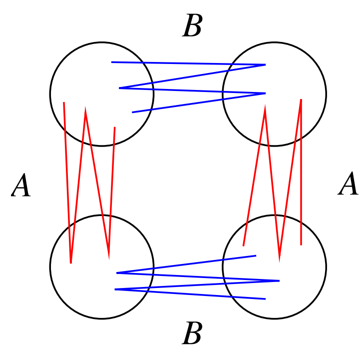
Important. We are not assuming that the graphs are connected!

[1] A lie!

1.2 Admissible Cartan Pairs

PROBLEM/DEFINITION. Classify all **admissible Cartan pairs** (A, B):

- A and B are simply-laced Dynkin diagrams on the same vertex set
- The vertex set may be partitioned into 4 blocks so that



- $AB = BA$.

NOTES.

- Disjoint unions of ACP's are ACP's, so w.l.o.g. assume connected.
- If (A, B) is connected, the vertex partition is unique.
- Checking $AB = BA$ amounts to choosing vertices x, y in diagonally opposite blocks and comparing 2-step paths $x \to y$.

DEFINITION. The **dual** of an ACP (A, B) is $(A, B)^* := (B, A)$.

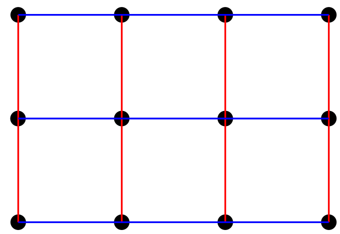
1.3 Examples

1. Tensor Product.

Choose **connected** Dynkin diagrams C, D.

Set $A := C \otimes 1$, $B := 1 \otimes D$. Write $(A, B) := C \otimes D$.

Example: $A_3 \otimes A_4$

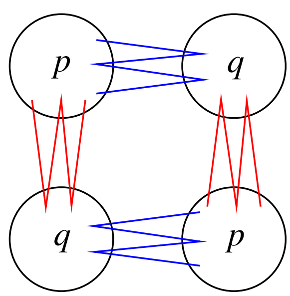


2. Twisted Product.

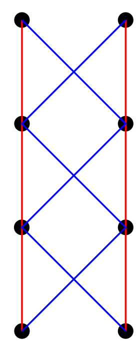
Choose a connected Dynkin diagram C with p black vertices, q white.

Construct four copies of C on $2p + 2q$ vertices:

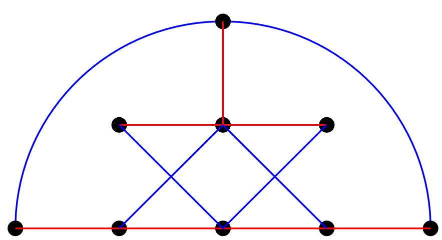
$C \times C$:=



$A_4 \times A_4$ =



3. Many other possibilities, such as



Note. Every ACP has a **type**. Example 3 has type $(A_5 D_4\, , A_3^3)$.

2.1 Why Do We Care?

The story begins with the theory of W-graphs.

Start with a Coxeter system (W, S), $S = \{s_1, \ldots, s_n\}$.

Ex: $W = S_{n+1}$, $s_i = (i, i+1)$.

Of primary interest are the finite Weyl groups.

Definition (Kazhdan-Lusztig). A W-graph is a triple (V, m, τ) s.t.

- V is a (finite) vertex set
- $m : V \times V \to \mathbb{Z}$ (matrix of edge weights; 0 means "no edge")
- $\tau : V \to \{\text{subsets of } S\}$ (each vertex has a "descent set")
- The following defines a W-action on $\mathbb{Z}V$:

$$s_i(v) = \begin{cases} v & \text{if } i \notin \tau(v), \\ -v + \sum_{u : i \notin \tau(u)} m(v \to u) u & \text{if } i \in \tau(v). \end{cases} \qquad (*)$$

Notes.

- We've set $q = 1$; the Hecke algebra action has been hidden.
- $s_i^2 = 1$ is automatic; $(*) \Leftrightarrow$ braid relations.
- If $\tau(v) \subseteq \tau(u)$ then $m(v \to u) := 0$ by convention.

The strongly connected components of a W-graph are called **cells**.

Cells are themselves W-graphs; they are the combinatorially irreducible W-graphs, but need not be algebraically irreducible.

2.2 The Kazhdan-Lusztig W-graph

The W-graphs that people care about are the ones that occur in representation theory (cf. the Kazhdan-Lusztig "Conjecture").

In the K-L construction, $\mathbb{Q}W$ has a distinguished basis $\{C_w : w \in W\}$.

The action of s_i on this basis has the structure of a W-graph, with

- vertex set W,
- $\tau(v) := \{i : \ell(s_i v) < \ell(v)\}$ for all $v \in W$,
- $m(u \to v) := \mu(u,v) + \mu(v,u)$ if $\tau(u) \not\subseteq \tau(v)$,

where $\mu(u,v) :=$ coefficient of $q^{(\ell(v)-\ell(u)-1)/2}$ in $P_{u,v}(q)$.

Remarks.

- For Weyl groups, we know that $P_{u,v}(q) \in \mathbb{Z}^{\geqslant 0}[q]$, so $m(u \to v) \geqslant 0$.
- $\mu(u,v)$ is hard to compute without first computing $P_{u,v}(q)$.
- The cells of this W-graph are "left K-L cells".
- Left and right actions of W on $\mathbb{Q}W$ yield a $W \times W$-graph.
- The cells of the $(W \times W)$-graph are "two-sided K-L cells".
- There exist analogous graphs and cells associated with representations of real groups via K-L-V polynomials.

Goals.

- Understand the structure of K-L and K-L-V cells.
- What are the essential combinatorial features of these cells?
- Can they be determined without K-L(-V) polynomials?

2.3 Admissible W-Graphs

Definition. A W-graph is **admissible** if it

- has $\mathbb{Z}^{\geqslant 0}$ weights,
- is bipartite (cf. $\ell(u) - \ell(v) - 1 \in 2\mathbb{Z}$), and
- is **edge-symmetric**: $m(u \to v) = m(v \to u)$ if $\tau(u) \not\subseteq\!\!\!\!\supseteq \tau(v)$.

Note. If $\tau(u) \not\subseteq\!\!\!\!\supseteq \tau(v)$, then $m(u \to v) = m(v \to u) \in \{0, 1\}$.

Main Contention. These axioms come close to capturing what is essential about K-L(-V) cells. There do exist admissible cells that are not K-L-V, but all admissible cells seem to be built from the same "molecules."

Problem. Classify all admissible W-cells.

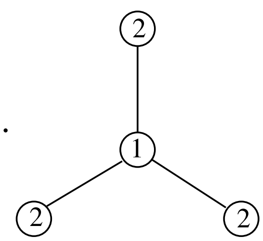
Solved: A_9, D_6, E_6, $H_3, \ldots$

Problem. Are there finitely many admissible W-cells?

What does all of this have to do with classifying ACP's?

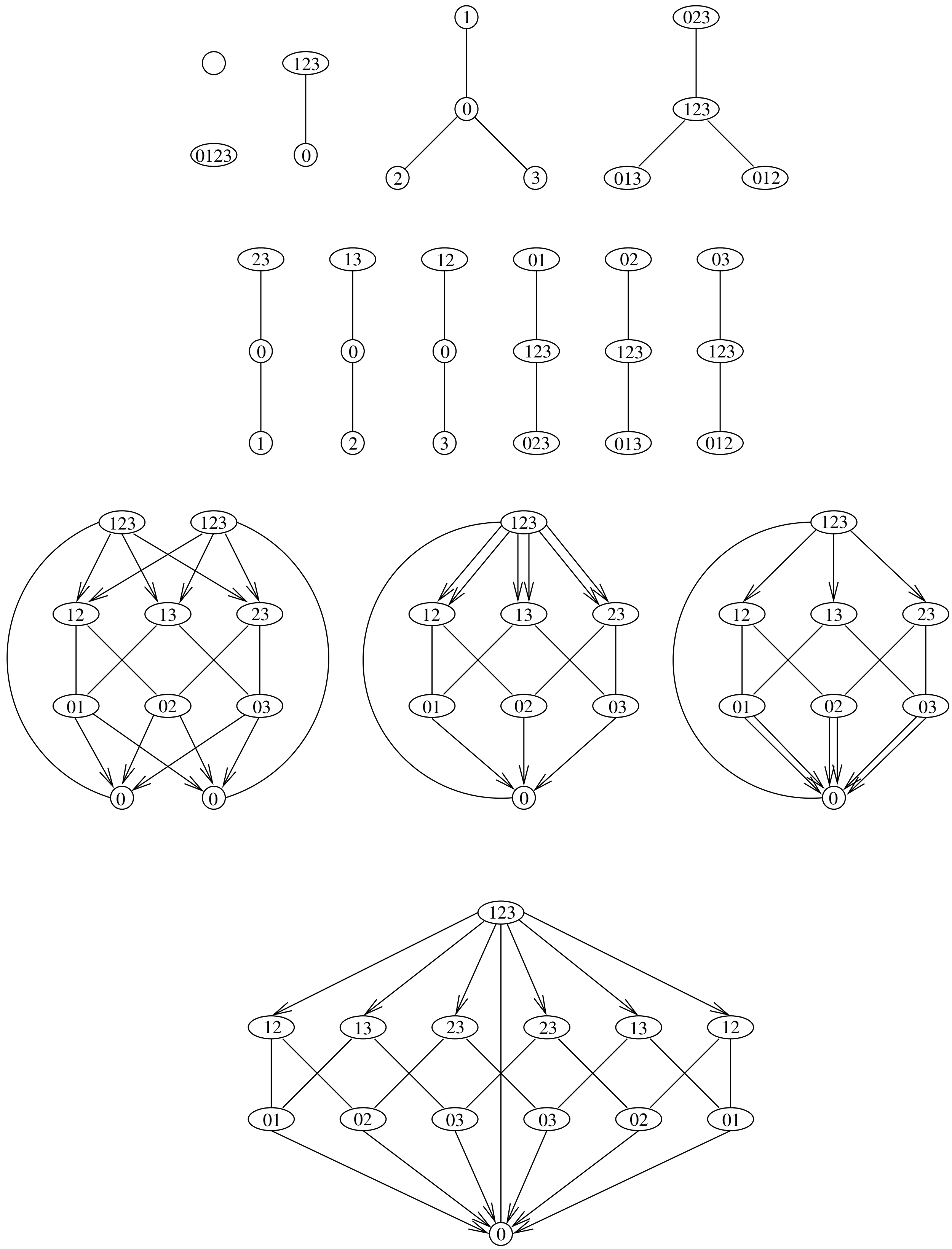
Theorem. *The nontrivial admissible $I_2(m)$-cells are the simply-laced Dynkin diagrams with Coxeter number $h \mid m$.*

Example. The D_4 diagram is a G_2-cell (Coxeter number 6).



Note. The Dynkin diagram A_{m-1} is the only nontrivial K-L $I_2(m)$-cell.

The admissible D_4-cells (three are not K-L):



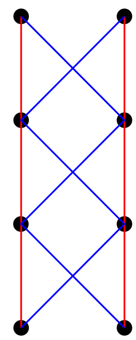
2.4 The Reducibility Issue

Obstacle. *It does not suffice to assume that W is irreducible!*

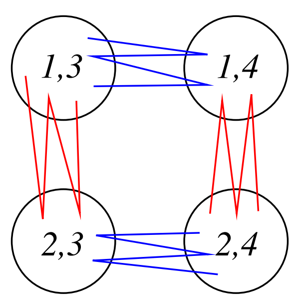
There exist "interesting" $(W_1 \times W_2)$-cells that are not tensor products.

Example. The nontrivial two-sided K-L cell for $I_2(m)$ is $A_{m-1} \times A_{m-1}$.

$A_4 \times A_4 =$



Proposition. The nontrivial admissible $(I_2(p) \times I_2(q))$-cells are the connected ACP's (A, B) such that each component of A has Coxeter number dividing p, and each component of B has Coxeter number dividing q.



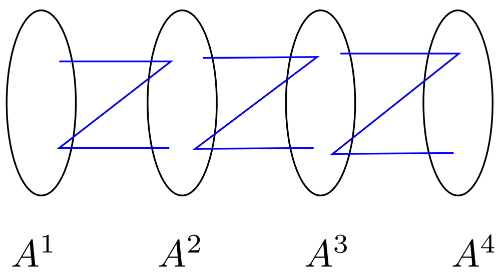
Remark. All components of A necessarily have the same Coxeter number.

3.1 Bindings

First, a mechanism for breaking connected ACP's (A, B) into smaller pieces.

Decompose A into connected components $A^1, \ldots, A^\ell$.

Set B_{ij} := edges of B that connect A^i and A^j.



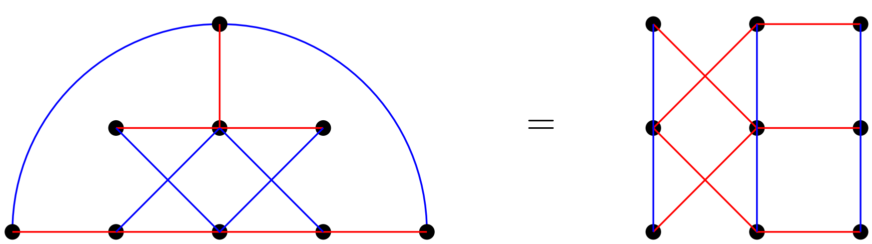
Fact. If $B_{ij} \neq \varnothing$, then $(A^i \cup A^j, B_{ij})$ is itself a connected ACP.

Proof. Commutativity is a local condition. □

Definition. If A has exactly two components A^1 and A^2, then (A, B) is a **binding** of A^1 and A^2.

Corollary. It suffices to classify all bindings of connected diagrams, and then determine all ways they can be combined into larger ACP's.

Note. The dual of a binding need not be a binding.



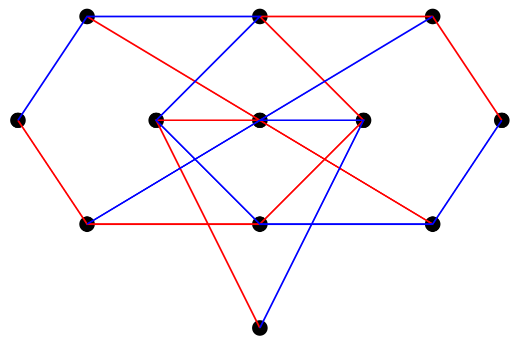
Here, the dual of a D_4, A_5 binding reduces to two A_3, A_3 bindings.

Definition. (A, B) is **irreducible** if A and B both have two components.

3.2 Classifying Bindings

Lemma 1. *The irreducible bindings are twisted products* $C \times C$, *along with two (self-dual) exceptional bindings:* $D_5 \boxtimes A_7$ *and* $E_7 \boxtimes D_{10}$.

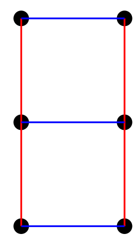
$D_5 \boxtimes A_7$:



What about other (reducible) bindings?

Definition. Parallel binding: $C \equiv C \ := \ C \otimes A_2$.

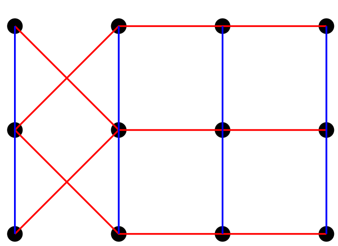
$A_3 \equiv A_3$:



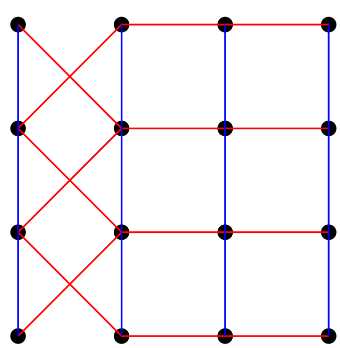
Lemma 2. *A complete list of bindings of A-D-E diagrams is as follows:*

- $C \times C, \quad C \equiv C, \quad D_5 \boxtimes A_7, \quad E_7 \boxtimes D_{10},$
- $D_{n+1} * A_{2n-1} \ := \ (A_3 \times A_3 \equiv A_3 \equiv \cdots \equiv A_3)^* \quad (n \geqslant 3),$
- $E_6 * E_6 \ := \ (A_3 \equiv A_3 \times A_3 \equiv A_3)^*,$
- $D_6 * D_6 \ := \ (A_4 \times A_4 \equiv A_4)^*,$
- $E_8 * E_8 \ := \ (A_4 \times A_4 \equiv A_4 \equiv A_4)^*.$

$D_5 * A_7$:



$E_8 * E_8$:



3.3 The Classification

THEOREM. *The connected ACP's are as follows:*

- $C \otimes D, \quad C \times C, \quad D_5 \boxtimes A_7, \quad E_7 \boxtimes D_{10},$
- $D_{n+1} \equiv \cdots \equiv D_{n+1} \equiv D_{n+1} * A_{2n-1},$
- $D_{n+1} * A_{2n-1} \equiv A_{2n-1} \equiv \cdots \equiv A_{2n-1},$
- $D_{n+1} \equiv D_{n+1} * A_{2n-1} \equiv A_{2n-1} \ = \ (E_6 * E_6 \equiv E_6 \equiv \cdots \equiv E_6)^*,$
- $D_6 * D_6, \quad D_6 * D_6 \equiv D_6, \quad D_6 * D_6 \equiv D_6 \equiv D_6 \ = \ (E_8 * E_8 \equiv E_8)^*,$
- $E_8 * E_8, \quad E_6 \equiv E_6 * E_6 \equiv E_6, \quad E_8 * E_8 \equiv E_8 \equiv E_8.$