Atlas of Lie Groups
and Representations

www.liegroups.org

# Atlas of Lie Groups and Representations

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August 4, 2018

Slides available at: www.liegroups.org

# Atlas Project

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# Overview

Atlas software:

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Computations in Lie theory

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Unless otherwise noted: (almost) everything in sight is complex and complex algebraic

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   1.5 Radical, center, derived group

# Real Groups

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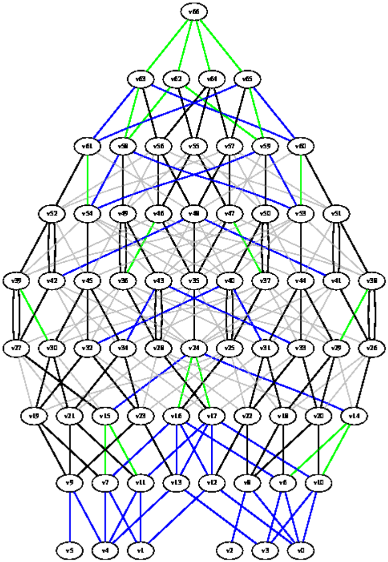
- **3.5** Real Cartan subgroups, (relative) Weyl groups

# Example: SO(4,4)



# Langlands classification

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4.1 Parameters for representations

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- 4.5 $M_\gamma = \{\gamma_1, \ldots, \gamma_n\}$ (parameters for irreducible representation with infinitesimal character $\gamma$)Harish-Chandra: finite set
- 4.6 Cuspidal data: $\pi = \overline{\mathrm{Ind}}_P^G(\pi_M)$
- 4.7 theta-stable data: $\pi = \overline{R_{\mathfrak{q}}^G}(\pi_L)$

# Kazhdan-Lusztig-Vogan Polynomials

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- 5.8 (Twisted KLV polynomials)

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- **9.4** Using: coherent continuation

Example $G = Sp(12, \mathbb{R})$, $M = GL(5, \mathbb{R}) \times SL(2, \mathbb{R})$

```
atlas> G:=Sp(12,R)
Value: connected split real group with Lie algebra 'sp(12,R)'
atlas> set real_parabolics=all_real_parabolics (G)
Variable real_parabolics: [KGPElt]
atlas> #real_parabolics
Value: 64
atlas> void:for P@i in real_parabolics do if
ss_rank (Levi(P))=5 then prints(i, " ", Levi(P)) fi od
31  sl(6,R).gl(1,R)
47  sl(5,R).sl(2,R).gl(1,R)
55  sl(4,R).sp(4,R).gl(1,R)
59  sl(3,R).sp(6,R).gl(1,R)
61  sl(2,R).sp(8,R).gl(1,R)
62  sp(10,R).gl(1,R)
atlas> set P=real_parabolics[47]
Variable P: KGPElt
atlas> real_induce_irreducible(trivial(Levi(P)),G)
Value:
1*parameter(x=4898,lambda=[6,5,4,3,2,1]/1,nu=[2,2,1,1,1,0]/1)
1*parameter(x=4117,lambda=[5,6,1,3,4,0]/1,nu=[3,4,0,2,3,0]/2)
1*parameter(x=4116,lambda=[5,6,1,3,4,0]/1,nu=[3,4,0,2,3,0]/2)
```

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