Unitary representations of reductive groups 1–5

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1. Why representations?
Fourier series
Finite-diml representations
Abstract harmonic analysis
Quadratic forms
2. Automorphic forms
Defining automorphic forms
Automorphic cohomology
3. Orbit method
Commuting algebras
Differential operator algebras
Hamiltonian G-spaces
4. Classical limit
Associated varieties
Deformation quantization
Howe's wavefront set
5. $(\mathfrak{g}, K)$-modules
Case of $SL(2, \mathbb{R})$
Definition of $(\mathfrak{g}, K)$-modules
Harish-Chandra algebraization theorems

Outline

1. Examples and applications of representation theory
   Fourier series
   Finite-diml representations
   Gelfand's abstract harmonic analysis
   Quadratic forms and reps of $GL(n)$

2. Examples from automorphic forms
   Defining automorphic forms
   Automorphic cohomology

3. Kirillov-Kostant orbit method
   Commuting algebras
   Differential operator algebras: how orbit method works
   Hamiltonian G-spaces: how Kostant does the orbit method

4. Classical limit: from group representations to symplectic geometry
   Associated varieties
   Deformation quantization
   Howe's wavefront set

5. Harish-Chandra's $(\mathfrak{g}, K)$-modules
   Case of $SL(2, \mathbb{R})$
   Definition of $(\mathfrak{g}, K)$-modules
   Harish-Chandra algebraization theorems

How does symmetry inform mathematics (I)?

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Example. $\int_{-\pi}^{\pi} \sin^5(t)dt = ?$ Zero!

Principle: group G acts on vector space V; decompose V using G; study each piece.

Here $G = \{1, -1\}$ acts on $V =$ functions on $\mathbb{R}$; pieces are even and odd functions.

How does symmetry inform mathematics (II)?

Example. Temp distn $T(t,\theta)$ on hot ring governed by

$$\partial T/\partial t = c^2 \partial^2 T/\partial\theta^2, \qquad T(0,\theta) = T_0(\theta).$$

Too hard for (algebraist) to solve; so look at special initial conditions with rotational (almost) symmetry:

$$T(0,\theta) = a_0/2 + a_m \cos(m\theta).$$

Diff eqn is symmetric, so hope soln is symmetric:

$$T(t,\theta) \stackrel{?}{=} a_0(t)/2 + a_m(t)\cos(m\theta).$$

Leads to ORDINARY differential equations

$$da_0/dt = 0, \qquad da_m/dt = -c^2 m^2 a_m.$$

These are well-suited to an algebraist:

$$T(t,\theta) = a_0/2 + a_m e^{-c^2 \cdot m^2 t} \cos(m\theta).$$

Generalize: Fourier series expansion of initial temp. . .

Principle: group G acts on vector space V; decompose V; study pieces separately. Here $G =$ rotations of ring acts on $V =$ functions on ring; decomposition is by frequency.

What's so good about sin and cos?

What's "$\cos(m\theta)$ is almost rotationally symmetric" mean?

If $f(\theta)$ any function on the circle ($f(\theta + 2\pi) = f(\theta)$), define rotation of f by ϕ to be new function $[\rho(\phi)f](\theta) = f(\theta - \phi)$.
Rotationally symm. $=_{\text{def}}$ unchgd by rotation $=_{\text{def}}$ constant.

$$c_m(\theta) =_{\text{def}} \cos(m\theta), \qquad s_m(\theta) =_{\text{def}} \sin(m\theta).$$

$$\begin{aligned}[\rho(\phi)c_m](\theta) &= c_m(\theta - \phi) = \cos(m\theta - m\phi) \\ &= \cos(m\theta)\cos(m\phi) + \sin(m\theta)\sin(m\phi). \\ &= [\cos(m\phi)c_m + \sin(m\phi)s_m](\theta).\end{aligned}$$

Rotation of c_m is a linear combination of c_m and s_m:
"almost rotationally symmetric."

Similar calculation for sin shows that

$$\rho(\phi)\begin{pmatrix} c_m \\ s_m \end{pmatrix} = \begin{pmatrix} \cos(m\phi) & \sin(m\phi) \\ -\sin(m\phi) & \cos(m\phi) \end{pmatrix} \begin{pmatrix} c_m \\ s_m \end{pmatrix}.$$

HARD transcendental rotation $\rightsquigarrow$ EASY linear algebra!

In which we meet the hero of our story…

$$\rho(\phi)\begin{pmatrix} c_m \\ s_m \end{pmatrix} = \begin{pmatrix} \cos(m\phi) & \sin(m\phi) \\ -\sin(m\phi) & \cos(m\phi) \end{pmatrix}\begin{pmatrix} c_m \\ s_m \end{pmatrix}.$$

Definition

A *representation* of a group G on a vector space V is a group homomorphism

$$\rho\colon G \to GL(V).$$

Equiv: action of G on V by linear transformations.

Equiv (if $V = \mathbb{C}^n$): each $g \in G \rightsquigarrow n \times n$ matrix $\rho(g)$,

$$\rho(gh) = \rho(g)\rho(h), \qquad \rho(e) = I_n.$$

HARD questions about G, (nonlinear) actions $\rightsquigarrow$ EASY linear algebra!

How does symmetry inform math (III)?

First two examples involved easy abelian G; usually understood without groups.

Fourier series provide a nice basis $\{\cos(m\theta), \sin(m\theta)\}$ for functions on the circle S^1.

What analogues are possible on the sphere S^2?

$G = O(3) =$ group of 3×3 real orthogonal matrices, the distance-preserving linear transformations of $\mathbb{R}^3$.

$$V = \text{functions on } S^2.$$

Seek small subspaces of V preserved by $O(3)$.

Example. $V_0 = \langle 1 \rangle =$ constant functions; 1-diml.

Example. $V_1 = \langle x, y, z \rangle =$ linear functions; 3-diml.

Example. $V_2 = \langle x^2, xy, \ldots, z^2 \rangle =$ quad fns; 6-diml.

Problem: $x^2 + y^2 + z^2 = 1$ on S^2: so $V_2 \supset V_0$.

Example. $V_m = \langle x^m, \ldots, z^m \rangle =$ deg m polys; $\binom{m+2}{2}$-diml.

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Polynomials and the group $O(3)$

$$\underbrace{S(\mathbb{R}^3)}_{\text{poly fns}} = \underbrace{V_0}_{\substack{\text{constants}\\ \text{dim}=1}} + \underbrace{V_1}_{\substack{\text{linear}\\ \text{dim}=3}} + \cdots + \underbrace{V_m}_{\substack{\text{degree } m\\ \text{dim}=\binom{m+2}{2}}} + \cdots$$

Want to understand restriction of these functions to

$$S^2 = \{(x,y,z) \mid r^2 = 1\} \quad (r^2 = x^2 + y^2 + z^2).$$

Algebraic geometry point of view (Q for *quotient*):

$$\text{nice fns on } S^2 =_{\text{def}} Q(S^2) = S(\mathbb{R}^3)/\langle r^2 - 1\rangle.$$

To study polynomials with finite-dimensional linear algebra, use the increasing filtration $S^{\le m}(\mathbb{R}^3)$; get

$$Q^{\le m}(S^2) = S^{\le m}(\mathbb{R}^3)/(r^2-1)S^{\le m-2}(\mathbb{R}^3).$$

$$S^{\le m}(\mathbb{R}^3)/S^{\le m-1}(\mathbb{R}^3) \simeq V_m,$$

$$Q^{\le m}(S^2)/Q^{\le m-1}(S^2) \simeq V_m/(r^2)V_{m-2}.$$

$O(3)$ has rep on V_m/r^2V_{m-2}, dim $= \binom{m+2}{2} - \binom{m}{2} = 2m+1$; sum over m gives all (polynomial) fns on S^2.

Polynomials and the group $O(3)$ (reprise)

$$\underbrace{S(\mathbb{R}^3)}_{\text{poly fns}} = \underbrace{V_0}_{\text{constants}} + \underbrace{V_1}_{\text{linear}} + \cdots + \underbrace{V_m}_{\text{degree } m} + \cdots$$

Want to understand restriction of these functions to S^2.

Analysis point of view $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

nice fns on S^2 = initial conditions for diff eq $\Delta F = 0$.

$$V_{m-2} \underset{\Delta}{\overset{\cdot r^2}{\rightleftarrows}} V_m; \qquad H_m =_{\text{def}} \ker(\Delta|_{V_m}).$$

Proposition

H_m is a complement for $r^2 V_{m-2}$ in V_m. Consequently

$$V_m / r^2 V_{m-2} \simeq H_m, \qquad (O(3) \text{ rep of } \dim = 2m+1).$$

$$V_m = H_m \oplus r^2 H_{m-2} \oplus r^4 H_{m-4} + \cdots .$$

functions on $S^2 \simeq H_0 \oplus H_1 \oplus H_2 \oplus \cdots$

Fourier series on S^2

Abstract representation theory: group $O(3)$ has two irr repns of each odd dim $2m+1$, namely

$$H_m = \text{harmonic polys of deg } m \simeq V_m/r^2 V_{m-2},$$

and $H_m \otimes \epsilon$; here

$$\epsilon\colon O(3) \to \{\pm 1\} \subset GL(1), \quad \text{sgn}(g) =_{\text{def}} \text{sgn}(\det(g)).$$

Schur's lemma: any invariant Hermitian pairing

$$\langle , \rangle\colon E \times F \to \mathbb{C}$$

between distinct irreducible representations of a compact group G must be zero. Consequence:

$$\text{subspaces } H_m \subset L^2(S^2) \text{ are orthogonal.}$$

Stone-Weierstrass: span(H_m) *dense* in $L^2(S^2)$.

Proposition

$L^2(S^2)$ is Hilbert space sum of the $2m+1$-diml subspaces H_m of harmonic polys of degree m.

$$f \in L^2(S^2) \to f_m \in H_m, \qquad f = \sum_{m=0}^{\infty} f_m.$$

Fourier coeff f_m *in* $2m+1$*-diml* $O(3)$ *rep.*

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Gelfand's abstract harmonic analysis

Topological grp G acts on X, have questions about X.

Step 1. Attach to X Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$).
Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest G-eqvt decomp $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$.
Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G: indecomposable action of G on a Hilbert space.

Step 3. Understand $\widehat{G}_u =$ all irreducible unitary representations of G: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Topic for these lectures: **Step 3** for Lie group G.

Mackey theory (normal subgps) $\rightsquigarrow$ case G reductive.

Making everything noncompact

Examples so far have compact spaces,groups...

$$\begin{aligned} D &= \text{pos def quad forms in } n \text{ vars} \\ &= n \times n \text{ real symm matrices, eigenvalues} > 0 \\ &= GL(n,\mathbb{R})/O(n). \end{aligned}$$

(invertible $n \times n$ real matrices modulo subgroup of orthogonal matrices.

$GL(n,\mathbb{R})$ acts on D by change of variables. In matrix realization, $g \cdot A = gA^t g$. Action is transitive; isotropy group at I_n is $O(n)$.

$C(D) =$ cont fns on D, $\quad [\lambda(g)f](x) = f(g^{-1} \cdot x) \quad (g \in GL(n,\mathbb{R}))$;

inf-diml rep of $G \rightsquigarrow$ action of G on D.

Seek (minimal = irreducible) $GL(n,\mathbb{R})$-invt subspaces inside $C(D)$, use them to "decompose" $L^2(D)$.

(V,ρ) any rep of $G = GL(n,\mathbb{R})$; write $K = O(n)$.

$$T \in \mathrm{Hom}_G(V, C(D)) \simeq \mathrm{Hom}_K(V,\mathbb{C}) = K\text{-fixed lin fnls on } V \ni \tau,$$

$$[T(v)](gK) = \tau(\rho(g^{-1}v)).$$

Study D by representation theory

$$G = GL(n,\mathbb{R}), \quad K = O(n)$$

$$D = \text{positive definite quadratic forms},$$

$$\mathrm{Hom}_G(V, C(D)) \simeq K\text{-fixed linear functionals on } V.$$

So seek to construct (irreducible) reps of G having nonzero K-fixed linear functionals.

Idea from Borel-Weil theorem for compact groups:

irr repns $\leftrightsquigarrow$ secs of line bdles on flag mflds.

Complete flag in m-diml E is chain of subspaces

$$\mathcal{F} = \{0 = F_0 \subset F_1 \subset \cdots \subset F_m = E\}, \quad \dim F_i = i.$$

Define $X(\mathbb{R})$ = complete flags in $\mathbb{R}^n$. Group G acts transitively on flags. Base point of $X(\mathbb{R})$ is std flag

$$\mathcal{F}^0 = \{\mathbb{R}^0 \subset \mathbb{R}^1 \subset \cdots \subset \mathbb{R}^n\}, G^{\mathcal{F}^0} = B,$$

B group of upper triangular matrices. Hence $X(\mathbb{R}) \simeq G/B$.

Get rep of G on $V = C(X(\mathbb{R}))$ (functions on flags); has K-fixed lin fnl τ = integration over $X(\mathbb{R})$. Get embedding

$$T\colon V \hookrightarrow C(D), \quad [Tv](gK) = \int_{x\in X(\mathbb{R})} v(g\cdot x)\,dx.$$

Study D by rep theory (continued)

$$G = GL(n,\mathbb{R}), \quad K = O(n), \quad B = \text{upper } \Delta$$
$$D = \text{pos def quad forms} \simeq G/K,$$
$$X(\mathbb{R}) = \text{complete flags in } \mathbb{R}^n \simeq G/B$$

Found embedding

$$T\colon C(X(\mathbb{R})) \hookrightarrow C(D), \quad [Tv](gK) = \int_{x\in X(\mathbb{R})} v(g\cdot x)\,dx.$$

To generalize, use G-eqvt real line bdle $\mathcal{L}_i$ on $X(\mathbb{R})$, $1 \le i \le n$; fiber at $\mathcal{F}$ is F_i/F_{i-1}.

$\mathbb{R}^\times \ni t \rightsquigarrow |t|^\nu \operatorname{sgn}(t)^\epsilon \in \mathbb{C}^\times$ (any $\nu \in \mathbb{C}$, $\epsilon \in \mathbb{Z}/2\mathbb{Z}$);

Similarly get G-eqvt cplx line bdle $\mathcal{L}^{\nu,\epsilon} = \mathcal{L}_1^{\nu_1,\epsilon_1} \otimes \cdots \otimes \mathcal{L}_n^{\nu_n,\epsilon_n}$.

$$V^{\nu,\epsilon} = C(X(\mathbb{R}), \mathcal{L}^{\nu,\epsilon}) = \text{continuous sections of } \mathcal{L}^{\nu,\epsilon}$$

family of reps $\rho^{\nu,\epsilon}$ of G: index n cplx numbers, n “parities.”

This is what “all” reps of “all” G look like; study more!

Case all $\epsilon_i = 0$: can make sense of

$$T^\nu\colon V^{\nu,0} \to C(D), \quad [T^\nu v](gK) = \int_{x\in X(\mathbb{R})} v(g\cdot x)\,dx.$$

Study D directly

$$G = GL(n,\mathbb{R}), \quad K = O(n)$$
$$D = \text{positive definite quadratic forms.}$$

Seek (minimal = irreducible) $GL(n,\mathbb{R})$-invt subspaces inside $C(D)$, use them to "decompose" $L^2(D)$.

If G acts on functions, how do you find invt subspaces?

Look at this in third lecture. For now, two ideas. . .

Can scale pos def quad forms (mult by nonzero pos real):

$$\begin{aligned} C(D) \supset C^{\lambda_1}(D) &= \text{fns homog of degree } \lambda_1 \in \mathbb{C}. \\ &= \{f \in C(D) \mid f(tx) = t_1^{\lambda} f(x) \quad (t \in \mathbb{R}^+, x \in D)\} \\ &= \{f \in C(D) \mid \Delta_1 f = \lambda_1 f\}, \end{aligned}$$

$\Delta_1 =$ Euler degree operator $= \sum_j x_j \partial/\partial x_j$.

D has G-invt Riemannian structure and therefore Laplace operator Δ_2 commuting with G.

$$\begin{aligned} C(D) \supset C^{\lambda_2}(D) &= \lambda_2\text{-eigenspace of } \Delta_2 \\ &= \{f \in C(D) \mid \Delta_2 f = \lambda_2 f \quad (\lambda_2 \in \mathbb{C})\}. \end{aligned}$$

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Study D directly (continued)

$$G = GL(n, \mathbb{R}), \quad K = O(n)$$

$$D = \text{positive definite quadratic forms.}$$

Seek (minimal = irreducible) $GL(n, \mathbb{R})$-invt subspaces.

So far: found eigenspaces of two G-invt diff ops (Euler degree op Δ_1, Laplace op Δ_2

Theorem (Harish-Chandra, Helgason)

Algebra $\mathcal{D}^G$ of G-invt diff ops on D is a (comm) poly ring, gens $\{\Delta_1, \Delta_2, \ldots, \Delta_n\}$, $\deg(\Delta_j) = j$.

Get nice G-invt spaces of (analytic) functions

$$C(D) \supset C^{\lambda}(D) = \text{joint eigenspace of all } \Delta_j$$
$$= \{f \in C(D) \mid \Delta_j f = \lambda_j f \quad (1 \le j \le n)\}.$$

Relation to rep-theoretic approach: had

$$T^{\nu} \colon V^{\nu,0} \to C(D), \quad [T^{\nu} v](gK) = \int_{x \in X(\mathbb{R})} v(g \cdot x)\, dx$$

Here V^{ν} = secs of bundle on flag variety $X(\mathbb{R})$; each V^{ν} maps to one eigenspace $\lambda(\nu)$.

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What's so great about automorphic forms?

Arithmetic questions (about ratl solns of poly eqns) hard: lack tools from analysis and geometry).

Cure: embed arithmetic questions in real ones...

Arithmetic: cardinality of $\{(p,q) \in \mathbb{Z}^2 \mid p^2 + q^2 \leq N\}$?

Geom: area of $\{(p,q) \in \mathbb{R}^2 \mid p^2 + q^2 \leq N\}$? Ans: $N\pi$.

Conclusion: answer to arithmetic question is "$N\pi +$ small error." Error $O(N^{131/416+\epsilon})$ (Huxley 2003); conjecturally $N^{1/4+\epsilon}$.

Similarly: counting solns of arithmetic eqns mod p^n $\leftrightsquigarrow$ analytic/geometric problems over $\mathbb{Q}_p$.

Model example: relationship among $\mathbb{Z}$, $\mathbb{R}$, circle.

Algebraic/counting problems live on $\mathbb{Z}$; analysis lives on $\mathbb{R}$; geometry lives on circle $\mathbb{R}/\mathbb{Z}$.

Automorphic forms provide parallel interaction among arithmetic, analysis, geometry.

What's so great about automorphic forms

Theorem

Write $\mathbb{A} = \mathbb{R} \times \prod_p' \mathbb{Q}_p$ *(restricted product). Then* $\mathbb{A}$ *is locally compact topological ring containing* $\mathbb{Q}$ *as a discrete subring, and* $\mathbb{A}/\mathbb{Q}$ *is compact.*

Corollary

1. $GL(n, \mathbb{A}) = GL(n, \mathbb{R}) \times \prod_p' GL(n, \mathbb{Q}_p)$ *is loc cpt grp.*
2. $GL(n, \mathbb{Q})$ *is a discrete subgroup.*
3. *Quotient space* $GL(n, \mathbb{A})/GL(n, \mathbb{Q})$ *is nearly compact.*

Conclusion: the space $GL(n, \mathbb{Q})\backslash GL(n, \mathbb{A})$ is a convenient place to relate arithmetic and analytic questions.

$\mathcal{A}(n)$ = automorphic forms on $GL(n)$ = functions on $GL(n, \mathbb{Q})\backslash GL(n, \mathbb{A})$ $(+$ technical growth conds).

Vector space $\mathcal{A}(n)$ is a representation of $GL(n, \mathbb{A})$.

Irr constituents of $\mathcal{A}(n)$ are *automorphic representations*; carry information about arithmetic.

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What's that mean really???

$K = O(n) \times \prod_p GL(n, \mathbb{Z}_p)$ is compact subgroup of $GL(n, \mathbb{A}) = GL(n, \mathbb{R}) \times \prod_p' GL(n, \mathbb{Q}_p)$.

Since representation theory for compact groups is nice, can look only at "almost K-invt" automorphic forms.

$$\mathcal{A}(n)^K = \text{ fns on } GL(n, \mathbb{Q})\backslash GL(n, \mathbb{A})/K.$$

Easy:

$$\begin{aligned} GL(n, \mathbb{Q})\backslash GL(n, \mathbb{A})/K &\supset GL(n, \mathbb{Z})\backslash GL(n, \mathbb{R}/O(n) \\ &= GL(n, \mathbb{Z})\backslash D \\ &= GL(n, \mathbb{Z})\backslash \text{pos def forms} \\ &= \{(\text{rk } n \text{ lattice, } \mathbb{R}\text{-val pos def form})\}/\sim \end{aligned}$$

Conclusion: automorphic form on $GL(n) \approx$ fn on isom classes of [rank n lattice w pos def $\mathbb{R}$-valued form].

More general automorphic forms:

$GL(n, \mathbb{Z}_p) \rightsquigarrow$ open subgp $\quad$ $GL(n, \mathbb{Z}) \rightsquigarrow$ cong subgp Γ

$O(n)$-invt $\rightsquigarrow$ rep E of $O(n)$ $\quad$ fns on $\Gamma\backslash D \rightsquigarrow$ secs of $\mathcal{E} \to \Gamma\backslash D$

G reductive group defined over $\mathbb{Q}$: replace $GL(n,$ by $G($.

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What representation theory can tell you (I)

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Automorphic forms $\mathcal{A}(n)$ for $GL(n)$...

Make "decomposition" as in Gelfand's abstract program

$$\mathcal{A}(n) = \int_{\pi \in \widehat{GL(n,\mathbb{A})}_u} V_\pi \otimes M(\pi, \mathcal{A}(n)).$$

V_π = rep space of π, M = multiplicity space.

Done by Langlands (1965).

$$K\text{-invt aut forms} = \mathcal{A}(n)^K = \int_{\pi \in \widehat{GL(n,\mathbb{A})}_u} V_\pi^K \otimes M(\pi, \mathcal{A}(n)).$$

Knowing which unitary reps π can have $V_\pi^K \neq 0$ restricts K-invt automorphic forms.

Knowing which unitary reps of $GL(n, \mathbb{R})$ can have $O(n)$-fixed vectors restricts $L^2(GL(n, \mathbb{Z}) \backslash D)$.

Questions answered (for $GL(n)$) by DV, Tadić in 1980s.

What representation theory can tell you (II)

Example. X compact (arithmetic) locally symmetric manifold of dim 128; $\dim\left(H^{28}(X,\mathbb{C})\right) = ?$ Eight!

Same as H^{28} for compact globally symmetric space.

Generalize: $X = \Gamma\backslash G/K$, $H^p(X,\mathbb{C}) = H^p_{\text{cont}}(G, L^2(\Gamma\backslash G))$. Decomp L^2:

$$L^2(\Gamma\backslash G) = \sum_{\pi \text{ irr rep of } G} m_\pi(\Gamma)\mathcal{H}_\pi \qquad (m_\pi = \text{dim of some aut forms})$$

Deduce $H^p(X,\mathbb{C}) = \sum_\pi m_\pi(\Gamma) \cdot H^p_{\text{cont}}(G, \mathcal{H}_\pi)$.

General principle: group G acts on vector space V; decompose V; study pieces separately.

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Time for something serious

Today: orbit method for predicting what irreducible representations look like.

Can't emphasize enough how important this idea is.

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What the orbit method does

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Gelfand's program says that to better understand problems involving Lie group G, should understand $\widehat{G}_u$, the set of equiv classes of irr unitary reps π of G.

Such π is homomorphism of G into group of unitary operators on (usually ∞-diml) Hilbert space $\mathcal{H}_\pi$: seems much more complicated than G; so what have we gained?

How should we think of an irr unitary representation?

Kirillov-Kostant idea: philosophy of coadjoint orbits...

$$\text{irr unitary rep} \rightsquigarrow \text{coadjoint orbit},$$

orbit of G on dual vector space $\mathfrak{g}_0^*$ of $\mathfrak{g}_0 = \mathrm{Lie}(G)$.

Case of $GL(n)$: says unitary rep is more or less a conj class of $n \times n$ matrices.

Will explain what this statement means, why it is reasonable, and how one can try to prove it.

Decomposing a representation

Given: interesting operators $\mathcal{A}$ on Hilbert space $\mathcal{H}$.
Goal: decompose $\mathcal{H}$ in $\mathcal{A}$-invt way.

Finite-dimensional case:

$V/\mathbb{C}$ fin-diml, $\mathcal{A} \subset \mathrm{End}(V)$ cplx semisimple algebra.

Classical (Wedderburn) structure theorem:

$W_1, \ldots, W_r$ list of all simple $\mathcal{A}$-modules; then

$$\mathcal{A} \simeq \mathrm{End}(W_1) \times \cdots \times \mathrm{End}(W_r) \quad V \simeq m_1 W_1 + \cdots + m_r W_r.$$

Positive integer m_i is *multiplicity* of W_i in V.

Slicker version: define *multiplicity space* $M_i = \mathrm{Hom}_{\mathcal{A}}(W_i, V)$; then $m_i = \dim M_i$, and

$$V \simeq M_1 \otimes W_1 + \cdots + M_r \otimes W_r.$$

Slickest version: COMMUTING ALGEBRAS...

Commuting algebras and all that

Theorem

$\mathcal{A}$ = semisimple algebra of ops on fin-diml V as above; define $\mathcal{Z} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{A})$, second semisimple alg of ops on V.

1. *Relation between $\mathcal{A}$ and $\mathcal{Z}$ is symmetric:*
$$\mathcal{A} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{Z}).$$
2. *There is a natural bijection between irr modules W_i for $\mathcal{A}$ and irr modules M_i for $\mathcal{Z}$, given by*
$$M_i \simeq \mathrm{Hom}_{\mathcal{A}}(W_i, V), \qquad W_i \simeq \mathrm{Hom}_{\mathcal{Z}}(M_i, V).$$
3. *$V \simeq \sum_i M_i \otimes W_i$ as a module for $\mathcal{A} \times \mathcal{Z}$.*

Example 1: finite G acts left and right on $V = \mathbb{C}[G]$.

Example 2: S_n and $GL(E)$ act on $V = T^n(E)$.

But those are stories for other days...

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A version for Lie algebras

Just to show that commuting algebra idea can be made to work... $\mathfrak{g} \supset \mathfrak{h}$ reductive in $\mathfrak{g}$.

$\mathcal{A} =_{\text{def}} U(\mathfrak{h})$, $\mathcal{Z} = \text{Cent}_{U(\mathfrak{g})}(\mathcal{A}) = U(\mathfrak{g})^{\mathfrak{h}}$.

Fix $V = U(\mathfrak{g})$-module. For (μ, E_μ) fin diml $\mathfrak{h}$-irr, set

$$M_\mu = \text{Hom}_{\mathcal{A}}(E_\mu, V) = \text{Hom}_{\mathfrak{h}}(E_\mu, V); \quad \text{then}$$

$$M_\mu \otimes E_\mu \hookrightarrow V \quad \text{(all copies of } \mu \text{ in } V);$$

and M_μ is $\mathcal{Z}$-module.

Theorem (Lepowsky-McCollum)

Suppose V irr for $\mathfrak{g}$, and action of $\mathfrak{h}$ locally finite. Then

$$V = \sum_{\mu \text{ for } \mathfrak{h}} M_\mu \otimes E_\mu.$$

Each M_μ is an irreducible module for $\mathcal{Z}$; and M_μ determines μ and V.

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Infinite-dimensional representations

Need framework to study ops on inf-diml V.

Dictionary

Fin-diml	$\leftrightarrow$	Inf-diml
finite-diml V	$\leftrightarrow$	$C^\infty(M)$
repn of G on V	$\leftrightarrow$	action of G on M
End(V)	$\leftrightarrow$	Diff(M)
$\mathcal{A} = \mathrm{im}(\mathbb{C}[G]) \subset \mathrm{End}(V)$	$\leftrightarrow$	$\mathcal{A} = \mathrm{im}(U(\mathfrak{g})) \subset \mathrm{Diff}(M)$
$\mathcal{Z} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{A})$	$\leftrightarrow$	$\mathcal{Z} = G$-invt diff ops

Suggests: G-irr $V \subset C^\infty(M) \leftrightsquigarrow$ simple modules E for $\mathrm{Diff}(M)^G$, $V \leftrightsquigarrow \mathrm{Hom}_{\mathrm{Diff}(M)^G}(E, C^\infty(M))$.

Suggests: G action on $C^\infty(M)$ irr $\leftrightsquigarrow \mathrm{Diff}(M)^G = \mathbb{C}$.

Not always true, but a good place to start.

Which differential operators commute with G?

Differential operators and symbols

$$\mathrm{Diff}_n(M) = \text{diff operators of order} \le n.$$

Increasing filtration, $(\mathrm{Diff}_p)(\mathrm{Diff}_q) \subset \mathrm{Diff}_{p+q}$.

Theorem (Symbol calculus)

1. *There is an isomorphism of graded algebras*
$$\sigma\colon \operatorname{gr}\mathrm{Diff}(M) \to \mathrm{Poly}(T^*(M))$$
to fns on $T^*(M)$ that are polynomial in fibers.
2. $$\sigma_n\colon \mathrm{Diff}_n(M)/\mathrm{Diff}_{n-1}(M) \to \mathrm{Poly}^n(T^*(M)).$$
3. *Commutator of diff ops $\rightsquigarrow$ Poisson bracket $\{,\}$ on $T^*(M)$: for $D \in \mathrm{Diff}_p(M)$, $D' \in \mathrm{Diff}_q(M)$,*
$$\sigma_{p+q-1}([D, D']) = \{\sigma_p(D), \sigma_q(D')\}.$$

Diff ops comm with G $\leftrightsquigarrow$ symbols Poisson-comm with $\mathfrak{g}$.

$\leftrightsquigarrow$: $\Longrightarrow$ is true, and $\Longleftarrow$ closer than you'd think.

Orig question which diff ops commute with G? becomes which functions on $T^*(M)$ Poisson-commute with $\mathfrak{g}$?

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Poisson structure and Lie group actions

To find fns on $T^*(M)$ Poisson-comm w $\mathfrak{g}$, generalize...

Poisson manifold X has Lie bracket $\{,\}$ on $C^\infty(M)$, such that $\{f,\cdot\}$ is a derivation of $C^\infty(M)$. Poisson bracket on $T^*(M)$ is an example.

Bracket with $f \rightsquigarrow \xi_f \in \mathrm{Vect}(X)$: $\xi_f(g) = \{f, g\}$.

Vector flds ξ_f called *Hamiltonian*; preserve $\{,\}$. Map $C^\infty(X) \to \mathrm{Vect}(X)$, $f \mapsto \xi_f$ is Lie alg homomorphism.

G acts on mfld $X \rightsquigarrow$ Lie alg hom $\mathfrak{g} \to \mathrm{Vect}(X)$, $Y \mapsto \xi_Y$.

Poisson X is *Hamiltonian G-space* if Lie alg action lifts

$$\begin{array}{ccc} & \nearrow & C^\infty(X,\mathbb{R}) \\ & & \downarrow \\ \mathfrak{g}_0 & \to & \mathrm{Vect}(X) \end{array} \qquad \begin{array}{ccc} & \nearrow & f_Y \\ & & \downarrow \\ Y & \to & \xi_Y \end{array}$$

A linear map $\mathfrak{g}_0 \to C^\infty(X,\mathbb{R})$ is the same thing as a smooth moment map $\mu\colon X \to \mathfrak{g}_0^*$.

Poisson structure and invt diff operators

X Hamiltonian G-space, moment map $\mu\colon X \to \mathfrak{g}_0^*$ G-eqvt map of Poisson mflds,

$$f_Y(x) = \langle \mu(x), Y\rangle \qquad (Y \in \mathfrak{g}_0, x \in X).$$

$$f \in C^\infty(X) \text{ Poisson-commutes with } \mathfrak{g}_0$$

$$\iff \xi_Y f = 0, \quad (Y \in \mathfrak{g}_0)$$

$$\iff f \text{ constant on } G \text{ orbits on } X.$$

Only $\mathbb{C}$ Poisson-comm with $\mathfrak{g}_0 \iff$ dense orbit on X.

Proves: dense orbit on $T^*(M) \implies \mathrm{Diff}(M)^G = \mathbb{C}$.

Suggests: G irr on $C^\infty(M) \iff$ dense orbit on $T^*(M)$.

Suggests to a visionary: Irr reps of G correspond to homogeneous Hamiltonian G-spaces.

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Method of coadjoint orbits

Recall: Hamiltonian G-space X comes with (G-equivariant) moment map $\mu\colon X \to \mathfrak{g}_0^*$.

Kostant's theorem: homogeneous Hamiltonian G-space $=$ covering of G-orbit on $\mathfrak{g}_0^*$.

Recall: commuting algebra formalism for diff operators suggests irreducible representations $\leftrightsquigarrow$ homogeneous Hamiltonian G-spaces.

Kirillov-Kostant philosophy of coadjt orbits suggests

$$\{\text{irr unitary reps of } G\} = \widehat{G}_u \leftrightsquigarrow \mathfrak{g}_0^*/G. \qquad (\star)$$

MORE PRECISELY… restrict right side to "admissible" orbits (integrality cond). Expect to find "almost all" of $\widehat{G}_u$: enough for interesting harmonic analysis.

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Evidence for orbit method

With the caveat about restricting to admissible orbits. . .

$$\widehat{G}_u \leftrightsquigarrow \mathfrak{g}^*/G. \qquad (\star)$$

$(\star)$ is true for G simply conn nilpotent (Kirillov).

$(\star)$ is true for G type I solvable (Auslander-Kostant).

$(\star)$ for algebraic G reduces to reductive G (Duflo).

Case of reductive G is still open.

Actually $(\star)$ is false for connected nonabelian reductive G.
But there are still theorems close to $(\star)$.

Two ways to do repn theory for reductive G:

1. start with coadjt orbit, look for repn. Hard: Lecture 5.
2. start with repn, look for coadjt orbit. Easy: Lecture 4.

Really need to do both things at once. Having started to do mathematics in the Ford administration, I find this challenging. (Gave up chewing gum at that time.)

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From $\mathfrak{g}$-modules to $\mathfrak{g}^*$

"Classical limit" direction of the orbit philosophy asks for a map (irr unitary reps) $\rightsquigarrow$ orbits in $\mathfrak{g}_0^*$.

V rep of complex Lie alg $\mathfrak{g}$.

Assume V is finitely generated: exists fin diml $V_0 \subset V$ so that $U(\mathfrak{g})V_0 = V$.

Define increasing family of subspaces
$V_0 \subset V_1 \subset V_2 \subset \cdots$, $V_m = U_m(\mathfrak{g})V_0$.

$V_m =$ span of $Y_1 \cdot Y_2 \cdots Y_{m'} \cdot v_0$, $(v_0 \in V_0, Y_i \in \mathfrak{g}, m' \leq m)$.

Action of $\mathfrak{g}$ gives $\mathfrak{g} \times V_m \to V_{m+1}, \quad (Y, v_m) \mapsto Y \cdot v_m$, and therefore a well-defined map
$\mathfrak{g} \times [V_m/V_{m-1}] \to [V_{m+1}/V_m], \quad (Y, v_m + V_{m-1}) \mapsto Y \cdot v_m + V_m.$
Actions of different elts of $\mathfrak{g}$ commute; so gr V is a graded $S(\mathfrak{g})$-module generated by the fin-diml subspace V_0.

Associated variety $\mathrm{Ass}(V) = \mathrm{supp}(\mathrm{gr}\ V) \subset \mathfrak{g}^*$ (defined by commutative algebra).

What's good about Ass(V)

V fin gen $/U(\mathfrak{g})$, $V_m = U_m(\mathfrak{g})V_0$, $\mathrm{Ass}(V) = \mathrm{supp}(\mathrm{gr}(V))$.

Commutative algebra tells you many things:

1. $\dim V_m = p_V(m)$, is a polynomial function of m.
2. The degree d of p_V is $\dim(\mathrm{Ass}(V))$. Define the Gelfand-Kirillov dimension of V to be $\mathrm{Dim}\, V = d$.
3. $I_{\mathrm{gr}} =_{\mathrm{def}} \mathrm{Ann}(\mathrm{gr}(V)) \subset S(\mathfrak{g})$, graded ideal; then $d = \dim(S(\mathfrak{g})/I_{\mathrm{gr}})$ (Krull dimension).
4. $I =_{\mathrm{def}} \mathrm{Ann}(V) \subset U(\mathfrak{g})$ 2-sided ideal; $\mathrm{gr}\, I \subset I_{\mathrm{gr}}$, usually $\neq$.

Example. $\mathfrak{g} = \mathrm{span}(p, q, z)$, $[p, q] = z$, $[z, p] = [z, q] = 0$.

$$V = \mathbb{C}[x], \quad p \cdot f = df/dx, \quad q \cdot f = xf, z \cdot f = f.$$

This is (irr) rep of $\mathfrak{g}$ generated by $V_0 = \mathbb{C}$.

$$V_m = \text{polys in } x \text{ of degree} \leq m, \quad \dim V_m = m + 1.$$

$\mathrm{gr}\, V \simeq \mathbb{C}[x]$; $p \rightsquigarrow$ mult by x; $q, z \rightsquigarrow$ zero; $I_{\mathrm{gr}} = \langle q, z\rangle \subset S(\mathfrak{g})$.

$I = \langle z - 1\rangle$, $\quad U(\mathfrak{g})/I \simeq$ Weyl algebra $\mathbb{C}[d/dx, x]$, $\mathrm{gr}\, I = \langle z\rangle$.

$\mathrm{Ass}(V) = \{\lambda \in \mathfrak{g}^* \mid \lambda(q) = \lambda(z) = 0\} \subset \mathrm{supp}(\mathrm{gr}\, I) = \{\lambda \mid \lambda(z) = 0\}$.

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What's bad about Ass(V)

For fin gen M over poly alg S, $I = \mathrm{Ann}(M) \subset S$,

$$\mathrm{Dim}(M) = \mathrm{Dim}\, S/I, \qquad \mathrm{supp}\, M = \mathrm{supp}(I).$$

For fin gen V over $U(\mathfrak{g})$, $I = \mathrm{Ann}(V)$, $I_{\mathrm{gr}} = \mathrm{Ann}(\mathrm{gr}(V))$,

$$\mathrm{Dim}(V) = \mathrm{Dim}\, S(\mathfrak{g})/I_{\mathrm{gr}}, \qquad \mathrm{Ass}(V) = \mathrm{supp}(I_{\mathrm{gr}}), \quad \text{but}$$
$$\mathrm{gr}(I) \subset I_{\mathrm{gr}}, \quad \mathrm{supp}(\mathrm{gr}\, I) \supset \mathrm{Ass}(V), \quad \mathrm{Dim}(S(\mathfrak{g})/\,\mathrm{gr}\, I) \geq \mathrm{Dim}(V);$$

containments and inequalities generally strict.

Closely related and worse: even if V related to nice rep of G, Ass(V) rarely preserved by G. Some good news. . .

Proposition

V fin gen $/U(\mathfrak{g})$ by V_0, V_0 preserved by $\mathfrak{h} \subset \mathfrak{g} \Longrightarrow$ $\mathrm{Ass}(V) \subset (\mathfrak{g}/\mathfrak{h})^*$ *stable under coadjt action of H.*

I 2-sided ideal in $U(\mathfrak{g}) \Longrightarrow$ $\mathrm{Ass}(\mathrm{gr}\, I)$ *G-stable.*

Ideal picture (correct for irr $(\mathfrak{g}, K)$-modules defined *infra*):

$$V = \text{irr } U(\mathfrak{g})\text{-module},$$
$$I = \mathrm{Ann}(V) = \text{2-sided prim ideal in } U(\mathfrak{g});$$
$$\mathrm{Ass}(I) = \text{aff alg Hamilt. } G\text{-space}, \qquad \dim \mathrm{Ass}(I) = 2d;$$
$$\mathrm{Ass}(V) = \text{coisotropic subvar of } X, \qquad \dim \mathrm{Ass}(V) = d.$$

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Deformation quantization and wishful thinking

Here is how orbit method might work for reductive groups.

$G(\mathbb{R})$ = real points of conn cplx reductive alg $G(\mathbb{C})$.

Start with $\mathcal{O}_0 \subset \mathfrak{g}_0^*$ coadjoint orbit for $G(\mathbb{R})$.

$$\mathcal{O}(\mathbb{C}) =_{\text{def}} G(\mathbb{C}) \cdot \mathcal{O}_0, \quad J_{\mathcal{O}} = \text{ideal of } \mathcal{O}(\mathbb{C}).$$

$\mathcal{O}_0 \subset \mathcal{O}(\mathbb{R})$ must be open, but may be proper subset.

Ring of functions $R_{\overline{\mathcal{O}}} = S(\mathfrak{g})/J_{\mathcal{O}}$ makes $\overline{\mathcal{O}}(\mathbb{C})$ affine alg Poisson variety, Hamiltonian G-space. (Better: normalize to slightly larger algebra $R(\mathcal{O}(\mathbb{C})$.)

Simplify: $\mathcal{O}(\mathbb{C})$ nilp; equiv, $J_{\mathcal{O}}$ and $R_{\overline{\mathcal{O}}}$ graded:

$$R_{\overline{\mathcal{O}}} = \sum_{p \geq 0} R^p, \quad R^p \cdot R^q \subset R^{p+q}, \quad \{R^p, R^q\} \subset R^{p+q-1}.$$

G-eqvt deformation quantization of $\overline{\mathcal{O}}$ is filtered algebra $D = \cup_{p \geq 0} D_p$, $G(\mathbb{C})$ action by alg auts, symbol calculus

$$\sigma_p \colon D_p/D_{p-1} \xrightarrow{\sim} R^p$$

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What deformation quantization looks like

$R_{\mathcal{O}} = \sum_{p\geq 0} R^p$ graded ring of fns on cplx nilpotent coadjt orbit, D_p "corresponding" filtered algebra with $G(\mathbb{C})$ action.

Since $G(\mathbb{C})$ reductive, can choose $G(\mathbb{C})$-stable complement C^p for D_{p-1} in D_p; then $\sigma_p\colon C^p \xrightarrow{\sim} R^p$ must be isom, so have $G(\mathbb{C})$-eqvt linear isoms

$$D_p = \sum_{q\leq p} C^p \xrightarrow{\sigma} \sum_{q\leq p} R^p, \qquad D \xrightarrow{\sigma} R.$$

Mult in D defines via isom σ new assoc product m on R:

$$m\colon R\times R \to R, \quad m(r,s) = \sigma\left(\sigma^{-1}(r)\cdot\sigma^{-1}(s)\right).$$

Filtration on D implies that for $r\in R^p$, $s\in R^q$,

$$m(r,s) = \sum_{k=0}^{p+q} m_k(r,s), \quad m_k(r,s)\in R^{p+q-k}.$$

Proposition

$G(\mathbb{C})$-eqvt deformation quantization of alg $R_{\mathcal{O}}$ (fns on a cplx nilp coadjt orbit) given by $G(\mathbb{C})$-eqvt bilinear maps $m_k\colon R^p\times R^q \to R^{p+q-k}$, subject to $m_0(r,s) = r\cdot s$, $m_1(r,s) = \{r,s\}$, and the reqt that $\sum_{k=0}^{\infty} m_k$ is assoc.

OPEN PROBLEM: PROVE DEFORMATIONS EXIST.

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Why this is reasonable

$P(\mathbb{C}) \subset G(\mathbb{C})$ parabolic, $M(\mathbb{C}) = G(\mathbb{C})/P(\mathbb{C})$ proj alg.
$G(\mathbb{C})$ has unique open orbit $\widetilde{\mathcal{O}}(\mathbb{C}) \subset T^*M(\mathbb{C})$, which by Kostant must be finite cover of nilp coadjt orbit $\mathcal{O}(\mathbb{C})$:

$$\begin{array}{ccccc} \widetilde{\mathcal{O}}(\mathbb{C}) & \subset & T^*M(\mathbb{C}) & & \\ \downarrow \mu_{\mathcal{O}} & & \downarrow \mu & & \\ \mathcal{O}(\mathbb{C}) & \subset & \overline{\mathcal{O}(\mathbb{C})} & \subset & \mathfrak{g}^* \end{array}$$

$\mu_{\mathcal{O}}$ is finite cover; μ is proper surjection. Put

$$D = \text{alg diff ops on } M(\mathbb{C}), \quad S = \text{alg fns on } T^*M(\mathbb{C})$$

$$R^{\text{norm}} = \text{alg fns on } \mathcal{O}(\mathbb{C}), \quad R = \text{alg fns on } \overline{\mathcal{O}(\mathbb{C})}.$$

1. Symbol calculus provides isom gr $D \xrightarrow{\sigma} S$.
2. Restriction provides isom $S \simeq$ alg fns on $\widetilde{\mathcal{O}}(\mathbb{C})$.
3. $\mu_{\mathcal{O}}^*$ isom $\Leftrightarrow$ cover triv $\Leftrightarrow$ μ is birational.
4. Inclusion exhibits R^{norm} as normalization of R.

Conclusion (Borho-Jantzen): D is nice deformation quantization of $\mathcal{O}(\mathbb{C}) \Leftrightarrow \mu$ birational with normal image.

Always true for $GL(n)$.

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Simple complex facts

$G(\mathbb{C})$ cplx conn reductive alg, $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{C}))$.

$\mathfrak{h} \subset \mathfrak{b} = \mathfrak{h} + \mathfrak{n} \subset \mathfrak{g}$ Cartan and Borel subalgebras.

$X_s \in \mathfrak{g}$ *semisimple* if following equiv conds hold:

1. $\mathrm{ad}(X_s)$ diagonalizable;
2. $\rho(X_s)$ diagonalizable, all $\rho\colon G(\mathbb{C}) \to GL(N, \mathbb{C})$ alg.
3. $G(\mathbb{C}) \cdot X_s$ is closed;
4. $G(\mathbb{C}) \cdot X_s$ meets $\mathfrak{h}$.
5. $G(\mathbb{C})^{X_s}$ is reductive.

$X_n \in \mathfrak{g}$ *nilpotent* if following equiv conds hold:

1. $\mathrm{ad}(X_n)$ nilpotent *and* $X_n \in [\mathfrak{g}, \mathfrak{g}]$;
2. $\rho(X_n)$ nilpotent, all $\rho\colon G(\mathbb{C}) \to GL(N, \mathbb{C})$ alg.
3. $G(\mathbb{C}) \cdot X_n$ closed under dilation;
4. $G(\mathbb{C}) \cdot X_n$ meets $\mathfrak{n}$.

Jordan decomposition: every $X \in \mathfrak{g}$ is uniquely
$X = X_s + X_n$ with X_s semisimple, X_n nilpotent, $[X_s, X_n] = 0$.

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Simple complex dual facts

$G(\mathbb{C})$ still cplx reductive, $\mathfrak{g}^* =$ complex dual space, Ad^* coadjoint action of $G(\mathbb{C})$.

There exists symm *Ad*-invt form on $\mathfrak{g}$; equiv, $\mathfrak{g} \simeq \mathfrak{g}^*$, $\mathrm{Ad} \simeq \mathrm{Ad}^*$. Can use to transfer previous slide to $\mathfrak{g}^*$.

THIS IS ALWAYS A BAD IDEA: $\mathfrak{g}^*$ is different.

$\lambda_s \in \mathfrak{g}^*$ *semisimple* if following equiv conds hold:

1. $G(\mathbb{C}) \cdot \lambda_s$ is closed;
2. $G(\mathbb{C})^{\lambda_s}$ is reductive.

$\lambda_n \in \mathfrak{g}^*$ *nilpotent* if following equiv conds hold:

1. $G(\mathbb{C}) \cdot \lambda_n$ closed under dilation;
2. λ_n vanishes on some Borel subalgebra of $\mathfrak{g}$.
3. For each $p \in S(\mathfrak{g})^{G(\mathbb{C})}$, $p(\lambda_n) = p(0)$.

Jordan decomposition: every $\lambda \in \mathfrak{g}^*$ is uniquely $\lambda = \lambda_s + \lambda_n$ with λ_s semisimple, λ_n nilpotent, and $\lambda_s + t\lambda_n \in G(\mathbb{C}) \cdot \lambda$ (all $t \in \mathbb{C}^\times$).

PROBLEM: extend these lists of equiv conds. Find analogue of Jacobson-Morozov for nilpotents in $\mathfrak{g}^*$.

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Back to associated varieties

$\mathfrak{Z}(\mathfrak{g})$ = center of $U(\mathfrak{g})$; at first $\mathfrak{g}$ is arbitrary.

Definition

Rep (π, V) of $\mathfrak{g}$ is *quasisimple* if $\pi(z)$ = scalar, all $z \in \mathfrak{Z}(\mathfrak{g})$. Alg homomorphism $\chi_V \colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ is the *infinitesimal character of V*. Write $J_V = \ker(\chi_V)$, maximal ideal in $\mathfrak{Z}(\mathfrak{g})$.

Easy fact: any irr V is quasisimple, so $I_V = \mathrm{Ann}(V) \supset J_V$, so $\mathrm{gr}\, I_V \supset \mathrm{gr}\, J_V$.

Another easy fact: $\mathrm{gr}\, \mathfrak{Z}(\mathfrak{g}) = S(\mathfrak{g})^{G(\mathbb{C})}$.

So $\mathrm{gr}\, J_V$ is graded maximal ideal in $S(\mathfrak{g})^{G(\mathbb{C})}$, so

$$\mathrm{gr}\, I_V \supset \mathrm{gr}\, J_V = \text{augmentation ideal in } S(\mathfrak{g})^{G(\mathbb{C})}.$$

$$\mathrm{Ass}(V) \subset \mathrm{Ass}(I_V) \subset \text{zeros of aug ideal in } S(\mathfrak{g})^{G(\mathbb{C})}.$$

Theorem

If V is fin gen quasisimple module for reductive $\mathfrak{g}$ (in particular, if V irreducible, then $\mathrm{Ass}(V)$ consists of nilpotent elts of $\mathfrak{g}^*$.

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Howe's wavefront set

...defined in Howe's beautiful paper, which you should read. Defined for unitary $(\pi, \mathcal{H}_\pi)$ of Lie gp G; def shows $\mathrm{WF}(\pi) \subset \mathfrak{g}_0^*$, closed cone preserved by coadjt action of G. Definition involves wavefront sets of certain distributions T on G constructed using matrix coeffs of π.

If π is quasisimple (automatic for irr unitary π, by thm of Segal in Lec 5) then such T has $(\partial(z) - \chi_\pi(z))T = 0$.

Distribution on right above is smooth, so wavefront set is zero. Basic smoothness thm: applying diff op D can decrease wavefront set only by zeros of $\sigma(D)$.

So $\mathrm{WF}(T) \subset$ zeros of $\sigma(z)$, all $z \in \mathfrak{Z}(\mathfrak{g})$ of pos deg:

$$\mathrm{WF}(\pi) \subset \text{zeros of augmentation ideal in } S(\mathfrak{g})^{G(\mathbb{C})}.$$

Same proof: $\mathrm{WF}(\pi) \subset \mathrm{Ass}(\mathrm{Ann}(\mathcal{H}_\pi))$.

So $\mathrm{WF}(\pi)$ gives G-invt subset of $\mathfrak{g}_0^*$ sharing many props of $\mathrm{Ass}(V_\pi) \overset{?}{\leadsto}$ *better* classical limit than $\mathrm{Ass}(V_\pi)$.

But for reductive G, $\mathrm{WF}(\pi)$, $\mathrm{Ass}(V_\pi)$ computable from each other (Schmid-Vilonen); so pick by preference.

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Principal series revisited

Recall complete flag in m-diml vector space E is

$$\mathcal{F} = \{0 = F_0 \subset F_1 \subset \cdots \subset F_m = E\}, \quad \dim F_i = i.$$

Recall construction of principal series representations:

$$G = GL(n,k) \supset B = \text{upper triangular matrices}$$

$$X_n(k) = \text{complete flags in } k^n \simeq G/B.$$

Fixing n characters (group homomorphisms) $\xi_j \colon k^\times \to \mathbb{C}^\times$ defines complex line bundle $\mathcal{L}^\xi$;

$$V^\xi = \text{secs of } \mathcal{L}^\xi \simeq \{f \colon G \to \mathbb{C} \mid f(gb) = \xi(b)^{-1} f(g)\ (b \in B)\},$$

$$\xi \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ & & \ddots & \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} = \xi_1(b_{11})\xi_2(b_{22})\cdots\xi_n(b_{nn}).$$

principal series rep of $GL(n,k)$ with param ξ.

Appropriate choice of topological vector space V^ξ (continuous, smooth, $L^2 \ldots$) depends on the problem.

$k = \mathbb{R}$: character ξ is $(\nu, \epsilon) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$, $t \mapsto |t|^\nu \operatorname{sgn}(t)^\epsilon$

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Principal series for $SL(2, \mathbb{R})$

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Want to understand principal series repns for ($GL(2, \mathbb{R})$ restricted to) $SL(2, \mathbb{R})$. Helpful to use different picture

$$W^{\nu,\epsilon} = \{f\colon (\mathbb{R}^2 - 0) \to \mathbb{C} \mid f(tx) = |t|^{-\nu} \operatorname{sgn}(t)^\epsilon f(x)\},$$

functions on the plane homog of degree $-(\nu, \epsilon)$.

Exercise: $V^{(\nu_1,\nu_2)(\epsilon_1,\epsilon_2)}|_{SL(2,\mathbb{R})} \simeq W^{\nu_1-\nu_2,\epsilon_1-\epsilon_2}$.

Lie algs easier than Lie gps $\rightsquigarrow$ write $\mathfrak{sl}(2, \mathbb{R})$ action, basis

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[D, E] = 2E, \quad [D, F] = -2F, \quad [E, F] = D.$$

action on functions on $\mathbb{R}^2$ is by

$$D = -x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \quad E = -x_2 \frac{\partial}{\partial x_1}, \quad F = -x_1 \frac{\partial}{\partial x_2}.$$

Now want to restrict to homogeneous functions. . .

Principal series for $SL(2, \mathbb{R})$ (continued)

Study homog fns on $\mathbb{R}^2 - 0$ by restr to $\{(\cos\theta, \sin\theta)\}$:

$$W^{\nu,\epsilon} \simeq \{w \colon S^1 \to \mathbb{C} \mid w(-s) = (-1)^\epsilon w(s)\},\ f(r,\theta) = r^{-\nu} w(\theta).$$

Compute Lie algebra action in polar coords using

$$\frac{\partial}{\partial x_1} = -x_2 \frac{\partial}{\partial \theta} + x_1 \frac{\partial}{\partial r}, \qquad \frac{\partial}{\partial x_2} = x_1 \frac{\partial}{\partial \theta} + x_2 \frac{\partial}{\partial r},$$

$$\frac{\partial}{\partial r} = -\nu, \qquad x_1 = \cos\theta, \qquad x_2 = \sin\theta.$$

Plug into formulas on preceding slide: get

$$\rho^\nu(D) = 2\sin\theta\cos\theta \frac{\partial}{\partial\theta} + (-\cos^2\theta + \sin^2\theta)\nu,$$

$$\rho^\nu(E) = \sin^2\theta \frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)\nu,$$

$$\rho^\nu(F) = -\cos^2\theta \frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)\nu.$$

Hard to make sense of. Clear: family of reps analytic (actually linear) in complex parameter ν.

Big idea: see how properties change as function of ν.

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A more suitable basis

Have family $\rho^{\nu,\epsilon}$ of reps of $SL(2,\mathbb{R})$ defined on functions on S^1 of homogeneity (or parity) ϵ:

$$\rho^\nu(D) = 2\sin\theta\cos\theta\frac{\partial}{\partial\theta} + (-\cos^2\theta + \sin^2\theta)\nu,$$
$$\rho^\nu(E) = \sin^2\theta\frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)\nu,$$
$$\rho^\nu(F) = -\cos^2\theta\frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)\nu.$$

Problem: $\{D, E, F\}$ adapted to wt vectors for diagonal Cartan subalgebra; rep $\rho^{\nu,\epsilon}$ has no such wt vectors.

But rotation matrix $E - F$ acts simply by $\partial/\partial\theta$.

Suggests new basis of the complexified Lie algebra:

$$H = -i(E - F), \quad X = \frac{1}{2}(D + iE + iF), \quad Y = \frac{1}{2}(D - iE - iF).$$

Same commutation relations as D, E, and F

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H$$

but complex conjugation is different: $\overline{H} = -H$, $\overline{X} = Y$.

$$\rho^\nu(H) = \frac{1}{i}\frac{\partial}{\partial\theta},\ \rho^\nu(X) = \frac{e^{2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i\nu\right),\ \rho^\nu(Y) = \frac{-e^{-2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i\nu\right).$$

Matrices for principal series, bad news

Have family $\rho^{\nu,\epsilon}$ of reps of $SL(2,\mathbb{R})$ defined on functions on S^1 of homogeneity (or parity) ϵ:

$$\rho^\nu(H) = \frac{1}{i}\frac{\partial}{\partial\theta}, \ \rho^\nu(X) = \frac{e^{2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i\nu\right), \ \rho^\nu(Y) = \frac{-e^{-2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i\nu\right).$$

These ops act simply on basis $w_m(\cos\theta, \sin\theta) = e^{im\theta}$:

$$\rho^\nu(H)w_m = mw_m,$$

$$\rho^\nu(X)w_m = \frac{1}{2}(m+\nu)w_{m+2},$$

$$\rho^\nu(Y)w_m = \frac{1}{2}(-m+\nu)w_{m-2}.$$

Suggests reasonable function space to consider:

$$W^{\nu,\epsilon,K} = \text{fns homog of deg } (\nu,\epsilon), \text{ finite under rotation}$$
$$= \text{span}(\{w_m \mid m \equiv \epsilon \pmod 2\}).$$

Space $W^{\nu,\epsilon,K}$ has beautiful rep of $\mathfrak{g}$: irr for most ν, easy submods otherwise. Not preserved by rep of $G = SL(2,\mathbb{R})$: $\exp(A) \in G \rightsquigarrow \sum A^k/k!$: A^k preserves $W^{\nu,\epsilon,K}$, sum need not.

Structure of principal series: good news

Original question was action of $G = SL(2,\mathbb{R})$ on

$$W^{\nu,\epsilon,\infty} = \{f \in C^\infty(\mathbb{R}^2 - 0) \mid f \text{ homog of deg } -(\nu,\epsilon)\}:$$

what are the closed G-invt subspaces. . . ?

Found nice subspace $W^{\nu,\epsilon,K}$, explicit basis, explicit action of Lie algebra $\rightsquigarrow$ easy to describe $\mathfrak{g}$–invt subspaces.

Theorem (Harish-Chandra tiny)

There is a one-to-one corr closed G-invt subspaces $S \subset W^{\nu,\epsilon,\infty}$ and $\mathfrak{g}$-invt subspaces $S^K \subset W^{\nu,\epsilon,K}$. Corr is $S \rightsquigarrow$ subspace of K-finite vectors, and $S^K \rightsquigarrow$ its closure:

$$S^K = \{s \in S \mid \dim \operatorname{span}(\rho^{\nu,\epsilon}(SO(2))s) < \infty)\}, \quad S = \overline{S^K}.$$

Content of thm: closure carries $\mathfrak{g}$-invt to G-invt.

Why this isn't obvious: $SO(2)$ acting by translation on $C^\infty(S^1)$. Lie alg acts by $\frac{d}{d\theta}$, so closed subspace

$$E = \{f \in C^\infty(S^1) \mid f(\cos\theta,\sin\theta) = 0, \theta \in (-\pi/2,\pi/2) + 2\pi\mathbb{Z}\}$$

is preserved by $\mathfrak{so}(2)$; *not* preserved by rotation.

Reason: Taylor series for in $f \in E$ doesn't converge to f.

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Same formalism, general G

Lesson of $SL(2,\mathbb{R})$ princ series: vecs finite under $SO(2)$ have nice/comprehensible/meaningful Lie algebra action.

General structure theory: $G = G(\mathbb{R})$ real pts of conn reductive complex algebraic group $\rightsquigarrow$ can embed

$$G \hookrightarrow GL(n,\mathbb{R}), \quad \text{stable by transpose}, \quad G/G_0 \text{ finite}.$$

Recall *polar decomposition*:

$$GL(n,\mathbb{R}) = O(n) \times (\text{pos def symmetric matrices})$$
$$= O(n) \times \exp(\text{symmetric matrices}).$$

Inherited by G as Cartan decomposition for G:

$$K = O(n) \cap G, \quad \mathfrak{s}_0 = \mathfrak{g}_0 \cap (\text{symm mats}), \quad S = \exp(\mathfrak{s}_0)$$

$$G = K \times S = K \times \exp(\mathfrak{s}_0).$$

(ρ, W) rep of G on complete loc cvx top vec W;

$$W^K = \{w \in W \mid \dim \operatorname{span}(\rho(K)w) < \infty\},$$
$$W^\infty = \{w \in W \mid G \to W, g \mapsto \rho(g)w \text{ smooth}\}.$$

Definition. The $(\mathfrak{g}, K)$-module of W is $W^{K,\infty}$. It is a representation of the Lie algebra $\mathfrak{g}$ and of the group K.

Category of $(\mathfrak{h}, L)$-modules

Setting: $\mathfrak{h} \supset \mathfrak{l}$ complex Lie algebras, L compact Lie group acting on $\mathfrak{h}$ by Lie alg auts Ad.

Definition

An $(\mathfrak{h}, L)$-module is complex vector space W endowed with reps of $\mathfrak{h}$ and of L, subject to following conds.

1. Each $w \in W$ belongs to fin-diml L-invt W_0, such that action of L on W_0 continuous (hence smooth).
2. Complexified differential of L action is $\mathfrak{l}$ action.
3. For $k \in L$, $Z \in \mathfrak{h}$, $w \in W$,
   $k \cdot (Z \cdot (k^{-1} \cdot w)) = [\mathrm{Ad}(k)(Z)] \cdot w$.

Proposition

Passage to smooth K-finite vectors defines a functor

$$\textit{(reps of } G \textit{ on complete loc cvx } W) \to (\mathfrak{g}, K)\textit{-mods } W^{K,\infty}$$

.

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Representations and R-modules

Rings and modules familiar and powerful $\rightsquigarrow$ try to make representation categories into module categories.

Category of reps of $\mathfrak{h}$ = category of $U(\mathfrak{h})$-modules.

Seek parallel for locally finite reps of compact L:

$$R(L) = \text{convolution alg of } \mathbb{C}\text{-valued } L\text{-finite msres on } L$$
$$\simeq \sum_{(\mu, E_\mu)\in \widehat{L}} \text{End}(E_\mu) \qquad \text{(Peter-Weyl)}$$

$1 \notin R(L)$ if L is infinite: convolution identity is delta function at $e \in L$; not L-finite.

$\alpha \subset \widehat{L}$ finite $\rightsquigarrow$ $1_\alpha =_{\text{def}} \sum_{\mu\in\alpha} \text{Id}_\mu$.

Elts 1_α are *approximate identity* in $R(L)$: for all $r \in R(L)$ there is $\alpha(r)$ finite so $1_\beta \cdot r = r \cdot 1_\beta = r$ if $\beta \supset \alpha(r)$.

$R(L)$-module M is *approximately unital* if for all $m \in M$ there is $\alpha(m)$ finite so $1_\beta \cdot m = m$ if $\beta \supset \alpha(m)$.

Loc fin reps of L = approx unital $R(L)$-modules.

If ring R has approx ident $\{1_\alpha\}_{\alpha\in S}$, write R-mod for category of approx unital R-modules.

Hecke algebras

Setting: $\mathfrak{h} \supset \mathfrak{l}$ complex Lie algebras, L compact Lie group acting on $\mathfrak{h}$ by Lie alg auts Ad.

Definition

The Hecke algebra $R(\mathfrak{h}, L)$ is

$$R(\mathfrak{h}, L) = U(\mathfrak{h}) \otimes_{U(\mathfrak{l})} R(L)$$
$$\simeq \text{[conv alg of } L\text{-finite } U(\mathfrak{h})\text{-valued msres on } L]/[U(\mathfrak{l}) \text{ action]}$$

$R(\mathfrak{h}, L)$ inherits approx identity from subalg $R(L)$.

Proposition

Category of $(\mathfrak{h}, L)$-modules is category $R(\mathfrak{h}, L)$-mod of approx unital modules for Hecke algebra $R(\mathfrak{h}, L)$.

Exercise: repeat with L cplx alg gp (not nec reductive).

Immediate corollary: category of $(\mathfrak{h}, L)$-modules has projective resolutions, so derived functors. . .

Lecture 7: use easy change-of-ring functors to construct $(\mathfrak{g}, K)$-modules.

Group reps and Lie algebra reps

G real reductive alg $\supset K$ max cpt, $\mathfrak{Z}(\mathfrak{g})$ = center of $U(\mathfrak{g})$.

Definition

Rep (π, V) of G on complete loc cvx V is *quasisimple* if $\pi^\infty(z)$ = scalar, all $z \in \mathfrak{Z}(\mathfrak{g})$. Alg hom $\chi_\pi \colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ is the *infinitesimal character of* π.

Make exactly same defn for $(\mathfrak{g}, K)$-modules.

Theorem (Segal, Harish-Chandra)

1. *Any irr $(\mathfrak{g}, K)$-module is quasisimple.*
2. *Any irr unitary rep of G is quasisimple.*
3. *Suppose V quasisimple rep of G. Then $W \mapsto W^{K,\infty}$ is bij [closed $W \subset V$] and [$W^{K,\infty} \subset V^{K,\infty}$].*
4. *Correspondence (irr quasisimple reps of G) $\rightsquigarrow$ (irr $(\mathfrak{g}, K)$-modules) is surjective. Fibers are* infinitesimal equiv classes *of irr quasisimple reps of G.*

Non-quasisimple irr reps exist if G' noncompact (Soergel), but are "pathological;" unrelated to harmonic analysis.

Idea of proof: $G/K \simeq \mathfrak{s}_0$, vector space. Describe anything analytic on G by Taylor exp along K.

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