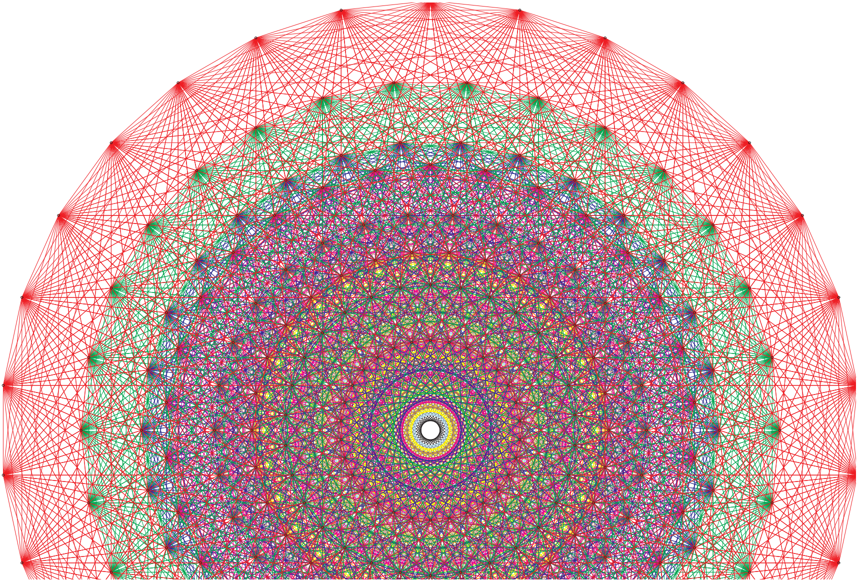
Atlas of Lie Groups
and Representations

www.liegroups.org



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Atlas Project Members, AIM, July 2007

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Example: $G = GL(n,\mathbb{R})$ - Vogan (1986)

Known Unitary Duals

red: known black: not known

Type A: $SL(n,\mathbb{R})$, $SL(n,\mathbb{H})$, $SU(n,1)$, $SU(n,2)$, $SL(n,\mathbb{C})$
$SU(p,q)$ $(p,q>2)$

Type B: $SO(2n,1)$, $SO(2n+1,2)$, $SO(2n+1,\mathbb{C})$

$SO(p,q)$ $(p,q\geq 3)$

Type C: $Sp(4,\mathbb{R})$, $Sp(n,1)$, $Sp(2n,\mathbb{C})$

$Sp(p,q)$ $(p,q\geq 2)$

Type D: $SO(2n+1,1)$, $SO(2n,2)$, $SO(2n,\mathbb{C})$

$SO(p,q)$ $(p,q\geq 3)$, $SO^*(2n)$ $(n\geq 4)$

Type E_6: $E_6(F_4)=SL(3,\text{Cayley})$

E_6(Hermitian), E_6(split), E_6(quaternionic), $E_6(\mathbb{C})$

Type F_4: $F_4(B_4)$

F_4(split), $F_4(\mathbb{C})$

Type G_2: G_2(split), $G_2(\mathbb{C})$

E_7/E_8: nothing known

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Atlas of Lie Groups and Representations:

Take this idea seriously

p-adic groups

Fix a p-adic group G.

Question: Is there a finite algorithm to compute:

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(So far the answer seems to be no...)

Goals of the Atlas Project

- Tools for education: teaching Lie groups to graduate students and researchers

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- Compute the unitary dual

Outline of the lecture

Constructing representations of Weyl Groups
Computing the signature of a quadratic form
Explicitly computing the admissible dual
KLV polynomials and the E_8 calculation
Unipotent representations and the future

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Example: The character table of every Weyl group W is known.

W=Weyl group, simple reflections $s_1, \dots, s_n$

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Fact: can use matrices with integral entries (Springer correspondence)

Character table of $W(E_8)$

Class		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Size		1	1	120	120	3150	3780	3780	37800	37800	113400	2240	4480	89600	268800	15120
Order		1	2	2	2	2	2	2	2	2	2	3	3	3	3	4
X.1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	+	1	1	-1	-1	1	1	1	-1	-1	1	1	1	1	1	1
X.3	+	8	-8	-6	6	0	4	-4	2	-2	0	5	-4	-1	2	0
X.4	+	8	-8	6	-6	0	4	-4	-2	2	0	5	-4	-1	2	0
X.5	+	28	28	14	14	-4	4	4	-2	-2	-4	10	10	1	1	4
X.6	+	28	28	-14	-14	-4	4	4	2	2	-4	10	10	1	1	4
X.7	+	35	35	21	21	3	11	11	5	5	3	14	5	-1	2	-5
X.8	+	35	35	-21	-21	3	11	11	-5	-5	3	14	5	-1	2	-5
X.9	+	50	50	20	20	18	10	10	4	4	2	5	5	-4	5	10
...																
X.100	+	4200	4200	0	0	104	40	40	0	0	8	-120	15	-12	6	-40
X.101	+	4200	4200	420	420	-24	40	40	4	4	8	-30	-30	15	-3	40
X.102	+	4480	4480	0	0	-128	0	0	0	0	0	-80	-44	-20	4	64
X.103	+	4536	-4536	-378	378	0	60	-60	30	-30	0	-81	0	0	0	0
X.104	+	4536	-4536	378	-378	0	60	-60	-30	30	0	-81	0	0	0	0
X.105	+	4536	4536	0	0	-72	-72	-72	0	0	24	0	81	0	0	-24
X.106	+	5600	-5600	0	0	0	-80	80	0	0	0	-10	-100	2	-4	0
X.107	+	5600	-5600	-280	280	0	-80	80	8	-8	0	20	20	11	2	0
X.108	+	5600	-5600	280	-280	0	-80	80	-8	8	0	20	20	11	2	0
X.109	+	5670	5670	0	0	-90	-90	-90	0	0	6	0	-81	0	0	6
X.110	+	6075	6075	405	405	27	-45	-45	-27	-27	-21	0	0	0	0	-45
X.111	+	6075	6075	-405	-405	27	-45	-45	27	27	-21	0	0	0	0	-45
X.112	+	7168	-7168	0	0	0	0	0	0	0	0	-128	16	-32	-8	0

Example: one matrix from a 30-dimensional representation of $W(E_6)$

```
1/2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1/4,0,0,0,0,0,0,0,0,0,
0,-1/8,0,0,0,-15/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3/8,0,0,0,0,0,0,0,0,
0,0,-1/8,0,0,0,-15/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3/8,0,0,0,0,0,0,0,
0,0,0,-1/8,0,0,0,-15/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3/8,0,0,0,0,0,0,
0,0,0,0,-1/8,0,0,0,-15/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3/8,0,0,0,0,0,
0,-3/8,0,0,0,3/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1/8,0,0,0,0,0,0,0,0,
0,0,-3/8,0,0,0,3/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1/8,0,0,0,0,0,0,0,
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3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1/2,0,0,0,0,0,0,0,0,0,
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Construct π by constructing its restriction to a subgroup, and building up.
John Stembridge: $\mathbb{Q}$-models including $W(E_8)$
(for $W(E_8)$, LCD(denominators)$\leq$594)

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Positive semidefinite:
1) $(v, v) = vMv^t \geq 0$ for all v
2) or all eigenvalues are ≥ 0
3) or det(all principal minors)≥ 0 (2^n of them)

What is wrong with computers

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix}$$

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$$\frac{11}{3} + \frac{23\,5^{\frac{2}{3}}}{3\left(241 + 9\,i\,\sqrt{34}\right)^{\frac{1}{3}}} + \frac{\left(5\left(241 + 9\,i\,\sqrt{34}\right)\right)^{\frac{1}{3}}}{3}$$

$$\frac{11}{3} - \frac{23\,5^{\frac{2}{3}}\left(1 + i\,\sqrt{3}\right)}{6\left(241 + 9\,i\,\sqrt{34}\right)^{\frac{1}{3}}} - \frac{\left(1 - i\,\sqrt{3}\right)\left(5\left(241 + 9\,i\,\sqrt{34}\right)\right)^{\frac{1}{3}}}{6}$$

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$$=\{10.79 + 0.i,\ -0.34 + 4.44 \times 10^{-16} i,\ 0.54 - 4.44 \times 10^{-16} i\}$$

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M $n \times n$ symmetric, rational

$\sigma(M) = (p, z, q)$ number of (positive, zero, negative) eigenvalues

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$\sigma(v) = (p, z, q)$:

p = number of sign changes: $(\dots a_i, 0, \dots, 0, a_j \dots)$ $\quad (a_i a_j < 0)$

z = number of zeroes at the beginning

q = number of sign changes using $f_M(-x)$

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Lemma (Descartes' rule of signs)

$$\sigma(M) = \sigma(f_M)$$

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Compute the characteristic polynomial mod p + Chinese Remainder Theorem $\rightarrow$ compute $\sigma(M)$

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7,168	1 cpu year (projected)

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Note: Embarassingly parallelizable

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What is wrong with computers II

$\int \sin^{10}(x)\cos(x)\,dx$ = [Mathematica]:

$$\frac{21}{512}\sin(x) - \frac{15}{512}\sin(3x) + \frac{15}{512}\sin(35x) - \frac{5}{1024}\sin(7x) + \frac{11}{11264}\sin(9x) + C$$

Spherical Unitary dual

G=classical real or split p-adic group
$\widehat{G}_{sph}$ = spherical unitary dual: irreducible unitary representations containing a K-fixed vector.

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Subset of $\mathfrak{A}(\mathbb{C})^*$ (reduces to $\mathfrak{A}(\mathbb{R})^* \simeq \mathbb{R}^n$)
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Barbasch/Ciubotaru: Also results for exceptional groups; confirmed by atlas computations

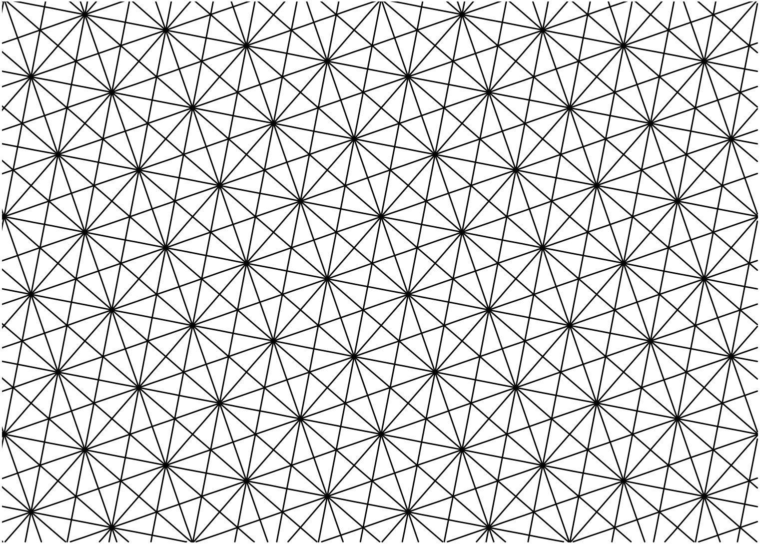
Spherical Unitary dual via atlas

G: split, p-adic

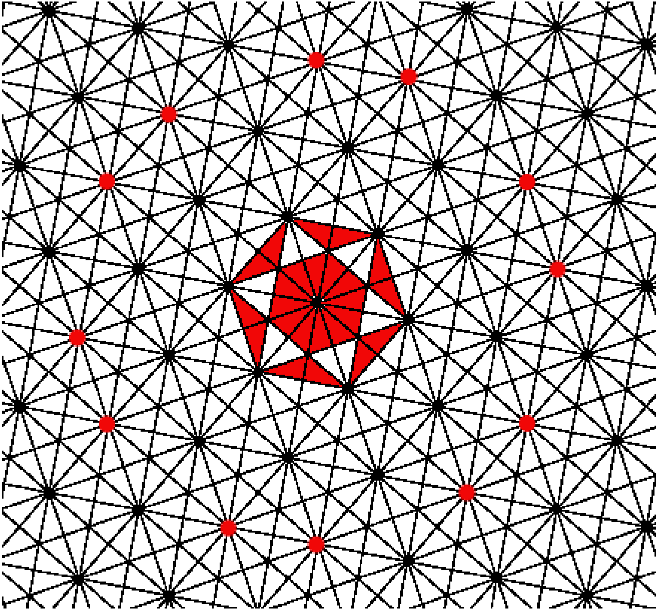
Atlas: computes the spherical unitary dual $\widehat{G}_{sph}$

Example G=G_2

```
(0,0,0)
(-3/8,-3/8,3/4)
(-1/4,-1/2,3/4)
(-1/6,-5/12,7/12)
(-1/2,-1/2,1)
(-1,-2,3)
(0,-1,1)
(-1/3,-1/3,2/3)
```



Example: Hyperplanes in $\mathfrak{a}(\mathbb{R})^*$ for G_2



Example: Spherical unitary dual of G_2 (Vogan, Barbasch, Atlas)

Unitary Dual

G = real reductive group

for example $GL(n,\mathbb{R})$, $Sp(2n,\mathbb{R})$, $Spin(p,q)$, $E_8(split)$,...)

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$$\widehat{G}_u=\{\text{irreducible unitary representations of } G\}/\sim$$

(unitary equivalence)

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Equivalently:

Definition: A $(\mathfrak{g}, K)$-module is a vector space V, with compatible representations of $\mathfrak{g}$ and K.

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Other Duals

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Hermitian Dual $\widehat{G}_h$: $(\mathfrak{g}, K)$-modules preserving a Hermitian form (not necessarily positive definite)

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Uncountably many π to test

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Note: $\langle\, , \,\rangle$ is not the usual one for $-1 \leq \nu \leq 1, \nu \neq 0$

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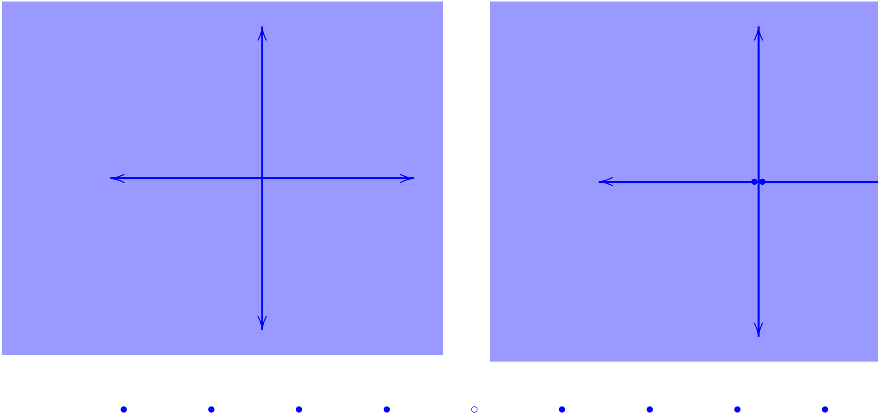

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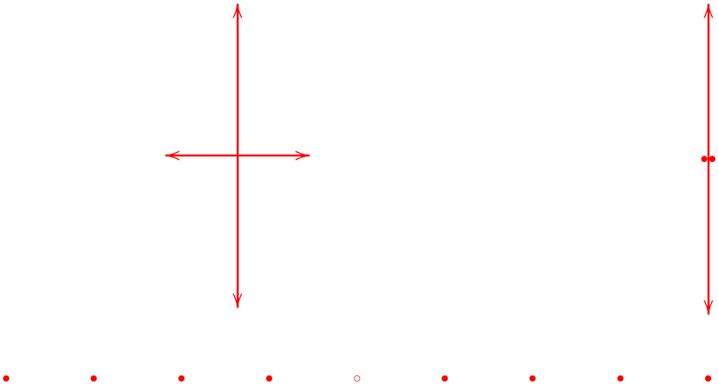

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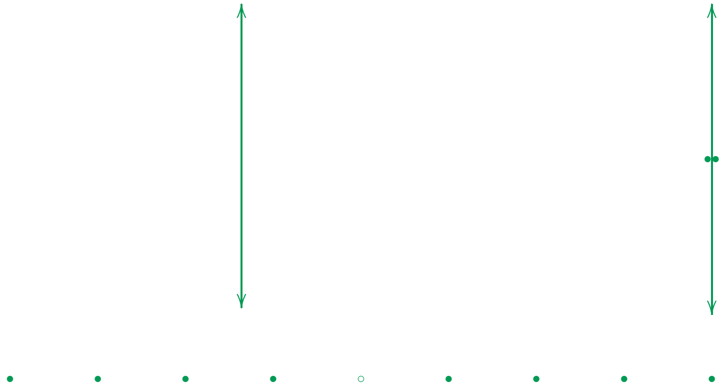

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Answer: 526,471

(2,157 of them = .41% are unitary)

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Fix an infinitesimal character λ.

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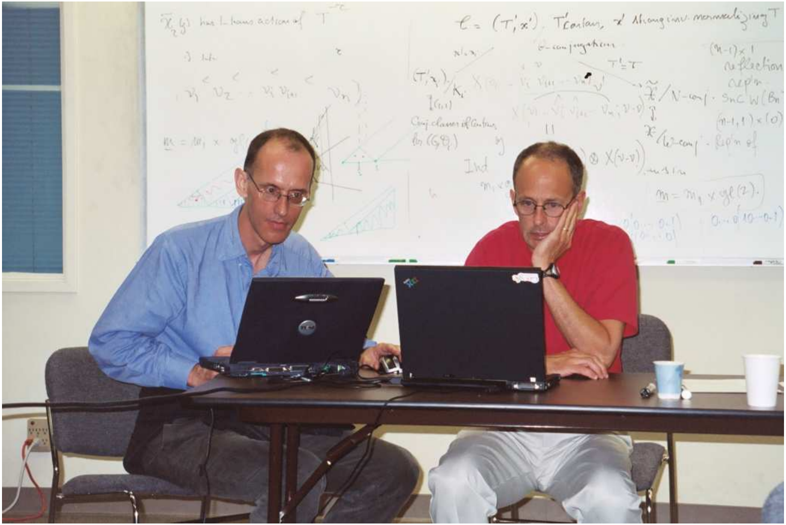
Requirement 2) comes down to:
3) make Cayley transforms and the cross action evident



Fokko du Cloux

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Abstract Mathematics	→	Algorithm	→	Software
Lie Groups		Combinatorial Set		C++ code
Representation Theory				

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Mathematics ⟵ Computer output

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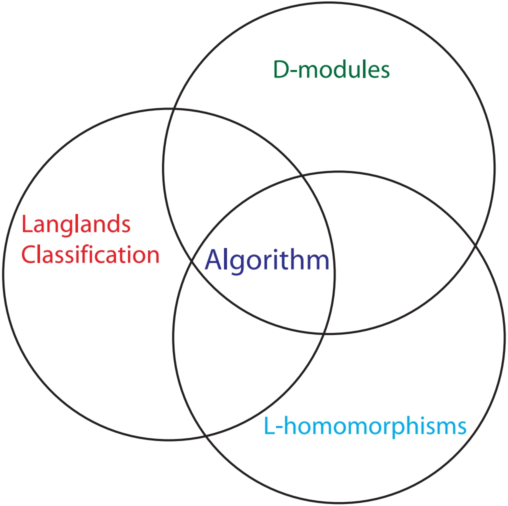
For now assume G is simply connected, adjoint and $\mathrm{Out}(G)=1$
(Examples: $G=G_2, F_4$ or E_8)


D-modules
Langlands
Classification
Algorithm
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Many contributors, including Beilinson, Bernstein, Zuckerman, Knapp, Vogan, Hecht/Miličić/Schmid/Wolf…(in particular relating these pictures)

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Definition:

$$\mathcal{C}(G(\mathbb{R}),\rho)=\{(H(\mathbb{R}),\chi)\}/G(\mathbb{R})$$

$H(\mathbb{R})$=Cartan subgroup
χ = character of $H(\mathbb{R})$ with $d\chi=\rho$

The Langlands Classification

$(H(\mathbb{R}),\chi) \rightarrow I(H(\mathbb{R}),\chi)$=standard module (induced from discrete series of $M(\mathbb{R})$)
$\rightarrow \pi(H(\mathbb{R}),\chi)$ (unique irreducible quotient)

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Theorem: The map $(H(\mathbb{R}),\chi) \rightarrow \pi(H(\mathbb{R}),\chi)$ induces a canonical bijection:

$$\boxed{\Pi(G(\mathbb{R}),\rho) \overset{1-1}{\longleftrightarrow} \mathcal{C}(G,\rho)}$$

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In particular:

$$|\Pi(G(\mathbb{R}), \rho)| = \sum_i |W/W(G(\mathbb{R}), H(\mathbb{R})_i)||H(\mathbb{R})/H(\mathbb{R})_i|$$

$H(\mathbb{R})_1, \dots, H(\mathbb{R})_n$ are representatives of the conjugacy classes of Cartan subgroups.

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$T = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos\theta \end{pmatrix} \simeq S^1$, $|H(\mathbb{R})/H(\mathbb{R})^0| = 1$, $W = 1$

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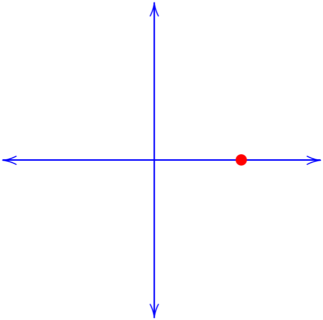
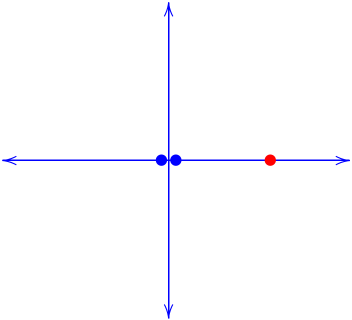
Example: $G = SL(2,\mathbb{R})$, infinitesimal character $=\rho$





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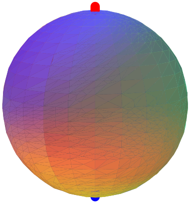
Example: $G = SL(2,\mathbb{C})$, $G(\mathbb{R}) = SL(2,\mathbb{R})$

$\mathcal{B}$ is the sphere = $\mathbb{C} \cup \infty$



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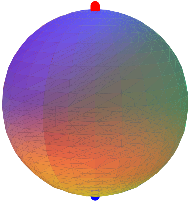

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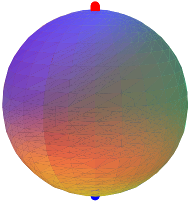

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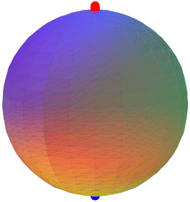

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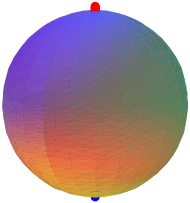

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Isotropy group: $1, 1, \mathbb{Z}/2\mathbb{Z} \to 4$ representations

L-homomorphisms

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Definition:

$$\mathcal{L}(G, \rho) = \{(\phi, \chi)\}/G^\vee$$

$\phi : W_{\mathbb{R}} \to G^\vee$, ($\phi(\mathbb{C}^\times)$ is semisimple, “infinitesimal character ρ”)
χ = local system on $\Omega^\vee = G^\vee \cdot \phi$
= character of $\mathrm{Stab}(\phi)/\mathrm{Stab}(\phi)^0$

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Theorem: There is a natural bijection

$$\boxed{\coprod_i \Pi(G(\mathbb{R})_i, \rho) \overset{1-1}{\longleftrightarrow} \mathcal{L}(G, \rho)}$$

where $G_1(\mathbb{R}), \dots, G_n(\mathbb{R})$ are the real forms of G.
(this version: book by A/Barbasch/Vogan)

Recapitulation

Recapitulation

(1) Character Data (orbits of $G(\mathbb{R})$ on Cartans):

$$\Pi(G(\mathbb{R}),\rho) \stackrel{1-1}{\longleftrightarrow} \mathcal{C}(G(\mathbb{R})) = \{(H(\mathbb{R}),\chi)\}/G(\mathbb{R})$$

(2) $\mathcal{D}$-modules (orbits $\mathcal{O}$ of K on G/B):

$$\Pi(G(\mathbb{R}),\rho) \stackrel{1-1}{\longleftrightarrow} \mathcal{D}(G,K,\rho) = \{(x,\chi)\}/K$$

(3) L-homomorphisms (orbits $\Omega^\vee$ of $G^\vee$ on L-homomorphisms):

$$\coprod_{i=1}^{n} \Pi(G(\mathbb{R})_i,\rho) \stackrel{1-1}{\longleftrightarrow} \mathcal{L}(G,\rho) = \{(\phi,\chi)\}/G^\vee$$

In each case there is some geometric data, and a character of a finite abelian group (two-group)

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We'd rather talk about orbits than characters of $(\mathbb{Z}/2\mathbb{Z})^n$
(Matching the three pictures: easy up to χ)

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Theorem (Vogan): The intersection of an L-packet and an R-packet is at most one element.

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Corollary: $\Pi(G(\mathbb{R}),\rho)$ is parametrized by a subset of pairs

$$(K \text{ orbit on } \mathcal{B}, G^\vee \text{ orbit on L-homomorphisms})$$

via

$$(\mathcal{O},\Omega^\vee) \rightarrow \Pi_R(G(\mathbb{R}),\mathcal{O}) \cap \Pi_L(G(\mathbb{R}),\Omega^\vee)$$

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Which pairs?. . .

K-orbits on the dual side

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Proposition: There is a natural bijection:

$$\mathcal{L} \stackrel{1-1}{\longleftrightarrow} \coprod_{i=1}^{n} K_i^\vee \backslash \mathcal{B}^\vee$$

Symmetric Picture

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Note: This symmetry is Vogan Duality.

This reduces the problem to:

$$\boxed{\text{Parametrize } K \text{ orbits on } \mathcal{B} = G/B}$$

(applied to G and $G^\vee$)

K orbits on G/B

Definition:

$$\mathcal{X} = \{x \in \mathrm{Norm}_G(H) \mid x^2 = 1\}/H$$

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(union over real forms, corresponding $K_1, \ldots, K_n$)

Sketch of Proof

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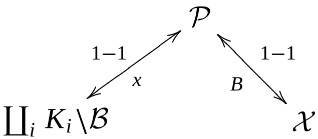
$$\mathcal{P} = \{(x, B)\}/G \ (x^2 = 1, \ B = \text{Borel})$$



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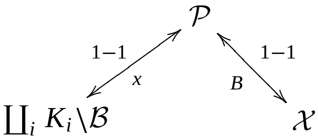

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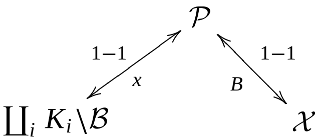

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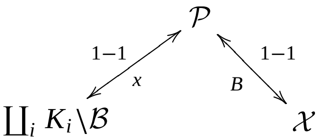

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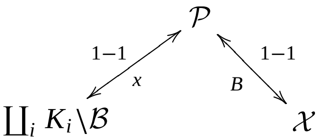

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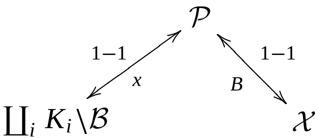

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$$(x, B) \sim_G (x', B_0) \to x' \in \mathcal{X} \quad (\text{wlog } x' \in \text{Norm}(H))$$

$K\backslash G/B$ for $Sp(4,\mathbb{R})$ and $SO(3,2)$:

$Sp(4,\mathbb{R})$:

```
 0:     1   2     6   4  [nn]  0  0
 1:     0   3     6   5  [nn]  0  0
 2:     2   0     *   4  [cn]  0  0
 3:     3   1     *   5  [cn]  0  0
 4:     8   4     *   *  [Cr]  2  1  2
 5:     9   5     *   *  [Cr]  2  1  2
 6:     6   7     *   *  [rC]  1  1  1
 7:     7   6    10   *  [nC]  1  2  2,1,2
 8:     4   9     *  10  [Cn]  2  2  1,2,1
 9:     5   8     *  10  [Cn]  2  2  1,2,1
10:    10  10     *   *  [rr]  3  3  1,2,1,2
```

$SO(3,2)$:

```
0:    0  1    3  2  [nn]  0  0
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3:    3  4    *  *  [rC]  1  1  1
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$$\mathcal{Z} \subset \coprod_i K_i \backslash \mathcal{B} \times \coprod_j K_j^\vee \backslash \mathcal{B}^\vee$$

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Theorem: There is a natural bijection:

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(Canonical up to characters of $G_{qs}(\mathbb{R})/G_{qs}(\mathbb{R})^0$, $G^\vee_{qs}(\mathbb{R})/G^\vee_{qs}(\mathbb{R})^0$)

General Groups

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For simplicity we assumed (recall $G = G(\mathbb{C})$):

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In general:

1. Fix an inner class of real forms
2. Need twists $G^\Gamma = G \rtimes \Gamma$, $G^\vee \rtimes \Gamma$ ($\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$)
3. Require $x^2 \in Z(G)$ (not $x^2 = 1$)
4. Need several infinitesimal characters
5. Need strong real forms

The General Algorithm

$$\mathcal{X} = \{x \in \mathrm{Norm}_{G^\Gamma \backslash G}(H) \mid x^2 \in Z(G)\}/H$$

$\mathcal{X}^\vee$ similarly, $\mathcal{Z} = \{(x, y) \mid \dots\} \subset \mathcal{X} \times \mathcal{X}^\vee$ as before.

Theorem: There is a natural bijection

$$\mathcal{Z} \overset{1-1}{\longleftrightarrow} \coprod_{i \in S} \Pi(G(\mathbb{R})_i, \Lambda)$$

Λ = certain set of infinitesimal characters
S is the set of "strong real forms"

Reference: Algorithms for Representation Theory of Real Reductive Groups, preprint (`www.liegroups.org`), Fokko du Cloux, A

Block of the trivial representation of $Sp(4,\mathbb{R})$

```
 0( 0,6):  0  0  [i1,i1]   1   2   ( 6, *)  ( 4, *)
 1( 1,6):  0  0  [i1,i1]   0   3   ( 6, *)  ( 5, *)
 2( 2,6):  0  0  [ic,i1]   2   0   ( *, *)  ( 4, *)
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 4( 4,4):  1  2  [C+,r1]   8   4   ( *, *)  ( 0, 2)   2
 5( 5,4):  1  2  [C+,r1]   9   5   ( *, *)  ( 1, 3)   2
 6( 6,5):  1  1  [r1,C+]   6   7   ( 0, 1)  ( *, *)   1
 7( 7,2):  2  1  [i2,C-]   7   6   (10,11)  ( *, *)   2,1,2
 8( 8,3):  2  2  [C-,i1]   4   9   ( *, *)  (10, *)   1,2,1
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10(10,0):  3  3  [r2,r1]  11  10   ( 7, *)  ( 8, 9)   1,2,1,2
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```

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Two natural ways of constructing new representations from old (Vogan): Cayley transforms and cross action

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Two natural ways of constructing new representations from old (Vogan): Cayley transforms and cross action

In our picture:

1) W acts by conjugation on $\mathcal{X}$ and $\mathcal{Z}$: cross action

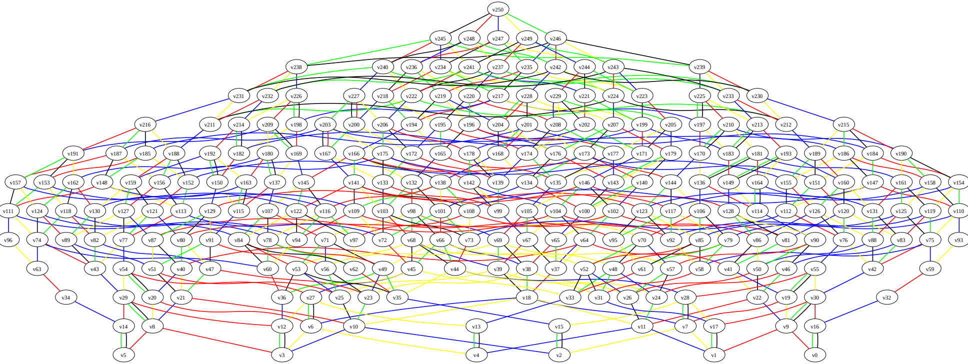
2) $w \in W_2, s_\alpha w = w s_\alpha$,

$$w \rightarrow w' = s_\alpha w \in W_2$$

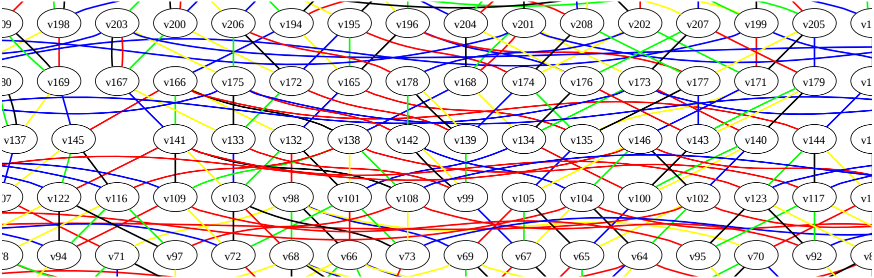
lifts to

$$x \rightarrow x' = \sigma_\alpha x$$

(Multivalued due to choice of σ_α: x' or $\{x_1', x_2'\}$)
This is the Cayley transform



$K\backslash G/B$ for $SO(5,5)$



Closeup of $SO(5,5)$ graph

Cayley Transforms and Cross Actions

Do this on $\mathcal{X}$ and $\mathcal{X}^\vee$, and $\mathcal{Z}\dots$

Cayley Transforms and Cross Actions

Do this on $\mathcal{X}$ and $\mathcal{X}^\vee$, and $\mathcal{Z}$…

Proposition Cayley transforms and cross actions are naturally computable in $\mathcal{X}$ and $\mathcal{Z}$

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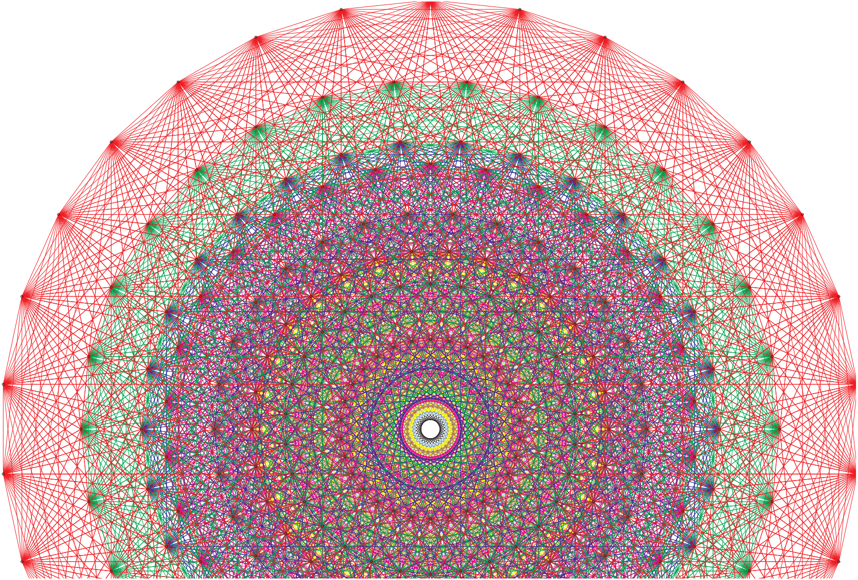
```

Kazhdan-Lusztig-Vogan Polynomials



Fokko du Cloux
December 20, 1954 - November 10, 2006

Marc van Leeuwen
Poitiers
LiE software

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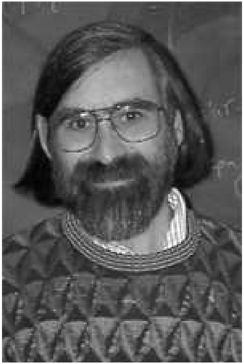
Poitiers
LiE software



David Vogan
MIT

Kazhdan-Lusztig-Vogan Polynomials

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(also: block $\mathcal{B}$ of representations at λ)

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Proposition (Langlands, Zuckerman): $\mathcal{M} = \mathbb{Z}\langle I(\gamma)\rangle \quad (\gamma \in \mathcal{P})$

Kazhdan-Lusztig-Vogan Polynomials

Change of Basis Matrices:

$I(\delta) = \sum_{\delta \in \mathcal{P}} m(\gamma, \delta)\pi(\gamma)$

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Character Table for $\mathcal{B}$

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Proposition: KLV polynomials + coherent continuation $\rightarrow$ compute character of any admissible representation in $\mathcal{B}$ as a function on the regular semisimple set.

Computable solely from output of `atlas` software

KL and KLV polynomials

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	original KL polynomials	KLV polynomials
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Note: David Vogan calls the polynomials for $G(\mathbb{R})$ Kazhdan-Lusztig (not Kazhdan-Lusztig-Vogan) polynomials

Definition of KLV polynomials

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$\mathcal{M} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\langle T_\gamma \rangle \quad (\gamma \in \mathcal{P})$

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(W-graph, Duality operation, self-dual elements, C_γ, $R_{\gamma,\delta}$, $P_{\gamma,\delta}$ as in the original Kazhdan-Lusztig paper)

Recursive Definition of KLV polynomials

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Roots are labelled C+,C-,rn,r1,r2,ic,i1,i2 (`atlas` output):
1303(952, 31): 13 7 [i2,C-,r2,C-,i1] 1303 1250 1304...

Recursive Definition of KLV polynomials

Length order: $\gamma \leq \delta$ if $\gamma = \delta$ or $\ell(\gamma) < \ell(\delta)$

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$$\mu(\gamma, \delta) = \text{ coefficient of } q^{\frac{1}{2}(\ell(\delta)-\ell(\gamma)-1)} \text{ in } P_{\gamma,\delta}$$

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$$U^{\alpha}_{\gamma,\delta} = \sum_{\gamma \le \zeta < \delta} \mu(\zeta,\delta) P_{\gamma,\zeta}$$

Recursive Definition of KLV polynomials

α w.r.t. δ	α w.r.t. γ	$P_{\gamma,\delta} =$
ic/C-/r1 *or* r2	i1 or i2	$\upsilon^{-1} P_{\gamma_\alpha,\delta}$ *or* $\upsilon^{-1}(P_{\gamma_\alpha^+,\delta} + P_{\gamma_\alpha^-,\delta})$
ic/C-/r1 or r2	C+	$\upsilon^{-1} P_{s_\alpha \times \gamma,\delta}$
C-	C-	$\upsilon P_{\gamma,s_\alpha\times\delta} + P_{s_\alpha\times\gamma,s_\alpha\times\delta} - U^\alpha_{\gamma,\delta}$
r1 or r2*	r1	$(\upsilon - \upsilon^{-1}) P_{\gamma,\delta_\alpha^+} + P_{\gamma_\alpha^+,\delta_\alpha^+} + P_{\gamma_\alpha^-,\delta_\alpha^+} - U^\alpha_{\gamma,\delta_\alpha^+}$
r1 or r2*	r2	$\upsilon P_{\gamma,\delta_\alpha} - \upsilon^{-1} P_{s_\alpha\times\gamma,\delta_\alpha} + P_{\gamma_\alpha,\delta_\alpha} - U^\alpha_{\gamma,\delta_\alpha}$

(*): formula is for $P_{\gamma,\delta} + P_{\gamma,s_\alpha\delta}$

Recursive Definition of KLV polynomials

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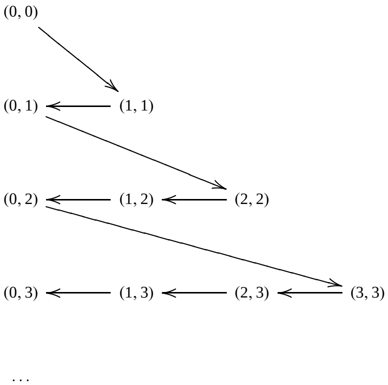
In each case the right formula in boxes involves
$P_{\gamma',\delta'}$ with
1) $\ell(\delta') < \ell(\delta)$ *or*
2) $\ell(\delta') = \ell(\delta), \ell(\gamma') > \ell(\gamma)$

Recursion Relations

$P_{\gamma,\gamma} = 1$

Compute $P_{\gamma,\delta}$ like this:



((i, j) is the $P_{\gamma,\delta}$ with $\ell(\gamma) = i, \ell(\delta) = j$)

Recursion Relations

(0, 0)					
(0, 1)	(1, 1)				
(0, 2)	(1, 2)	(2, 2)			
(0, 3)	(1, 3)	(2, 3)	(3, 3)		
(0, 4)	(1, 4)	(2, 4)	(3, 4)	(4, 4)	
(0, 5)	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)

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To compute $P_{\gamma,\delta}$ with $\ell(\gamma) = 3, \ell(\delta) = 5$, need potentially all of the $P_{\gamma,\delta}$ from the blue region.

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(E_8: $U^{\alpha}_{\gamma,\delta}$ has 150 terms on average)

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See:

David Vogan's narrative, October Notices

Marc van Leeuwen's technical discussion

`www.liegroups.org/talks`

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Challenge: Compute KLV for (the large block) of E_8

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$|\mathcal{P}| = 453,060$
$\deg(P_{\gamma,\delta}) \leq 31$
Big Problem: we did not have a good idea of the size of the answer beforehand.
$a_i \geq 2^{16} = 65{,}535$ (almost certainly)
$a_i \leq 2^{32}$=4.3 billion (we hope?)

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Crude estimates: need about 1 terabyte of RAM (=1,000 gigabytes)
(1 gigabyte = 1 billion bytes = RAM in typical home computer)
Typical computational machine (not a cluster): 4-8 gigabytes of RAM

Many of the polynomials are equal for obvious reasons.
Hope: number of distinct polynomials $\leq$ 200 million.
Store only the distinct polynomials (cost of pointers)
Hope: average degree = 20
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Experiments (Birne Binegar and Dan Barbasch):
About 800 billion distinct polynomials $\rightarrow$ 65 billion bytes

William Stein at Washington lent us SAGE, with 64 gigabytes of RAM (all accessible from one processor)

Noam Elkies: have to think harder

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$2^{32} < 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 = 100$ billion

You then get the answer mod 100,280,245,065 using the Chinese Remainder theorem (cost: running the calculation 9 times)

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This gets us down to about $15 + 4 = 19$ billion bytes

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Dec. 26	255	complete	12 hours

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Dec. 27	253	crash	
Jan. 3	253	complete	12 hours

The final result

Combine the answers using the Chinese Remainder Theorem.
Answer is correct if the biggest coefficient is less than 4,145,475,840
Total time (on SAGE): 77 hours

Some Statistics

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$$152q^{22} + 3,472q^{21} + 38,791q^{20} + 293,021q^{19} + 1,370,892q^{18} + 4,067,059q^{17} + 7,964,012q^{16} + 11,159,003q^{15} + 11,808,808q^{14} + 9,859,915q^{13} + 6,778,956q^{12} + 3,964,369q^{11} + 2,015,441q^{10} + 906,567q^{9} + 363,611q^{8} + 129,820q^{7} + 41,239q^{6} + 11,426q^{5} + 2,677q^{4} + 492q^{3} + 61q^{2} + 3q$$

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Value of this polynomial at q=1: 60,779,787

Number of coefficients in distinct polynomials: 13,721,641,221 (13.9 billion)

Unipotent Representations

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Proposition: From the output of `atlas` one can list the special unipotent representations associated to a given nilpotent orbit.

Sketch

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*) Fix nilpotent orbit $\mathcal{O}$ for $\mathfrak{g}^\vee$. Let $S = \{i_1, \ldots, i_r\}$ be the nodes of Dynkin diagram labelled 2. Let λ = corresponding infinitesimal character.

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5) Push these to λ

David Vogan has carried this out for E_8
(70 nilpotent orbits; 20 even ones; 143 unipotent representations with integral infinitesimal character for $E_8(split)$)

Conjecture (Arthur): These representations are unitary.

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Stay tuned. . .