Atlas of Lie Groups
and Representations

www.liegroups.org

The Unitary Dual

Conference in Honor of Jim Arthur
Fields Institute
August 11, 2025

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University of Maryland
Institute for Defense Analysis

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Alfred Noel	Gregg Zuckerman
Tatiana Howard	Alessandra Pantano

Big Picture

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Atlas of Lie Groups and Representations (2002): study this with the aid of a computer

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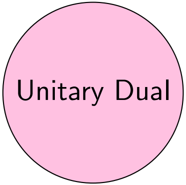
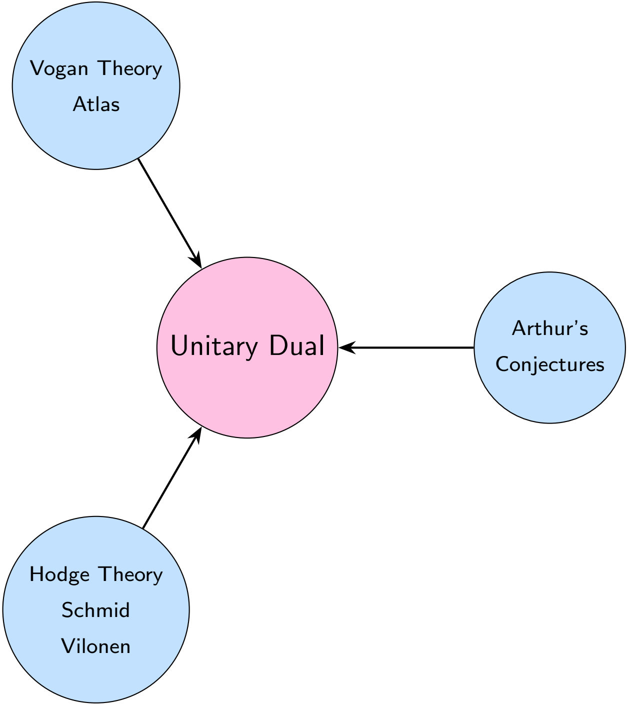

Unitary Dual


Vogan Theory
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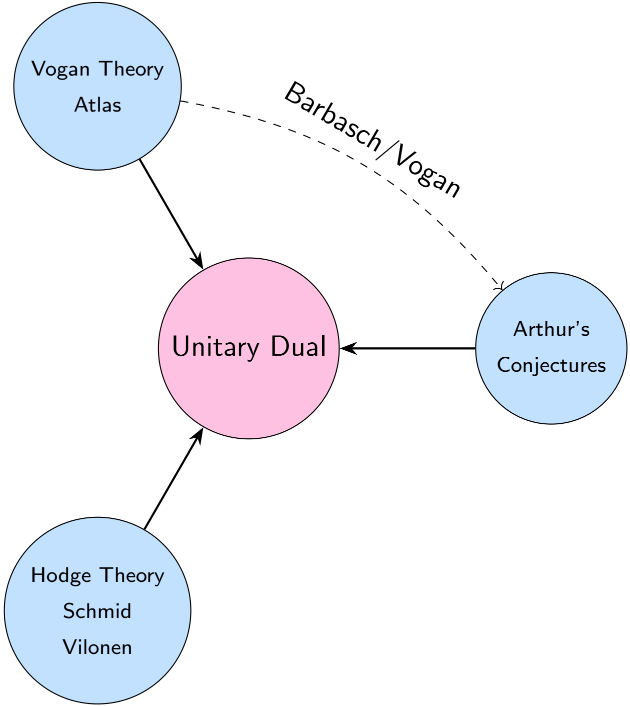
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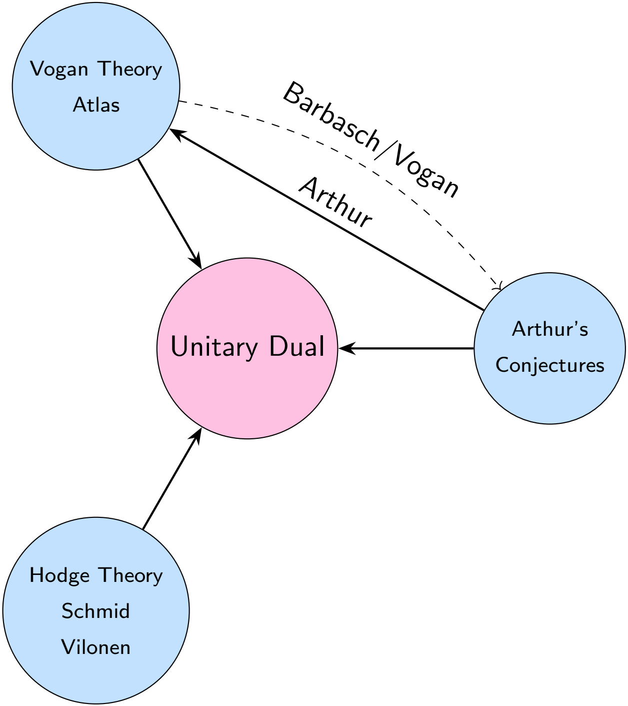
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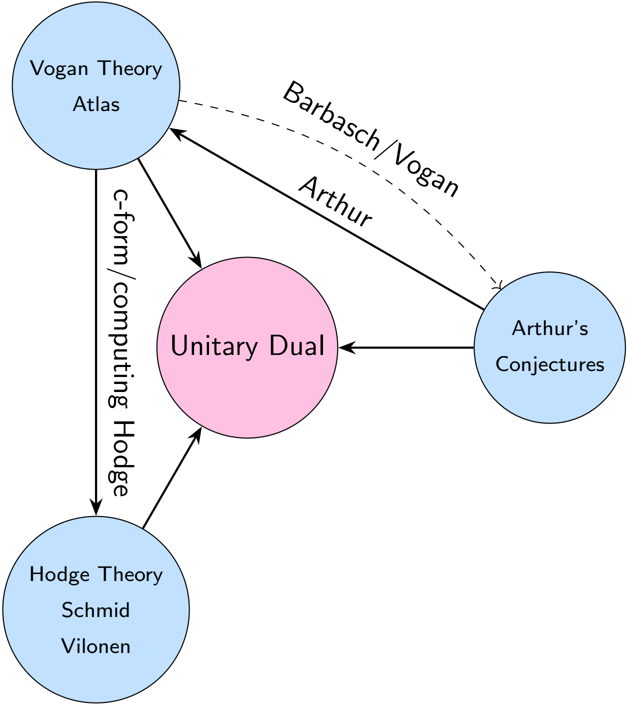
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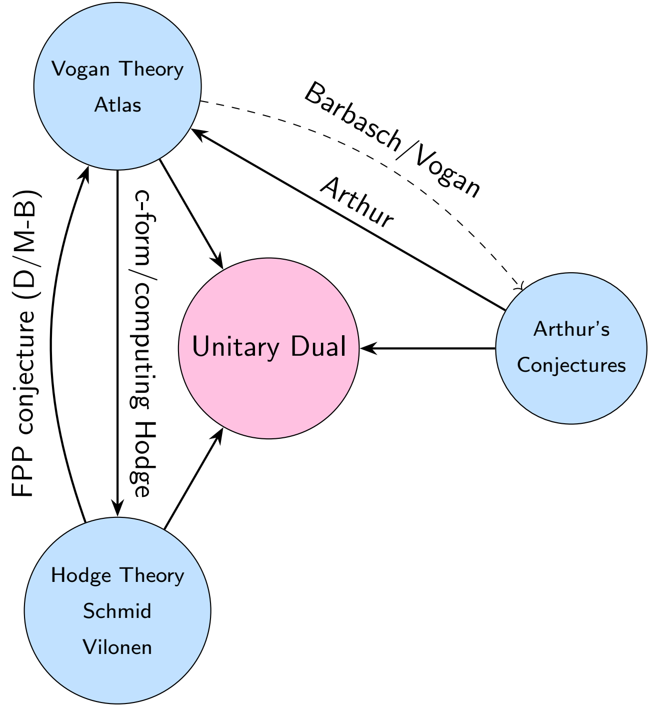
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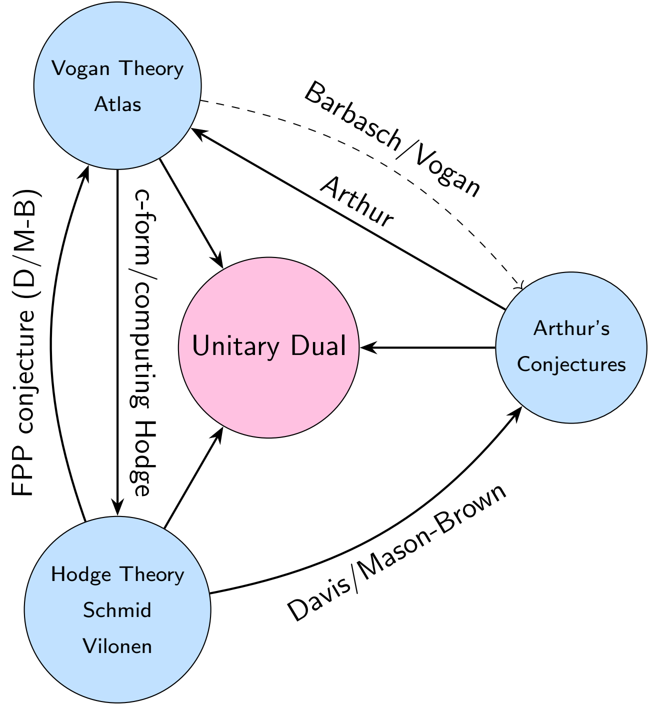

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(π, V) vector space; compatible actions of $\mathfrak{g}, K$

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Note: Only *real* infinitesimal character ($\gamma \in X^*(H) \otimes_{\mathbb{R}}$).

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Summary: $\boxed{\Gamma = (\text{finite set, vector, rational vector}) \mapsto \widehat{H(\mathbb{R})_\rho}}$

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How do you compute the signature of a Hermitian form on an infinite dimensional vector space?

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2 lowest K-types ± 1, any invariant form has opposite signs on them.

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So: Write $I_c(\Gamma), J_c(\Gamma)$ for these representations, equipped with their canonical c-Hermitian forms

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Idea: the Hodge filtration (parametrized by $\mathbb{Z}$) reduced mod 2 gives the c-Hermitian form (a $\mathbb{Z}/2\mathbb{Z}$ object)

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$$I_c((1+\epsilon)t = I_c((1-\epsilon)t) - \sum_{\tau<\gamma} s^{\ell_0(\gamma)-\ell_0(\tau)}$$

$$* \left[\sum_{\tau<\delta<\gamma} (-1)^{\ell(\delta)-\ell(\tau)} s^{\ell(\gamma)-\ell(\delta)} P_{\tau,\delta}(s) Q_{\delta,\gamma}(s) \right] I_c(\delta)$$

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$$J(\gamma) \text{ is unitary} \Leftrightarrow \text{ all } z_i' \in \mathbb{Z} \quad (z_i' = a_i + 0 * s)$$

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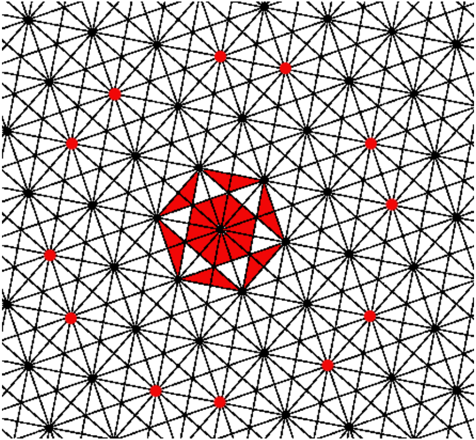

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("weakly fair range")

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Together with the reduction provided by the FPP Theorem this gives a description of the unitary dual.

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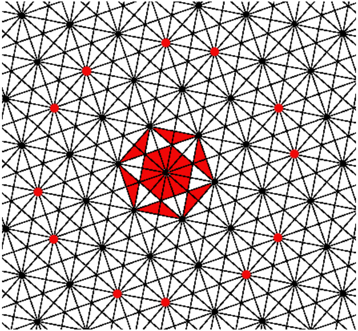

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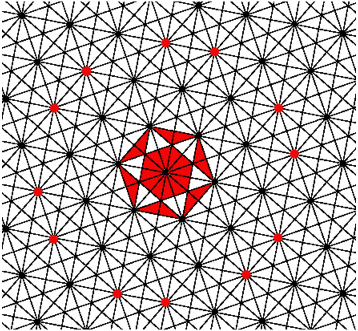
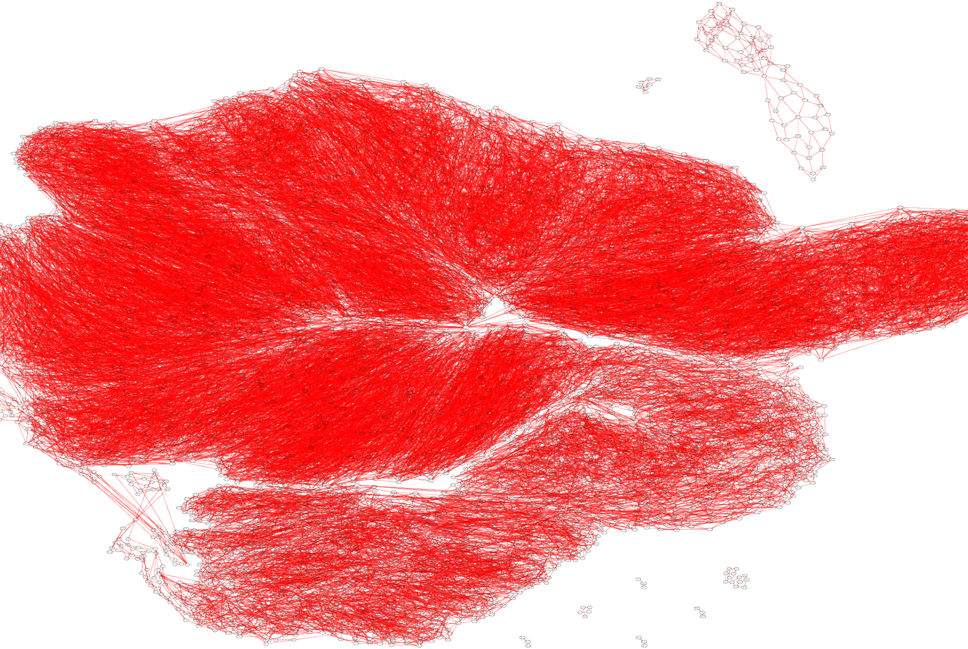
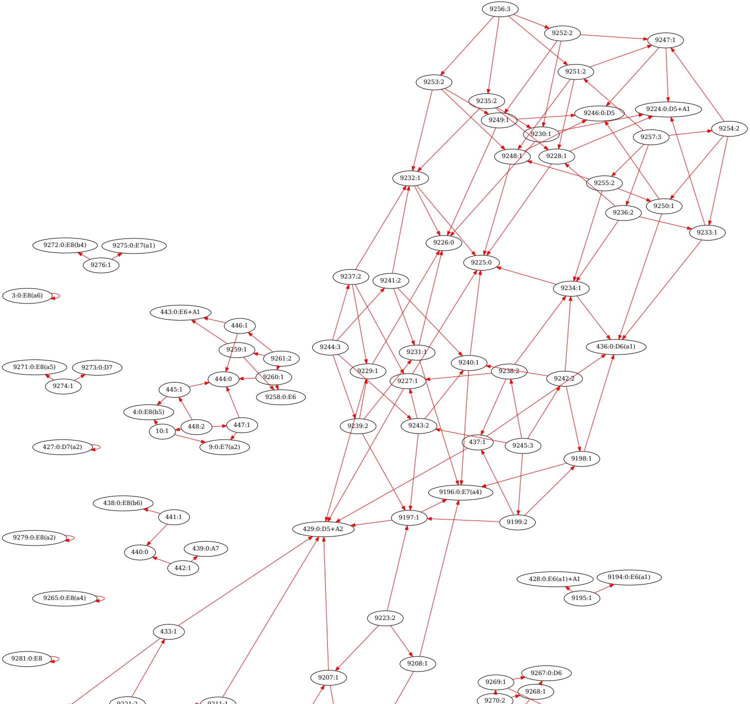


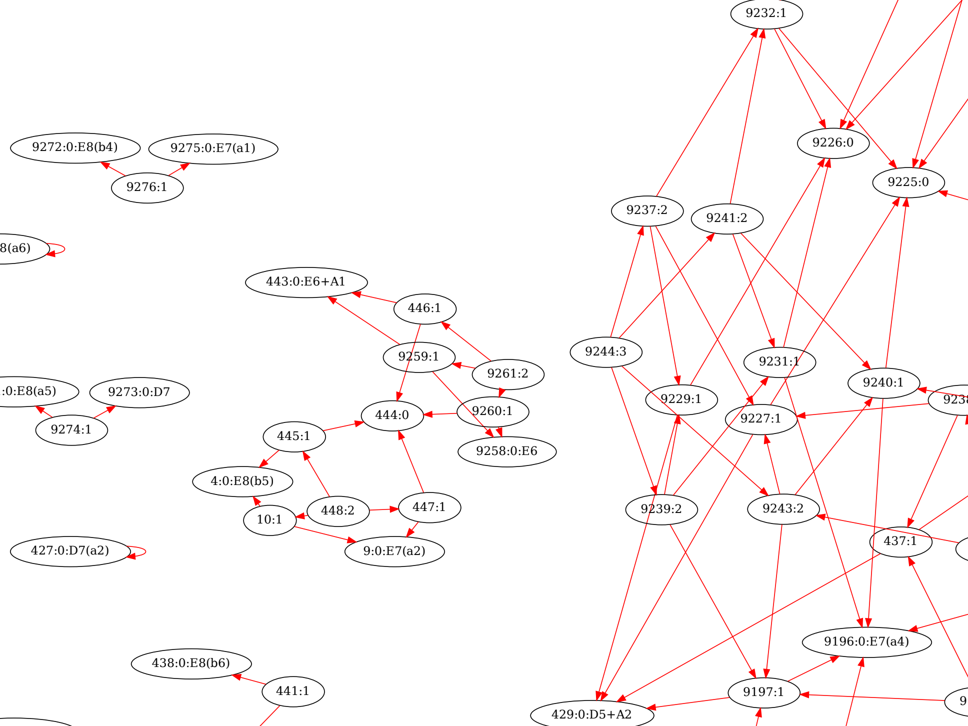
Note: The pictures for different (x, λ) interact in a complicated way.

Spherical representations of E_8

Here is a graph of the closure relations among the 9,282 spherical unitary representations of E_8 (split)




9256:3
9252:2
9247:1
9251:2
9253:2
9235:2
9246:0:D5
9224:0:D5+A1
9249:1
9254:2
9230:1
9257:3
9248:1
9228:1
9232:1
9255:2
9236:2
9250:1
9233:1
9226:0
9225:0
9272:0:E8(b4)
9275:0:E7(a1)
9276:1
9237:2
9241:2
9234:1
3:0:E8(a6)
443:0:E6+A1
446:1
9259:1
9261:2
9244:3
9231:1
436:0:D6(a1)
9271:0:E8(a5)
9273:0:D7
9240:1
9229:1
9238:2
9242:2
9260:1
444:0
9227:1
9274:1
445:1
9258:0:E6
4:0:E8(b5)
448:2
447:1
9239:2
9243:2
10:1
437:1
9245:3
427:0:D7(a2)
9:0:E7(a2)
9198:1
9196:0:E7(a4)
438:0:E8(b6)
441:1
9197:1
9199:2
429:0:D5+A2
9279:0:E8(a2)
440:0
439:0:A7
442:1
428:0:E6(a1)+A1
9194:0:E6(a1)
9195:1
9265:0:E8(a4)
9223:2
433:1
9281:0:E8
9208:1
9207:1
9267:0:D6
9269:1
9268:1
9221:2
9211:1
9270:2



9232:1
9272:0:E8(b4)
9275:0:E7(a1)
9276:1
9226:0
9225:0
8(a6)
9237:2
9241:2
443:0:E6+A1
446:1
9244:3
9259:1
9261:2
9231:1
9240:1
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9229:1
9260:1
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9274:1
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9239:2
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G complex, connected, reductive, $\theta \in \mathrm{Aut}(G)$, $\theta^2 = 1, K, G(\mathbb{R})$

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The FPP has a finite facet decomposition; unitarity is constant on facets; there is a finite calculation to compute $\widehat{L(\mathbb{R})_{\mathsf{FPP}}}$ for each of the (finitely many) Q.

Some computer results

Groups up to rank 6 are quite fast on a laptop. . .

group	#	time	group	#	time	group	#	time
su(2)	1	0.000	sl(2,R)	7	0.007	su(3)	1	0.000
su(2,1)	20	0.015	sl(3,R)	9	0.014	so(5)	1	0.000
so(4,1)	12	0.024	so(3,2)	46	0.044	g2	1	0.000
g2(R)	60	0.039	su(4)	1	0.000	su(3,1)	40	0.036
su(2,2)	126	0.072	sl(2,H)	8	0.020	sl(4,R)	47	0.067
so(7)	1	0.000	so(6,1)	17	0.069	so(5,2)	129	0.349
so(4,3)	207	1.029	sp(3)	1	0.001	sp(2,1)	33	0.202
sp(6,R)	319	1.330	su(5)	1	0.000	su(4,1)	67	0.338
su(3,2)	458	1.985	sl(5,R)	66	0.513	so(9)	1	0.001
so(8,1)	22	0.316	so(7,2)	231	1.682	so(6,3)	668	5.029
so(5,4)	1244	13.061	sp(4)	1	0.000	sp(3,1)	66	0.665
sp(2,2)	252	1.542	sp(8,R)	2043	17.548	so(8)	1	0.001
so(6,2)	225	1.286	so*(8)[0,1]	225	1.216	so*(8)[1,0]	224	1.300
so(4,4)	1062	5.259	so(7,1)	11	0.166	so(5,3)	215	1.993
f4	1	0.000	f4(so(9))	51	0.746	f4(R)	1864	39.995
su(6)	1	0.000	su(5,1)	101	0.760	su(4,2)	1243	7.609
su(3,3)	2786	11.500	sl(3,H)	37	0.409	sl(6,R)	286	3.569
so(11)	1	0.001	so(10,1)	27	0.897	so(9,2)	352	4.871
so(8,3)	1376	19.230	so(7,4)	5094	108.205	so(6,5)	6485	172.78
sp(5)	1	0.007	sp(4,1)	111	2.167	sp(3,2)	907	14.038
sp(10,R)	13768	295.383	so(10)	1	0.008	so(8,2)	343	3.149
so*(12)[1,0]	6305	142.027	so*(12)[0,1]	6413	114.670	so(8,4)	10365	334.10
so(6,6)	30309	912.176	so(11,1)	17	1.394	so(9,3)	1124	35.933
so(7,5)	8427	544.170	e6	1	0.052	e6(so(10).u(1))	3413	98.846
e6(q)	19831	648.611	e6(f4)	58	1.918	e6(R)	2217	98.264
su(8)	1	0.001	su(7,1)	190	4.614	su(6,2)	5242	109.93
su(5,3)	37314	836.892	su(4,4)	70237	1137.030	sl(4,H)	221	5.268
sl(8,R)	1775	121.184	so(15)	1	0.013	so(14,1)	37	9.604
so(13,2)	651	40.589	so(12,3)	3700	216.952	so(11,4)	24725	3584.46
so(10,5)	74867	12576.352	so(9,6)	194538	90513.295	sp(7)	1	0.002
sp(6,1)	237	19.414	sp(5,2)	5389	495.628	sp(4,3)	24722	3007.44

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$E_6/E_7/E_8$

group	$\#(x, lambda)$	#unitary	time (secs)
$E_6(split)$	26,325	2,217	98
$E_6(quat)$	74,459	19,831	662.316
$E_7(split)$	2,025,526	237,641	$\sim$ 16 hours
$E_8(split)$	$\sim$60 M	3,075,281	?

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2) General Arthur packets: In many cases $\Pi(\Psi)$ is known to be unitary (see the following slide)

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Davis/Mason-Brown uniform proof: 1): all complex groups; many cases for real groups (Hodge theory). Plus Adams/Ionov/Mason-Brown/Vogan (unpublished): 1) in all cases, and 2) under a certain genericity condition.

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Please stay tuned for the next talk.

Thank you Jim!