

# Atlas of Lie Groups and Representations



[www.liegroups.org](http://www.liegroups.org)

# Computing Unipotent Representations

Jeffrey Adams  
Joint Meetings, Denver  
January 17, 2020

Slides available at: [www.liegroups.org](http://www.liegroups.org)

# ATLAS PROJECT

Jeffrey Adams

Annegret Paul

Peter Trapa

Marc van Leeuwen

David Vogan

[Stephen Miller](#)

Dan Barbasch

Birne Binengar

Fokko du Cloux

Alfred Noel

Susana Salamanca

Siddhartha Sahi

John Stembridge

# THE ATLAS PROJECT

In short: compute the unitary dual of any real reductive group.

- ▶ **Tools for education**: teaching Lie groups to graduate students and researchers
- ▶ **Tools for non-specialists** who apply Lie groups in other areas
- ▶ **Tools for studying** other problems in Lie groups
- ▶ Deepen our understanding of the mathematics
- ▶ **Compute the unitary dual**

**Basic Principle**: You only really understand something if you can implement it on a computer

## EXAMPLE: SPEH REPRESENTATIONS

Birgit Speh [Inventiones 1983]:

$$G = GL(2n, \mathbb{R}), \pi(k) \quad k \in \{1, 2, 3, \dots\},$$

$\pi(k)$  is a very interesting, small, irreducible unitary representation with nontrivial  $(\mathfrak{g}, K)$ -cohomology

Unitarity: (Speh 1981)  $\pi(k)$  occurs in the non-cuspidal discrete spectrum of  $GL(2n, \mathbb{R})/\Gamma$

Later [Vogan/Zuckerman 1984]:

$\pi(k)$  is cohomologically induced from a unitary character of  $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$

## EXAMPLE: SPEH REPRESENTATIONS IN ATLAS

```
atlas> set pi=speh(8,1) {Speh representation pi(1) of GL(8,R)}
(x=188,lambda=[7,5,3,1,-1,-3,-5,-7]/2,nu=[3,1,-1,-3,3,1,-1,-3]/2
atlas> pi.infinitesimal_character
Value: [ 7, 5, 3, 1, -1, -3, -5, -7 ]/2
atlas> is_unitary(pi)
Value: true
atlas> test_line(pi)
reducibility points: [1/3,1/2,2/3,1/1]
t      lambda                                     unitary
0      [ 0, 0, 0, 0, 0, 0, 0, 0 ]                 true
1/6    [ 3, 1, -1, -3, 3, 1, -1, -3 ]/12          true
1/3    [ 3, 1, -1, -3, 3, 1, -1, -3 ]/6           true
5/12   [ 15, 5, -5, -15, 15, 5, -5, -15 ]/24     false
1/2    [ 3, 1, -1, -3, 3, 1, -1, -3 ]/4           true
7/12   [ 21, 7, -7, -21, 21, 7, -7, -21 ]/24     false
2/3    [ 3, 1, -1, -3, 3, 1, -1, -3 ]/3           false
5/6    [ 15, 5, -5, -15, 15, 5, -5, -15 ]/12     false
1/1    [ 3, 1, -1, -3, 3, 1, -1, -3 ]/2           true
```

# LOCAL LANGLANDS CONJECTURE

Local Langlands conjecture:

$G$ : connected reductive complex group, defined over  $\mathbb{R}$

$$W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad jzj^{-1} = \bar{z}, j^2 = -1$$

$G^{\vee}$ : complex dual group of  $G$

$${}^L G = G^{\vee} \rtimes \text{Gal}(\bar{\mathbb{C}}/\mathbb{R})$$

Local Langlands Conjecture:

$$\phi : W_{\mathbb{R}} \rightarrow {}^L G \mapsto \Pi_{\phi} \subset \Pi(G)_{adm}$$

( $\phi$  admissible)

$\Pi_{\phi}$  is an L-packet

$$\Pi(G)_{adm} = \cup_{\{\phi\}/G^{\vee}} \Pi(\phi)$$

# ARTHUR'S CONJECTURES

~ 1980: Jim Arthur: also consider

$$\Psi : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \rightarrow {}^L G$$

such that  $\Psi|_{W_{\mathbb{R}}}$  is bounded.

Arthur conjectures: for each such  $\Psi$  there should be a finite set

$$\Pi(\Psi) \subset \Pi(G)_{adm}$$

satisfying various properties

$$\phi_{\Psi} : W_{\mathbb{R}} \rightarrow {}^L G : \quad \phi_{\Psi}(w) = \Psi(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}) \Rightarrow$$

$\Pi(\phi_{\Psi}) \subset \Pi(\psi)$  (Arthur packets contain L-packets)

$\Pi(\Psi)$  consists of **unitary** representations

# ARTHUR'S UNIPOTENT REPRESENTATIONS

Unipotent case:  $W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle$   $\Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathrm{SL}(2, \mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \rightarrow {}^L G$

(Jacobson-Morozov)  $\Psi|_{\mathrm{SL}(2, \mathbb{C})} \leftrightarrow$  **Unipotent Orbit**  $\mathcal{O}^{\vee}$  in  $G^{\vee}$

Ignore  $\mathbb{Z}/2\mathbb{Z}$ : there should be a map

$$\mathcal{O}^{\vee} \mapsto \Pi(\mathcal{O}^{\vee}) = \{\pi_1, \dots, \pi_n\}$$

a finite set of irreducible unitary representations of  $G(\mathbb{R})$ .

We refer to  $\Pi(\mathcal{O}^{\vee})$  as a **weak Arthur packet**. It is a union of **honest Arthur packets**:

$$\Pi(\mathcal{O}^{\vee}) = \cup_x \Pi(\Psi_x)$$

$\Psi_x|_{\mathrm{SL}(2, \mathbb{C})} = \Psi$ , and  $\Psi_x(j) = x \in \mathrm{Cent}(\mathcal{O}^{\vee})_2$  (modulo conjugacy by  $\mathrm{Cent}(\mathcal{O}^{\vee})$ ).

# OVERVIEW

Today:  $\Pi(\mathcal{O}^\vee)$

**Note:** Arthur did not give a definition of these packets, just some properties they should satisfy (which are not sufficient to determine them).

A precise definition of  $\Pi(\mathcal{O}^\vee)$  is given in Barbasch-Vogan's 1985 Annals paper, and the 1992 book by Adams, Barbasch and Vogan (aka [ABV]).

**Remark** For classical groups there is a very different definition in Arthur's 2013 book. This definition agrees with [ABV] (forthcoming work by Adams, Arancibia and Mezo)

**Goal:** compute  $\Pi(\mathcal{O}^\vee)$

# ASSOCIATED VARIETY

Associated to an irreducible representation  $\pi$  of  $G(\mathbb{R})$  is a nilpotent  $G(\mathbb{C})$ -orbit in  $\mathfrak{g}$ .

$$I = \text{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$$\pi \mapsto I \mapsto \text{gr}(I) \subset \text{gr}(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g}) \mapsto \mathcal{V}(\text{gr}(I)) \subset \mathfrak{g}^*$$

**Definition:**  $\text{AV}(\text{Ann}(\pi)) = \mathcal{V}(\text{gr}(I)) \subset$  the nilpotent cone

**Fact:** (Borho/Brylinski/Joseph)  $\mathcal{V}(\pi)$  is the closure of a single nilpotent orbit  $\mathcal{O}$ .

**Definition:**  $\text{AV}_{\text{ann}}(\pi) = \overline{\mathcal{O}}$

# INFINITESIMAL CHARACTER

Infinitesimal character of a dual orbit:

$\mathcal{O}^\vee$  is a nilpotent orbit for the complex dual group  $G^\vee$ .

Jacobson-Morozov:

$$\mathcal{O}^\vee \rightarrow \{H, E, F\} \rightarrow H \in \mathfrak{h}^\vee \simeq \mathfrak{h}^* \rightarrow \lambda(\mathcal{O}^\vee) = \frac{1}{2}H \in \mathfrak{h}^*$$

# WEAK ARTHUR PACKETS

**Definition** (weak Arthur packet):

Suppose  $\mathcal{O}^\vee$  is a nilpotent orbit of  $G^\vee$ . Assume  $\mathcal{O}^\vee$  is even ( $\lambda(\mathcal{O}^\vee)$  is in the root lattice; every node in the Dynkin diagram of  $\mathcal{O}^\vee$  has label 0, 2). Let  $\mathcal{O}$  be the (special) dual  $G$ -orbit (via Lusztig-Spaltenstein duality of nilpotent orbits).

Then  $\Pi(\mathcal{O}^\vee)$  consists of the irreducible representations  $\pi$  of  $G(\mathbb{R})$  satisfying:

- (a) the infinitesimal character of  $\pi$  is  $\lambda(\mathcal{O}^\vee) \in \mathfrak{h}^*$
- (b)  $AV_{\text{ann}}(\pi) = \overline{\mathcal{O}}$

This is a weak Arthur packet of **special unipotent representations**

# WEAK ARTHUR PACKETS

**Note:** If  $\mathcal{O}^\vee$  is not even, the construction involves

$$H^\vee = \text{Cent}_{G^\vee}(e^{2\pi i\lambda(\mathcal{O}^\vee)})^0 \subset G^\vee$$

.

For simplicity stick to even orbits for this talk.

**Note:** The definition of **honest** Arthur packets involves the element  $\Psi(j)$ , and the  $G(\mathbb{R})$  orbits in  $\mathcal{O} \cap \mathfrak{g}_0$  (equivalently: the  $K(\mathbb{C})$  orbits in  $\mathcal{O} \cap \mathfrak{g}^{-\theta}$ ). There is a definition in [ABV], and now an explicit algorithm, but the implementation is more difficult.

# COMPUTATIONAL INGREDIENTS

1) Explicit Langlands classification

Fix a regular infinitesimal character  $\lambda$ :

$M_\lambda = \mathbb{Z}$ -span of the irreducible representations with inf. char.  $\lambda$   
(finite dimensional)

An explicit parameter space  $S_\lambda$ :

$$M_\lambda = \mathbb{Z}\langle\{J(\gamma) \mid \gamma \in S_\lambda\}\rangle$$

$J(\gamma)$  is irreducible; the unique irreducible quotient of a standard module  $I(\gamma)$ , and

$$M_\lambda = \mathbb{Z}\langle\{I(\gamma) \mid \gamma \in S_\lambda\}\rangle$$

2) Kazhdan-Lusztig-Vogan polynomials (change of basis matrix for  $M_\gamma$  for the  $J(\gamma)$  and  $I(\gamma)$  bases)

# COMPUTATIONAL INGREDIENTS

3) Coherent continuation (Zuckerman):  $M_\lambda$  is a representation of the Weyl group  $W$ . This is easily computed in the basis of standard modules  $I(\gamma)$ :  $s_\alpha$  takes  $I(\gamma)$  to a sum of a small number of other  $I(\gamma')$ , with small integer coefficients.

The KLV polynomials then give this action in the bases of irreducibles  $J(\gamma)$ .

$M_\lambda$  is the direct sum of **cell representations** (defined combinatorially using the KLV polynomials). Each cell  $\mathcal{C}$  contains a unique special representation  $Sp(\mathcal{C})$  of  $W$  with multiplicity one.

4) Parametrization of complex nilpotent orbits  $\mathcal{N}/G$

5) The Springer correspondence  $\mathcal{N}/G \leftrightarrow \hat{W} : \mathcal{O} \mapsto \text{Springer}(\mathcal{O})$

# ALGORITHM

Fix  $G(\mathbb{C})$ ,  $G(\mathbb{R})$ . For simplicity assume  $G(\mathbb{C})$  is simply connected, and  $G(\mathbb{R})$  is inner to a split group.

- 1) Compute the character table of  $W = W(G(\mathbb{C}))$ .
- 2) Explicitly compute  $S_\rho, M_\rho$ .
- 3) Compute the representation of  $W$  on  $M_\rho$ , and its decomposition into cells.
- 4) For each cell  $\mathcal{C}$  compute the special representation  $Sp(\mathcal{C})$ .

# ALGORITHM

Now: fix a complex even nilpotent orbit  $\mathcal{O}^\vee$   
Initialize an empty list of parameters  $WP(\mathcal{O}^\vee)$ .

5) Let  $\sigma = \text{Springer}(\mathcal{O}^\vee) \otimes \text{sgn}$

6) Run over the cells  $\mathcal{C}$ . For each cell  $\mathcal{C}$  check if  $Sp(\mathcal{C}) = \sigma$ . If not, ignore it. If so: apply translation to all of the irreducible representation  $J(\gamma) \in \mathcal{C}$ , to translate from  $\rho$  to  $\lambda(\mathcal{O}^\vee)$  (which is dominant, but usually singular). This operation takes an irreducible to irreducible or 0. Add the non-zero terms to  $WP(\mathcal{O}^\vee)$ .

7) After running over all cells  $WP(\mathcal{O}^\vee)$  is the weak packet defined by  $\mathcal{O}^\vee$ .

# CLASSICAL GROUPS

For classical groups the weak packets (and more) were computed, at least in small examples, by Jonathan Fernandes (UMD thesis, 2019).

```
atlas> show_nilpotent_orbits(G2_s)
```

i	H	diagram	dim	BC Levi	Cent	A(0)
0	[0,0]	[0,0]	0	2T1	G2	[1]
1	[1,2]	[0,1]	6	A1+T1	A1	[1]
2	[2,3]	[1,0]	8	A1+T1	A1	[1]
3	[2,4]	[0,2]	10	G2	e	[1,2,3]
4	[6,10]	[2,2]	12	G2	e	[1]

```
B=block/C=Cell
```

orbit	B	C	parameters	inf. char.
0	0	0	(x=9,lambda=[1,1]/1,nu=[0,0]/1)	[ 0, 0 ]/1
0	1	0	(x=0,lambda=[0,0]/1,nu=[0,0]/1)	[ 0, 0 ]/1
1	0	1	(x=9,lambda=[1,1]/1,nu=[1,0]/2)	[ 1, 0 ]/2
1	0	2	(x=9,lambda=[2,1]/1,nu=[1,0]/2)	[ 1, 0 ]/2
2	0	1	(x=9,lambda=[1,2]/1,nu=[0,1]/2)	[ 0, 1 ]/2
2	0	2	(x=9,lambda=[1,1]/1,nu=[0,1]/2)	[ 0, 1 ]/2
3	0	1	(x=4,lambda=[1,0]/1,nu=[2,-1]/2)	[ 1, 0 ]/1
3	0	1	(x=8,lambda=[3,0]/1,nu=[1,0]/1)	[ 1, 0 ]/1
3	0	2	(x=2,lambda=[1,0]/1,nu=[0,0]/1)	[ 1, 0 ]/1
3	0	2	(x=6,lambda=[4,-1]/1,nu=[3,-1]/2)	[ 1, 0 ]/1
3	0	2	(x=9,lambda=[1,1]/1,nu=[1,0]/1)	[ 1, 0 ]/1
4	0	3	(x=9,lambda=[1,1]/1,nu=[1,1]/1)	[ 1, 1 ]/1

## SOME UNIPOTENT REPRESENTATIONS

Group	#orbs.	#unip.	#unip by orbit
$SL(2, \mathbb{R})$	2	4	3,1
$SU(2)$	1	1	
$Sp(4, \mathbb{R})$	4	16	5,2,8,1
$Sp(1, 1)$	3	3	1,1,1
$Sp(2)$	1	1	
$Sp(6, \mathbb{R})$	7	47	7,6,16,2,7,8,1

---

$G_2$ (split)	5	12	2,2,2,5,1
$G_2$ (cpt)	1	1	
$F_4$ (split)	16	75	3,4,10,5,2,4,2,7,2,8,14,2,4,3,4,1
$F_4(B_4)$	3	3	1,1,1
$F_4$ (cpt)	1	1	
$E_6$ (split)	21	68	3,4,5,4,7,4,2,3,4,2,4,7,3,2,2,2,2,4,2,1,1
$E_6(F_4)$	3	3	1,1,1

# SOME UNIPOTENT REPRESENTATIONS

Group	#orbs.	#unip.	#unip by orbit
$E_7$ (split)	45	252	6,7,8,3,8,16,4,12,9,7,6,2,3,6,8, 17,4,4,7,7,12,2,3,2,8,8,5,6,4,2, 12,2,4,5,2,5,2,4,8,2,3,4,1,1,1
$E_7(E_6 T)$	13	28	3,1,1,5,1,2,2,1,5,1,3,2,1
$E_7(D_6 A_1 T)$	25	56	2,2,2,1,3,2,4,1,1,3,3,2 1,2,3,7,1,1,2,1,2,6,2,1,1
$E_7$ (compact)	1	1	

$E_8$  has 70 unipotent orbits, of which 27 are even.

Calculating the KLV polynomials for  $E_8$  now takes just under three hours.

Storing the KLV polynomials requires about 100 gigabytes of storage. However storing the information needed to compute the cell representations only requires 42 megabytes.

There are 104 cells sizes (average size: 4,356):

1,8,35,196,196,260,560,260,560,560,1100,567,3752,1100,4025,3240,3192,1100,  
 2625,3240,3240,3240,3240,3640,3240,3640,8192,3640,7560,3240,8192,5040,  
 4536,4536,4536,525,3500,6075,7560,2835,4536,8800,3500,6075,6075,4200,  
 4200,8800,46676,22778,4200,4200,38766,4200,2100,8800,4536,4200,8800,4200,  
 6075,6075,2835,4536,4536,4200,7560,4536,3640,6075,7560,5040,3500,8192,  
 3640,3240,3240,3240,1100,3500,8192,3640,3240,3240,525,3240,3240,4025,  
 3752,2625,3192,1100,1100,560,567,560,560,260,196,260,196,35,8,1

We need to compute the character of a representation of each of these dimensions, up to 46,676. That is: multiply up to 120 matrices of size 46,676 (and take the trace). Each matrix is sparse (the action of a simple reflection). So: multiply an arbitrary matrix \* sparse matrix.

Example:

$E_8(\text{split})$ : 27 even orbits  $\mapsto$  112 unipotent representations

1,4,6,5,1,4,6,4,4,4,18,1,4,4,4,1,6,5,5,5,6,3,4,4,1,1,1

Thank you