Atlas of Lie Groups and Representations

www.liegroups.org

Computing Unipotent Representations

Jeffrey Adams
Joint Meetings, Denver
January 17, 2020

Slides available at: www.liegroups.org

Atlas Project

Jeffrey Adams

Annegret Paul

Peter Trapa

Marc van Leeuwen

David Vogan

Stephen Miller

Dan Barbasch

Birne Binegar

Fokko du Cloux

Alfred Noel

Susana Salamanca

Siddhartha Sahi

John Stembridge

The Atlas project

The Atlas project

In short: compute the unitary dual of any real reductive group.

The Atlas project

In short: compute the unitary dual of any real reductive group.

- Tools for education: teaching Lie groups to graduate students and researchers

The Atlas project

In short: compute the unitary dual of any real reductive group.

- Tools for education: teaching Lie groups to graduate students and researchers
- Tools for non-specialists who apply Lie groups in other areas

In short: compute the unitary dual of any real reductive group.

- Tools for education: teaching Lie groups to graduate students and researchers
- Tools for non-specialists who apply Lie groups in other areas
- Tools for studying other problems in Lie groups

The Atlas project

In short: compute the unitary dual of any real reductive group.

- Tools for education: teaching Lie groups to graduate students and researchers
- Tools for non-specialists who apply Lie groups in other areas
- Tools for studying other problems in Lie groups
- Deepen our understanding of the mathematics

The Atlas project

In short: compute the unitary dual of any real reductive group.

- Tools for education: teaching Lie groups to graduate students and researchers
- Tools for non-specialists who apply Lie groups in other areas
- Tools for studying other problems in Lie groups
- Deepen our understanding of the mathematics
- Compute the unitary dual

In short: compute the unitary dual of any real reductive group.

- Tools for education: teaching Lie groups to graduate students and researchers
- Tools for non-specialists who apply Lie groups in other areas
- Tools for studying other problems in Lie groups
- Deepen our understanding of the mathematics
- Compute the unitary dual

Basic Principle: You only really understand something if you can implement it on a computer

Example: Speh Representations

Example: Speh Representations

Birgit Speh [Inventiones 1983]:

Example: Speh Representations

Birgit Speh [Inventiones 1983]:

$G = GL(2n, \mathbb{R})$

Example: Speh Representations

Birgit Speh [Inventiones 1983]:

$G = GL(2n, \mathbb{R})$, $\pi(k)$ $k \in \{1, 2, 3, \dots\}$,

Example: Speh Representations

Birgit Speh [Inventiones 1983]:

$G = GL(2n, \mathbb{R})$, $\pi(k)$ $k \in \{1, 2, 3, \dots\}$,

$\pi(k)$ is a very interesting, small, irreducible unitary representation with nontrivial $(\mathfrak{g}, K)$-cohomology

Example: Speh Representations

Birgit Speh [Inventiones 1983]:

$G = GL(2n, \mathbb{R})$, $\pi(k)$ $k \in \{1, 2, 3, \ldots\}$,

$\pi(k)$ is a very interesting, small, irreducible unitary representation with nontrivial $(\mathfrak{g}, K)$-cohomology

Unitarity: (Speh 1981) $\pi(k)$ occurs in the non-cuspidal discrete spectrum of $GL(2n, \mathbb{R})/\Gamma$

Example: Speh Representations

Birgit Speh [Inventiones 1983]:

$G = GL(2n, \mathbb{R})$, $\pi(k)$ $k \in \{1, 2, 3, \dots\}$,

$\pi(k)$ is a very interesting, small, irreducible unitary representation with nontrivial $(\mathfrak{g}, K)$-cohomology

Unitarity: (Speh 1981) $\pi(k)$ occurs in the non-cuspidal discrete spectrum of $GL(2n, \mathbb{R})/\Gamma$

Later [Vogan/Zuckerman 1984]:

Example: Speh Representations

Birgit Speh [Inventiones 1983]:

$G = GL(2n, \mathbb{R})$, $\pi(k)$ $k \in \{1, 2, 3, \dots\}$,

$\pi(k)$ is a very interesting, small, irreducible unitary representation with nontrivial $(\mathfrak{g}, K)$-cohomology

Unitarity: (Speh 1981) $\pi(k)$ occurs in the non-cuspidal discrete spectrum of $GL(2n, \mathbb{R})/\Gamma$

Later [Vogan/Zuckerman 1984]:

$\pi(k)$ is cohomologically induced from a unitary character of $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$

Example: Speh Representations in Atlas

Example: Speh Representations in Atlas

```
atlas> set pi=speh(8,1)  {Speh representation pi(1) of GL(8,R)}
```

Example: Speh Representations in Atlas

```
atlas> set pi=speh(8,1)  {Speh representation pi(1) of GL(8,R)}
(x=188,lambda=[7,5,3,1,-1,-3,-5,-7]/2,nu=[3,1,-1,-3,3,1,-1,-3]/2
```

Example: Speh Representations in Atlas

```
atlas> set pi=speh(8,1)  {Speh representation pi(1) of GL(8,R)}
(x=188,lambda=[7,5,3,1,-1,-3,-5,-7]/2,nu=[3,1,-1,-3,3,1,-1,-3]/2
atlas> pi.infinitesimal_character
Value: [  7,  5,  3,  1, -1, -3, -5, -7 ]/2
```

Example: Speh Representations in Atlas

```
atlas> set pi=speh(8,1)  {Speh representation pi(1) of GL(8,R)}
(x=188,lambda=[7,5,3,1,-1,-3,-5,-7]/2,nu=[3,1,-1,-3,3,1,-1,-3]/2
atlas> pi.infinitesimal_character
Value: [  7,  5,  3,  1, -1, -3, -5, -7 ]/2
atlas> is_unitary(pi)
Value: true
```

Example: Speh Representations in Atlas

```
atlas> set pi=speh(8,1)  {Speh representation pi(1) of GL(8,R)}
(x=188,lambda=[7,5,3,1,-1,-3,-5,-7]/2,nu=[3,1,-1,-3,3,1,-1,-3]/2
atlas> pi.infinitesimal_character
Value: [  7,  5,  3,  1, -1, -3, -5, -7 ]/2
atlas> is_unitary(pi)
Value: true
atlas> test_line(pi)
reducibility points: [1/3,1/2,2/3,1/1]
t     lambda                                                unitary
0     [  0,  0,  0,  0,  0,  0,  0,  0 ]                    true
1/6   [  3,  1, -1, -3,  3,  1, -1, -3 ]/12                 true
1/3   [  3,  1, -1, -3,  3,  1, -1, -3 ]/6                  true
5/12  [  15,   5,  -5, -15,  15,   5,  -5, -15 ]/24         false
1/2   [  3,  1, -1, -3,  3,  1, -1, -3 ]/4                  true
7/12  [  21,   7,  -7, -21,  21,   7,  -7, -21 ]/24         false
2/3   [  3,  1, -1, -3,  3,  1, -1, -3 ]/3                  false
5/6   [  15,   5,  -5, -15,  15,   5,  -5, -15 ]/12         false
1/1   [  3,  1, -1, -3,  3,  1, -1, -3 ]/2                  true
```

Local Langlands Conjecture

Local Langlands Conjecture

Local Langlands conjecture:

Local Langlands Conjecture

Local Langlands conjecture:

G: connected reductive complex group, defined over $\mathbb{R}$

Local Langlands Conjecture

Local Langlands conjecture:

G: connected reductive complex group, defined over $\mathbb{R}$

$W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad jzj^{-1} = \overline{z}, j^2 = -1$

Local Langlands Conjecture

Local Langlands conjecture:

G: connected reductive complex group, defined over $\mathbb{R}$

$W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad jzj^{-1} = \overline{z}, j^2 = -1$

$G^\vee$: complex dual group of G

Local Langlands Conjecture

Local Langlands conjecture:

G: connected reductive complex group, defined over $\mathbb{R}$

$W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad jzj^{-1} = \overline{z}, j^2 = -1$

$G^\vee$: complex dual group of G

${}^L G = G^\vee \rtimes \mathrm{Gal}(\overline{\mathbb{C}}/\mathbb{R})$

Local Langlands Conjecture

Local Langlands conjecture:

G: connected reductive complex group, defined over $\mathbb{R}$

$W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad jzj^{-1} = \overline{z}, j^2 = -1$

$G^\vee$: complex dual group of G

${}^LG = G^\vee \rtimes \mathrm{Gal}(\overline{\mathbb{C}}/\mathbb{R})$

Local Langlands Conjecture:

$$\phi : W_{\mathbb{R}} \to {}^LG$$

Local Langlands Conjecture

Local Langlands conjecture:

G: connected reductive complex group, defined over $\mathbb{R}$

$W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad jzj^{-1} = \overline{z}, j^2 = -1$

$G^\vee$: complex dual group of G

${}^LG = G^\vee \rtimes \mathrm{Gal}(\overline{\mathbb{C}}/\mathbb{R})$

Local Langlands Conjecture:

$$\phi : W_{\mathbb{R}} \to {}^LG \mapsto \Pi_\phi \subset \Pi(G)_{adm}$$

Local Langlands Conjecture

Local Langlands conjecture:

G: connected reductive complex group, defined over $\mathbb{R}$

$W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad jzj^{-1} = \overline{z}, j^2 = -1$

$G^\vee$: complex dual group of G

${}^L G = G^\vee \rtimes \mathrm{Gal}(\overline{\mathbb{C}}/\mathbb{R})$

Local Langlands Conjecture:

$$\phi : W_{\mathbb{R}} \to {}^L G \mapsto \Pi_\phi \subset \Pi(G)_{adm}$$

(ϕ admissible)

Local Langlands Conjecture

Local Langlands conjecture:

G: connected reductive complex group, defined over $\mathbb{R}$

$W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad jzj^{-1} = \overline{z}, j^2 = -1$

$G^\vee$: complex dual group of G

${}^L G = G^\vee \rtimes \mathrm{Gal}(\overline{\mathbb{C}}/\mathbb{R})$

Local Langlands Conjecture:

$$\phi : W_{\mathbb{R}} \rightarrow {}^L G \mapsto \Pi_\phi \subset \Pi(G)_{adm}$$

(ϕ admissible)

Π_ϕ is an L-packet

Local Langlands Conjecture

Local Langlands conjecture:

G: connected reductive complex group, defined over $\mathbb{R}$

$W_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad jzj^{-1} = \overline{z}, j^2 = -1$

$G^\vee$: complex dual group of G

${}^L G = G^\vee \rtimes \mathrm{Gal}(\overline{\mathbb{C}}/\mathbb{R})$

Local Langlands Conjecture:

$$\phi : W_{\mathbb{R}} \to {}^L G \mapsto \Pi_\phi \subset \Pi(G)_{adm}$$

(ϕ admissible)

Π_ϕ is an L-packet

$$\Pi(G)_{adm} = \dot\cup_{\{\phi\}/G^\vee} \Pi(\phi)$$

Arthur's Conjectures

Arthur's Conjectures

$\sim$ 1980: Jim Arthur: also consider

$$\Psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \rightarrow {}^L G$$

such that $\Psi|_{W_{\mathbb{R}}}$ is bounded.

Arthur's Conjectures

$\sim$ 1980: Jim Arthur: also consider

$$\Psi : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \to {}^L G$$

such that $\Psi|_{W_{\mathbb{R}}}$ is bounded.

Arthur conjectures: for each such Ψ there should be a finite set

Arthur's Conjectures

$\sim$ 1980: Jim Arthur: also consider

$$\Psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \rightarrow {}^L G$$

such that $\Psi|_{W_{\mathbb{R}}}$ is bounded.

Arthur conjectures: for each such Ψ there should be a finite set

$$\Pi(\Psi) \subset \Pi(G)_{adm}$$

satisfying various properties

Arthur's Conjectures

$\sim$ 1980: Jim Arthur: also consider

$$\Psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \rightarrow {}^L G$$

such that $\Psi|_{W_{\mathbb{R}}}$ is bounded.

Arthur conjectures: for each such Ψ there should be a finite set

$$\Pi(\Psi) \subset \Pi(G)_{adm}$$

satisfying various properties

$\phi_{\Psi} : W_{\mathbb{R}} \rightarrow {}^L G$:

Arthur's Conjectures

$\sim$ 1980: Jim Arthur: also consider

$$\Psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^L G$$

such that $\Psi|_{W_{\mathbb{R}}}$ is bounded.

Arthur conjectures: for each such Ψ there should be a finite set

$$\Pi(\Psi) \subset \Pi(G)_{adm}$$

satisfying various properties

$$\phi_\Psi : W_{\mathbb{R}} \to {}^L G : \quad \phi_\Psi(w) = \Psi(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \Rightarrow$$

Arthur's Conjectures

$\sim$ 1980: Jim Arthur: also consider

$$\Psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^L G$$

such that $\Psi|_{W_{\mathbb{R}}}$ is bounded.

Arthur conjectures: for each such Ψ there should be a finite set

$$\Pi(\Psi) \subset \Pi(G)_{adm}$$

satisfying various properties

$$\phi_\Psi : W_{\mathbb{R}} \to {}^L G : \quad \phi_\Psi(w) = \Psi(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \Rightarrow$$

$\Pi(\phi_\psi) \subset \Pi(\psi)$

Arthur's Conjectures

$\sim$ 1980: Jim Arthur: also consider

$$\Psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^L G$$

such that $\Psi|_{W_{\mathbb{R}}}$ is bounded.

Arthur conjectures: for each such Ψ there should be a finite set

$$\Pi(\Psi) \subset \Pi(G)_{adm}$$

satisfying various properties

$$\phi_{\Psi} : W_{\mathbb{R}} \to {}^L G : \quad \phi_{\Psi}(w) = \Psi(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \Rightarrow$$

$\Pi(\phi_{\psi}) \subset \Pi(\psi)$ (Arthur packets contain L-packets)

Arthur's Conjectures

$\sim$ 1980: Jim Arthur: also consider

$$\Psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \rightarrow {}^L G$$

such that $\Psi|_{W_{\mathbb{R}}}$ is bounded.

Arthur conjectures: for each such Ψ there should be a finite set

$$\Pi(\Psi) \subset \Pi(G)_{adm}$$

satisfying various properties

$$\phi_{\Psi} : W_{\mathbb{R}} \rightarrow {}^L G : \quad \phi_{\Psi}(w) = \Psi(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \Rightarrow$$

$\Pi(\phi_{\psi}) \subset \Pi(\psi)$ (Arthur packets contain L-packets)

$\Pi(\Psi)$ consists of unitary representations

Arthur's Unipotent Representations

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_\mathbb{R} = \langle \mathbb{C}^*, j \rangle$

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathrm{SL}(2, \mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \rightarrow {}^L G$

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathrm{SL}(2, \mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \to {}^L G$

(Jacobson-Morozov) $\Psi|_{\mathrm{SL}(2,\mathbb{C})} \leftrightarrow$ Unipotent Orbit $\mathcal{O}^\vee$ in $G^\vee$

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathrm{SL}(2, \mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \to {}^L G$

(Jacobson-Morozov) $\Psi|_{\mathrm{SL}(2,\mathbb{C})} \leftrightarrow$ Unipotent Orbit $\mathcal{O}^\vee$ in $G^\vee$

Ignore $\mathbb{Z}/2\mathbb{Z}$: there should be a map

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathrm{SL}(2,\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \to {}^L G$

(Jacobson-Morozov) $\Psi|_{\mathrm{SL}(2,\mathbb{C})} \leftrightarrow$ Unipotent Orbit $\mathcal{O}^\vee$ in $G^\vee$

Ignore $\mathbb{Z}/2\mathbb{Z}$: there should be a map

$$\mathcal{O}^\vee$$

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

$\Psi : SL(2,\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \rightarrow {}^L G$

(Jacobson-Morozov) $\Psi|_{SL(2,\mathbb{C})} \leftrightarrow$ Unipotent Orbit $\mathcal{O}^\vee$ in $G^\vee$

Ignore $\mathbb{Z}/2\mathbb{Z}$: there should be a map

$$\mathcal{O}^\vee \mapsto \Pi(\mathcal{O}^\vee) = \{\pi_1, \ldots, \pi_n\}$$

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathrm{SL}(2, \mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \rightarrow {}^L G$

(Jacobson-Morozov) $\Psi|_{\mathrm{SL}(2,\mathbb{C})} \leftrightarrow$ Unipotent Orbit $\mathcal{O}^\vee$ in $G^\vee$

Ignore $\mathbb{Z}/2\mathbb{Z}$: there should be a map

$$\mathcal{O}^\vee \mapsto \Pi(\mathcal{O}^\vee) = \{\pi_1, \dots, \pi_n\}$$

a finite set of irreducible unitary representations of $G(\mathbb{R})$.

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathrm{SL}(2, \mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \to {}^L G$

(Jacobson-Morozov) $\Psi|_{\mathrm{SL}(2,\mathbb{C})} \leftrightarrow$ Unipotent Orbit $\mathcal{O}^\vee$ in $G^\vee$

Ignore $\mathbb{Z}/2\mathbb{Z}$: there should be a map

$$\mathcal{O}^\vee \mapsto \Pi(\mathcal{O}^\vee) = \{\pi_1, \ldots, \pi_n\}$$

a finite set of irreducible unitary representations of $G(\mathbb{R})$.

We refer to $\Pi(\mathcal{O}^\vee)$ as a weak Arthur packet.

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

$\Psi : SL(2, \mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \to {}^L G$

(Jacobson-Morozov) $\Psi|_{SL(2,\mathbb{C})} \leftrightarrow$ Unipotent Orbit $\mathcal{O}^\vee$ in $G^\vee$

Ignore $\mathbb{Z}/2\mathbb{Z}$: there should be a map

$$\mathcal{O}^\vee \mapsto \Pi(\mathcal{O}^\vee) = \{\pi_1, \dots, \pi_n\}$$

a finite set of irreducible unitary representations of $G(\mathbb{R})$.

We refer to $\Pi(\mathcal{O}^\vee)$ as a weak Arthur packet. It is a union of honest Arthur packets:

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

$\Psi : SL(2, \mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \rightarrow {}^L G$

(Jacobson-Morozov) $\Psi|_{SL(2,\mathbb{C})} \leftrightarrow$ Unipotent Orbit $\mathcal{O}^\vee$ in $G^\vee$

Ignore $\mathbb{Z}/2\mathbb{Z}$: there should be a map

$$\mathcal{O}^\vee \mapsto \Pi(\mathcal{O}^\vee) = \{\pi_1, \ldots, \pi_n\}$$

a finite set of irreducible unitary representations of $G(\mathbb{R})$.

We refer to $\Pi(\mathcal{O}^\vee)$ as a weak Arthur packet. It is a union of honest Arthur packets:

$$\Pi(\mathcal{O}^\vee) = \cup_x \Pi(\Psi_x)$$

Arthur's Unipotent Representations

Unipotent case: $\mathbb{W}_{\mathbb{R}} = \langle \mathbb{C}^*, j \rangle \quad \Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathrm{SL}(2,\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \to {}^L G$

(Jacobson-Morozov) $\Psi|_{\mathrm{SL}(2,\mathbb{C})} \leftrightarrow$ Unipotent Orbit $\mathcal{O}^\vee$ in $G^\vee$

Ignore $\mathbb{Z}/2\mathbb{Z}$: there should be a map

$$\mathcal{O}^\vee \mapsto \Pi(\mathcal{O}^\vee) = \{\pi_1, \dots, \pi_n\}$$

a finite set of irreducible unitary representations of $G(\mathbb{R})$.

We refer to $\Pi(\mathcal{O}^\vee)$ as a weak Arthur packet. It is a union of honest Arthur packets:

$$\Pi(\mathcal{O}^\vee) = \cup_x \Pi(\Psi_x)$$

$\Psi_x|_{\mathrm{SL}(2,\mathbb{C})} = \Psi$, and $\Psi_x(j) = x \in \mathrm{Cent}(\mathcal{O}^\vee)_2$ (modulo conjugacy by $\mathrm{Cent}(\mathcal{O}^\vee)$).

Overview

Overview

Today: $\Pi(\mathcal{O}^\vee)$

Overview

Today: $\Pi(\mathcal{O}^\vee)$

Note: Arthur did not give a definition of these packets, just some properties they should satisfy (which are not sufficient to determine them).

Overview

Today: $\Pi(\mathcal{O}^\vee)$

Note: Arthur did not give a definition of these packets, just some properties they should satisfy (which are not sufficient to determine them).

A precise definition of $\Pi(\mathcal{O}^\vee)$ is given in Barbasch-Vogan's 1985 Annals paper, and the 1992 book by Adams, Barbasch and Vogan (aka [ABV]).

Overview

Today: $\Pi(\mathcal{O}^\vee)$

Note: Arthur did not give a definition of these packets, just some properties they should satisfy (which are not sufficient to determine them).

A precise definition of $\Pi(\mathcal{O}^\vee)$ is given in Barbasch-Vogan's 1985 Annals paper, and the 1992 book by Adams, Barbasch and Vogan (aka [ABV]).

Remark For classical groups there is a very different definition in Arthur's 2013 book. This definition agrees with [ABV] (forthcoming work by Adams, Arancibia and Mezo)

Overview

Today: $\Pi(\mathcal{O}^\vee)$

Note: Arthur did not give a definition of these packets, just some properties they should satisfy (which are not sufficient to determine them).

A precise definition of $\Pi(\mathcal{O}^\vee)$ is given in Barbasch-Vogan's 1985 Annals paper, and the 1992 book by Adams, Barbasch and Vogan (aka [ABV]).

Remark For classical groups there is a very different definition in Arthur's 2013 book. This definition agrees with [ABV] (forthcoming work by Adams, Arancibia and Mezo)

Goal: compute $\Pi(\mathcal{O}^\vee)$

Associated Variety

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

π

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$\pi \mapsto I$

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$$\pi \mapsto I \mapsto \mathrm{gr}(I)$$

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$$\pi \mapsto I \mapsto \mathrm{gr}(I) \subset \mathrm{gr}(\mathcal{U}(\mathfrak{g}))$$

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$$\pi \mapsto I \mapsto \mathrm{gr}(I) \subset \mathrm{gr}(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g})$$

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$$\pi \mapsto I \mapsto \mathrm{gr}(I) \subset \mathrm{gr}(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g}) \mapsto \mathcal{V}(\mathrm{gr}(I)) \subset \mathfrak{g}^*$$

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$$\pi \mapsto I \mapsto \mathrm{gr}(I) \subset \mathrm{gr}(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g}) \mapsto \mathcal{V}(\mathrm{gr}(I)) \subset \mathfrak{g}^*$$

Definition: $\mathrm{AV}(\mathrm{Ann}(\pi)) = \mathcal{V}(\mathrm{gr}(I))$

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$$\pi \mapsto I \mapsto \mathrm{gr}(I) \subset \mathrm{gr}(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g}) \mapsto \mathcal{V}(\mathrm{gr}(I)) \subset \mathfrak{g}^*$$

Definition: $\mathrm{AV}(\mathrm{Ann}(\pi)) = \mathcal{V}(\mathrm{gr}(I)) \subset$ the nilpotent cone

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$$\pi \mapsto I \mapsto \mathrm{gr}(I) \subset \mathrm{gr}(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g}) \mapsto \mathcal{V}(\mathrm{gr}(I)) \subset \mathfrak{g}^*$$

Definition: $\mathrm{AV}(\mathrm{Ann}(\pi)) = \mathcal{V}(\mathrm{gr}(I)) \subset$ the nilpotent cone

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$$\pi \mapsto I \mapsto \mathrm{gr}(I) \subset \mathrm{gr}(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g}) \mapsto \mathcal{V}(\mathrm{gr}(I)) \subset \mathfrak{g}^*$$

Definition: $AV(\mathrm{Ann}(\pi)) = \mathcal{V}(\mathrm{gr}(I)) \subset$ the nilpotent cone

Fact: (Borho/Brylinski/Joseph) $\mathcal{V}(\pi)$ is the closure of a single nilpotent orbit $\mathcal{O}$.

Associated Variety

Associated to an irreducible representation π of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

$$I = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)$$

$$\pi \mapsto I \mapsto \mathrm{gr}(I) \subset \mathrm{gr}(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g}) \mapsto \mathcal{V}(\mathrm{gr}(I)) \subset \mathfrak{g}^*$$

Definition: $\mathrm{AV}(\mathrm{Ann}(\pi)) = \mathcal{V}(\mathrm{gr}(I)) \subset$ the nilpotent cone

Fact: (Borho/Brylinski/Joseph) $\mathcal{V}(\pi)$ is the closure of a single nilpotent orbit $\mathcal{O}$.

Definition: $\mathrm{AV}_{\mathrm{ann}}(\pi) = \overline{\mathcal{O}}$

Infinitesimal character

Infinitesimal character

Infinitesimal character of a dual orbit:

Infinitesimal character

Infinitesimal character of a dual orbit:

$\mathcal{O}^\vee$ is a nilpotent orbit for the complex dual group $G^\vee$.

Infinitesimal character

Infinitesimal character of a dual orbit:

$\mathcal{O}^\vee$ is a nilpotent orbit for the complex dual group $G^\vee$.

Jacobson-Morozov:

Infinitesimal character

Infinitesimal character of a dual orbit:

$\mathcal{O}^\vee$ is a nilpotent orbit for the complex dual group $G^\vee$.

Jacobson-Morozov:

$$\mathcal{O}^\vee \to \{H, E, F\} \to H \in \mathfrak{h}^\vee \simeq \mathfrak{h}^* \to \lambda(\mathcal{O}^\vee) = \frac{1}{2}H \in \mathfrak{h}^*$$

Infinitesimal character

Infinitesimal character of a dual orbit:

$\mathcal{O}^\vee$ is a nilpotent orbit for the complex dual group $G^\vee$.

Jacobson-Morozov:

$$\mathcal{O}^\vee \to \{H, E, F\} \to H \in \mathfrak{h}^\vee \simeq \mathfrak{h}^* \to \lambda(\mathcal{O}^\vee) = \frac{1}{2}H \in \mathfrak{h}^*$$

Weak Arthur Packets

Weak Arthur Packets

Definition (weak Arthur packet):

Weak Arthur Packets

Definition (weak Arthur packet):

Suppose $\mathcal{O}^\vee$ is a nilpotent orbit of $G^\vee$. Assume $\mathcal{O}^\vee$ is even

Weak Arthur Packets

Definition (weak Arthur packet):

Suppose $\mathcal{O}^\vee$ is a nilpotent orbit of $G^\vee$. Assume $\mathcal{O}^\vee$ is even ($\lambda(\mathcal{O}^\vee)$ is in the root lattice;

Weak Arthur Packets

Definition (weak Arthur packet):

Suppose $\mathcal{O}^\vee$ is a nilpotent orbit of $G^\vee$. Assume $\mathcal{O}^\vee$ is even ($\lambda(\mathcal{O}^\vee)$ is in the root lattice; every node in the Dynkin diagram of $\mathcal{O}^\vee$ has label $0, 2$).

Weak Arthur Packets

Definition (weak Arthur packet):

Suppose $\mathcal{O}^\vee$ is a nilpotent orbit of $G^\vee$. Assume $\mathcal{O}^\vee$ is even ($\lambda(\mathcal{O}^\vee)$ is in the root lattice; every node in the Dynkin diagram of $\mathcal{O}^\vee$ has label $0, 2$).Let $\mathcal{O}$ be the (special) dual G-orbit (via Lusztig-Spaltenstein duality of nilpotent orbits).

Weak Arthur Packets

Definition (weak Arthur packet):

Suppose $\mathcal{O}^\vee$ is a nilpotent orbit of $G^\vee$. Assume $\mathcal{O}^\vee$ is even ($\lambda(\mathcal{O}^\vee)$ is in the root lattice; every node in the Dynkin diagram of $\mathcal{O}^\vee$ has label $0, 2$).Let $\mathcal{O}$ be the (special) dual G-orbit (via Lusztig-Spaltenstein duality of nilpotent orbits).

Then $\Pi(\mathcal{O}^\vee)$ consists of the irreducible representations π of $G(\mathbb{R})$ satisfying:

Weak Arthur Packets

Definition (weak Arthur packet):

Suppose $\mathcal{O}^\vee$ is a nilpotent orbit of $G^\vee$. Assume $\mathcal{O}^\vee$ is even ($\lambda(\mathcal{O}^\vee)$ is in the root lattice; every node in the Dynkin diagram of $\mathcal{O}^\vee$ has label $0, 2$).Let $\mathcal{O}$ be the (special) dual G-orbit (via Lusztig-Spaltenstein duality of nilpotent orbits).

Then $\Pi(\mathcal{O}^\vee)$ consists of the irreducible representations π of $G(\mathbb{R})$ satisfying:

(a) the infinitesimal character of π is $\lambda(\mathcal{O}^\vee) \in \mathfrak{h}^*$

Weak Arthur Packets

Definition (weak Arthur packet):

Suppose $\mathcal{O}^\vee$ is a nilpotent orbit of $G^\vee$. Assume $\mathcal{O}^\vee$ is even ($\lambda(\mathcal{O}^\vee)$ is in the root lattice; every node in the Dynkin diagram of $\mathcal{O}^\vee$ has label $0, 2$).Let $\mathcal{O}$ be the (special) dual G-orbit (via Lusztig-Spaltenstein duality of nilpotent orbits).

Then $\Pi(\mathcal{O}^\vee)$ consists of the irreducible representations π of $G(\mathbb{R})$ satisfying:

(a) the infinitesimal character of π is $\lambda(\mathcal{O}^\vee) \in \mathfrak{h}^*$

(b) $\mathrm{AV}_{\mathrm{ann}}(\pi) = \overline{\mathcal{O}}$

Weak Arthur Packets

Definition (weak Arthur packet):

Suppose $\mathcal{O}^\vee$ is a nilpotent orbit of $G^\vee$. Assume $\mathcal{O}^\vee$ is even ($\lambda(\mathcal{O}^\vee)$ is in the root lattice; every node in the Dynkin diagram of $\mathcal{O}^\vee$ has label $0, 2$).Let $\mathcal{O}$ be the (special) dual G-orbit (via Lusztig-Spaltenstein duality of nilpotent orbits).

Then $\Pi(\mathcal{O}^\vee)$ consists of the irreducible representations π of $G(\mathbb{R})$ satisfying:

(a) the infinitesimal character of π is $\lambda(\mathcal{O}^\vee) \in \mathfrak{h}^*$

(b) $\mathrm{AV}_{\mathrm{ann}}(\pi) = \overline{\mathcal{O}}$

This is a weak Arthur packet of special unipotent representations

Weak Arthur Packets

Definition (weak Arthur packet):

Suppose $\mathcal{O}^\vee$ is a nilpotent orbit of $G^\vee$. Assume $\mathcal{O}^\vee$ is even ($\lambda(\mathcal{O}^\vee)$ is in the root lattice; every node in the Dynkin diagram of $\mathcal{O}^\vee$ has label $0, 2$).Let $\mathcal{O}$ be the (special) dual G-orbit (via Lusztig-Spaltenstein duality of nilpotent orbits).

Then $\Pi(\mathcal{O}^\vee)$ consists of the irreducible representations π of $G(\mathbb{R})$ satisfying:

(a) the infinitesimal character of π is $\lambda(\mathcal{O}^\vee) \in \mathfrak{h}^*$

(b) $\mathrm{AV}_{\mathrm{ann}}(\pi) = \overline{\mathcal{O}}$

This is a weak Arthur packet of special unipotent representations

Weak Arthur Packets

Note: If $\mathcal{O}^\vee$ is not even, the construction involves

$$H^\vee = \mathrm{Cent}_{G^\vee}(e^{2\pi i \lambda(\mathcal{O}^\vee)})^0 \subset G^\vee$$

.

Weak Arthur Packets

Note: If $\mathcal{O}^\vee$ is not even, the construction involves

$$H^\vee = \mathrm{Cent}_{G^\vee}(e^{2\pi i\lambda(\mathcal{O}^\vee)})^0 \subset G^\vee$$

.

For simplicity stick to even orbits for this talk.

Weak Arthur Packets

Note: If $\mathcal{O}^\vee$ is not even, the construction involves

$$H^\vee = \mathrm{Cent}_{G^\vee}(e^{2\pi i\lambda(\mathcal{O}^\vee)})^0 \subset G^\vee$$

.

For simplicity stick to even orbits for this talk.

Note: The definition of honest Arthur packets involves the element $\Psi(j)$, and the $G(\mathbb{R})$ orbits in $\mathcal{O} \cap \mathfrak{g}_0$ (equivalently: the $K(\mathbb{C})$ orbits in $\mathcal{O} \cap \mathfrak{g}^{-\theta}$). There is a defintion in [ABV], and now an explicit algorithm, but the implementation is more difficult.

Computational Ingredients

Computational Ingredients

1) Explicit Langlands classification

Computational Ingredients

1) Explicit Langlands classification

Fix a regular infinitesimal character λ:

Computational Ingredients

1) Explicit Langlands classification

Fix a regular infinitesimal character λ:

$M_\lambda = \mathbb{Z}$-span of the irreducible representations with inf. char. λ

(finite dimensional)

Computational Ingredients

1) Explicit Langlands classification

Fix a regular infinitesimal character λ:

$M_\lambda = \mathbb{Z}$-span of the irreducible representations with inf. char. λ

(finite dimensional)

An explicit parameter space S_λ:

$$M_\lambda = \mathbb{Z}\langle\{J(\gamma) \mid \gamma \in S_\lambda\}\rangle$$

Computational Ingredients

1) Explicit Langlands classification

Fix a regular infinitesimal character λ:

$M_\lambda = \mathbb{Z}$-span of the irreducible representations with inf. char. λ

(finite dimensional)

An explicit parameter space S_λ:

$$M_\lambda = \mathbb{Z}\langle\{J(\gamma) \mid \gamma \in S_\lambda\}\rangle$$

$J(\gamma)$ is irreducible;

Computational Ingredients

1) Explicit Langlands classification

Fix a regular infinitesimal character λ:

$M_\lambda = \mathbb{Z}$-span of the irreducible representations with inf. char. λ

(finite dimensional)

An explicit parameter space S_λ:

$$M_\lambda = \mathbb{Z}\langle\{J(\gamma) \mid \gamma \in S_\lambda\}\rangle$$

$J(\gamma)$ is irreducible; the unique irreducible quotient of a standard module $I(\gamma)$, and

Computational Ingredients

1) Explicit Langlands classification

Fix a regular infinitesimal character λ:

$M_\lambda = \mathbb{Z}$-span of the irreducible representations with inf. char. λ

(finite dimensional)

An explicit parameter space S_λ:

$$M_\lambda = \mathbb{Z}\langle\{J(\gamma) \mid \gamma \in S_\lambda\}\rangle$$

$J(\gamma)$ is irreducible; the unique irreducible quotient of a standard module $I(\gamma)$, and

$$M_\lambda = \mathbb{Z}\langle\{I(\gamma) \mid \gamma \in S_\lambda\}\rangle$$

Computational Ingredients

1) Explicit Langlands classification

Fix a regular infinitesimal character λ:

$M_\lambda = \mathbb{Z}$-span of the irreducible representations with inf. char. λ

(finite dimensional)

An explicit parameter space S_λ:

$$M_\lambda = \mathbb{Z}\langle\{J(\gamma) \mid \gamma \in S_\lambda\}\rangle$$

$J(\gamma)$ is irreducible; the unique irreducible quotient of a standard module $I(\gamma)$, and

$$M_\lambda = \mathbb{Z}\langle\{I(\gamma) \mid \gamma \in S_\lambda\}\rangle$$

2) Kazhdan-Lusztig-Vogan polynomials (change of basis matrix for M_γ for the $J(\gamma)$ and $I(\gamma)$ bases)

Computational Ingredients

3) Coherent continuation (Zuckerman): M_λ is a representation of the Weyl group W. This is easily computed in the basis of standard modules $I(\gamma)$: s_α takes $I(\gamma)$ to a sum of a small number of other $I(\gamma')$, with small integer coefficients.
The KLV polynomials then give this action in the bases of irreducibles $J(\gamma)$.

Computational Ingredients

3) Coherent continuation (Zuckerman): M_λ is a representation of the Weyl group W. This is easily computed in the basis of standard modules $I(\gamma)$: s_α takes $I(\gamma)$ to a sum of a small number of other $I(\gamma')$, with small integer coefficients.
The KLV polynomials then give this action in the bases of irreducibles $J(\gamma)$.

M_λ is the direct sum of cell representations (defined combinatorially using the KLV polynomials). Each cell $\mathcal{C}$ contains a unique special representation $Sp(\mathcal{C})$ of W with multiplicity one.

Computational Ingredients

3) Coherent continuation (Zuckerman): M_λ is a representation of the Weyl group W. This is easily computed in the basis of standard modules $I(\gamma)$: s_α takes $I(\gamma)$ to a sum of a small number of other $I(\gamma')$, with small integer coefficients.
The KLV polynomials then give this action in the bases of irreducibles $J(\gamma)$.

M_λ is the direct sum of cell representations (defined combinatorially using the KLV polynomials). Each cell $\mathcal{C}$ contains a unique special representation $Sp(\mathcal{C})$ of W with multiplicity one.

4) Parametrization of complex nilpotent orbits $\mathcal{N}/G$

Computational Ingredients

3) Coherent continuation (Zuckerman): M_λ is a representation of the Weyl group W. This is easily computed in the basis of standard modules $I(\gamma)$: s_α takes $I(\gamma)$ to a sum of a small number of other $I(\gamma')$, with small integer coefficients.
The KLV polynomials then give this action in the bases of irreducibles $J(\gamma)$.

M_λ is the direct sum of cell representations (defined combinatorially using the KLV polynomials). Each cell $\mathcal{C}$ contains a unique special representation $Sp(\mathcal{C})$ of W with multiplicity one.

4) Parametrization of complex nilpotent orbits $\mathcal{N}/G$

5) The Springer correspondence $\mathcal{N}/G \hookrightarrow \hat{W}$:

Computational Ingredients

3) Coherent continuation (Zuckerman): M_λ is a representation of the Weyl group W. This is easily computed in the basis of standard modules $I(\gamma)$: s_α takes $I(\gamma)$ to a sum of a small number of other $I(\gamma')$, with small integer coefficients.
The KLV polynomials then give this action in the bases of irreducibles $J(\gamma)$.

M_λ is the direct sum of cell representations (defined combinatorially using the KLV polynomials). Each cell $\mathcal{C}$ contains a unique special representation $Sp(\mathcal{C})$ of W with multiplicity one.

4) Parametrization of complex nilpotent orbits $\mathcal{N}/G$

5) The Springer correspondence $\mathcal{N}/G \hookrightarrow \hat{W} : \mathcal{O} \mapsto \text{Springer}(\mathcal{O})$

Algorithm

Algorithm

Fix $G(\mathbb{C})$, $G(\mathbb{R})$.

Algorithm

Fix $G(\mathbb{C})$, $G(\mathbb{R})$. For simplicity assume $G(\mathbb{C})$ is simply connected, and $G(\mathbb{R})$ is inner to a split group.

Algorithm

Fix $G(\mathbb{C})$, $G(\mathbb{R})$. For simplicity assume $G(\mathbb{C})$ is simply connected, and $G(\mathbb{R})$ is inner to a split group.

1) Compute the character table of $W = W(G(\mathbb{C}))$.

Algorithm

Fix $G(\mathbb{C})$, $G(\mathbb{R})$. For simplicity assume $G(\mathbb{C})$ is simply connected, and $G(\mathbb{R})$ is inner to a split group.

1) Compute the character table of $W = W(G(\mathbb{C}))$.

2) Explicitly compute S_ρ, M_ρ.

Algorithm

Fix $G(\mathbb{C})$, $G(\mathbb{R})$. For simplicity assume $G(\mathbb{C})$ is simply connected, and $G(\mathbb{R})$ is inner to a split group.

1) Compute the character table of $W = W(G(\mathbb{C}))$.

2) Explicitly compute S_ρ, M_ρ.

3) Compute the representation of W on M_ρ, and its decomposition into cells.

Algorithm

Fix $G(\mathbb{C})$, $G(\mathbb{R})$. For simplicity assume $G(\mathbb{C})$ is simply connected, and $G(\mathbb{R})$ is inner to a split group.

1) Compute the character table of $W = W(G(\mathbb{C}))$.

2) Explicitly compute S_ρ, M_ρ.

3) Compute the representation of W on M_ρ, and its decomposition into cells.

4) For each cell $\mathcal{C}$ compute the special representation $Sp(\mathcal{C})$.

Algorithm

Algorithm

Now: fix a complex even nilpotent orbit $\mathcal{O}^\vee$
Initialize an empty list of parameters $WP(\mathcal{O}^\vee)$.

Algorithm

Now: fix a complex even nilpotent orbit $\mathcal{O}^\vee$
Initialize an empty list of parameters $WP(\mathcal{O}^\vee)$.

5) Let $\sigma = \text{Springer}(\mathcal{O}^\vee) \otimes \text{sgn}$

ALGORITHM

Now: fix a complex even nilpotent orbit $\mathcal{O}^\vee$
Initialize an empty list of parameters $WP(\mathcal{O}^\vee)$.

5) Let $\sigma = \text{Springer}(\mathcal{O}^\vee) \otimes \text{sgn}$

6) Run over the cells $\mathcal{C}$. For each cell $\mathcal{C}$ check if $Sp(\mathcal{C}) = \sigma$. If not, ignore it. If so: apply translation to all of the irreducible representation $J(\gamma) \in \mathcal{C}$, to translate from ρ to $\lambda(\mathcal{O}^\vee)$ (which is dominant, but usually singular). This operation takes an irreducible to irreducible or 0. Add the non-zero terms to $WP(\mathcal{O}^\vee)$.

Algorithm

Now: fix a complex even nilpotent orbit $\mathcal{O}^\vee$
Initialize an empty list of parameters $WP(\mathcal{O}^\vee)$.

5) Let $\sigma = \text{Springer}(\mathcal{O}^\vee) \otimes \text{sgn}$

6) Run over the cells $\mathcal{C}$. For each cell $\mathcal{C}$ check if $Sp(\mathcal{C}) = \sigma$. If not, ignore it. If so: apply translation to all of the irreducible representation $J(\gamma) \in \mathcal{C}$, to translate from ρ to $\lambda(\mathcal{O}^\vee)$ (which is dominant, but usually singular). This operation takes an irreducible to irreducible or 0. Add the non-zero terms to $WP(\mathcal{O}^\vee)$.

7) After running over all cells $WP(\mathcal{O}^\vee)$ is the weak packet defined by $\mathcal{O}^\vee$.

Classical groups

For classical groups the weak packets (and more) were computed, at least in small examples, by Jonathan Fernandes (UMD thesis, 2019).

```
atlas> show_nilpotent_orbits(G2_s)
i  H        diagram  dim  BC Levi   Cent  A(O)
0  [0,0]    [0,0]    0    2T1       G2    [1]
1  [1,2]    [0,1]    6    A1+T1     A1    [1]
2  [2,3]    [1,0]    8    A1+T1     A1    [1]
3  [2,4]    [0,2]    10   G2        e     [1,2,3]
4  [6,10]   [2,2]    12   G2        e     [1]

B=block/C=Cell
orbit   B  C  parameters                            inf. char.
0       0  0  (x=9,lambda=[1,1]/1,nu=[0,0]/1)       [ 0, 0 ]/1
0       1  0  (x=0,lambda=[0,0]/1,nu=[0,0]/1)       [ 0, 0 ]/1
1       0  1  (x=9,lambda=[1,1]/1,nu=[1,0]/2)       [ 1, 0 ]/2
1       0  2  (x=9,lambda=[2,1]/1,nu=[1,0]/2)       [ 1, 0 ]/2
2       0  1  (x=9,lambda=[1,2]/1,nu=[0,1]/2)       [ 0, 1 ]/2
2       0  2  (x=9,lambda=[1,1]/1,nu=[0,1]/2)       [ 0, 1 ]/2
3       0  1  (x=4,lambda=[1,0]/1,nu=[2,-1]/2)      [ 1, 0 ]/1
3       0  1  (x=8,lambda=[3,0]/1,nu=[1,0]/1)       [ 1, 0 ]/1
3       0  2  (x=2,lambda=[1,0]/1,nu=[0,0]/1)       [ 1, 0 ]/1
3       0  2  (x=6,lambda=[4,-1]/1,nu=[3,-1]/2)     [ 1, 0 ]/1
3       0  2  (x=9,lambda=[1,1]/1,nu=[1,0]/1)       [ 1, 0 ]/1
4       0  3  (x=9,lambda=[1,1]/1,nu=[1,1]/1)       [ 1, 1 ]/1
```

Some Unipotent Representations

Group	#orbs.	#unip.	#unip by orbit
$SL(2,\mathbb{R})$	2	4	3,1
$SU(2)$	1	1	
$Sp(4,\mathbb{R})$	4	16	5,2,8,1
$Sp(1,1)$	3	3	1,1,1
$Sp(2)$	1	1	
$Sp(6,\mathbb{R})$	7	47	7,6,16,2,7,8,1
G_2 (split)	5	12	2,2,2,5,1
G_2 (cpt)	1	1	
F_4 (split)	16	75	3,4,10,5,2,4,2,7,2,8,14,2,4,3,4,1
$F_4(B_4)$	3	3	1,1,1
F_4 (cpt)	1	1	
E_6 (split)	21	68	3,4,5,4,7,4,2,3,4,2,4,7,3,2,2,2,2,4,2,1,1
$E_6(F_4)$	3	3	1,1,1

Some Unipotent Representations

Group	#orbs.	#unip.	#unip by orbit
E_7 (split)	45	252	6,7,8,3,8,16,4,12,9,7,6,2,3,6,8,
			17,4,4,7,7,12,2,3,2,8,8,5,6,4,2,
			12,2,4,5,2,5,2,4,8,2,3,4,1,1,1
$E_7(E_6 T)$	13	28	3,1,1,5,1,2,2,1,5,1,3,2,1
$E_7(D_6 A_1 T)$	25	56	2,2,2,1,3,2,4,1,1,3,3,2
			1,2,3,7,1,1,2,1,2,6,2,1,1
E_7 (compact)	1	1	

E8

E_8 has 70 unipotent orbits, of which 27 are even.

E8

E_8 has 70 unipotent orbits, of which 27 are even.

Calculating the KLV polynomials for E_8 now takes just under three hours.

E8

E_8 has 70 unipotent orbits, of which 27 are even.

Calculating the KLV polynomials for E_8 now takes just under three hours.

Storing the KLV polynomials requires about 100 gigabytes of storage.

E8

E_8 has 70 unipotent orbits, of which 27 are even.

Calculating the KLV polynomials for E_8 now takes just under three hours.

Storing the KLV polynomials requires about 100 gigabytes of storage. However storing the information needed to compute the cell representations only requires 42 megabytes.

E8

E_8 has 70 unipotent orbits, of which 27 are even.

Calculating the KLV polynomials for E_8 now takes just under three hours.

Storing the KLV polynomials requires about 100 gigabytes of storage. However storing the information needed to compute the cell representations only requires 42 megabytes.

There are 104 cells sizes (average size: 4,356):
1,8,35,196,196,260,560,260,560,560,1100,567,3752,1100,4025,3240,3192,1100,
2625,3240,3240,3240,3240,3640,3240,3640,8192,3640,7560,3240,8192,5040,
4536,4536,4536,525,3500,6075,7560,2835,4536,8800,3500,6075,6075,4200,
4200,8800,46676,22778,4200,4200,38766,4200,2100,8800,4536,4200,8800,4200,
6075,6075,2835,4536,4536,4200,7560,4536,3640,6075,7560,5040,3500,8192,
3640,3240,3240,3240,1100,3500,8192,3640,3240,3240,525,3240,3240,4025,
3752,2625,3192,1100,1100,560,567,560,560,260,196,260,196,35,8,1

E8

We need to compute the character of a representation of each of these dimensions, up to 46,676. That is: multiply up to 120 matrices of size 46,676 (and take the trace). Each matrix is sparse (the action of a simple reflection). So: multiply an arbitrary matrix * sparse matrix.

E8

We need to compute the character of a representation of each of these dimensions, up to 46,676. That is: multiply up to 120 matrices of size 46,676 (and take the trace). Each matrix is sparse (the action of a simple reflection). So: multiply an arbitrary matrix * sparse matrix.

Example:

E8

We need to compute the character of a representation of each of these dimensions, up to 46,676. That is: multiply up to 120 matrices of size 46,676 (and take the trace). Each matrix is sparse (the action of a simple reflection). So: multiply an arbitrary matrix * sparse matrix.

Example:

E_8(split): 27 even orbits $\mapsto$ 112 unipotent representations

We need to compute the character of a representation of each of these dimensions, up to 46,676. That is: multiply up to 120 matrices of size 46,676 (and take the trace). Each matrix is sparse (the action of a simple reflection). So: multiply an arbitrary matrix * sparse matrix.

Example:

E_8(split): 27 even orbits $\mapsto$ 112 unipotent representations

1,4,6,5,1,4,6,4,4,4,18,1,4,4,4,1,6,5,5,5,6,3,4,4,1,1,1

Thank you