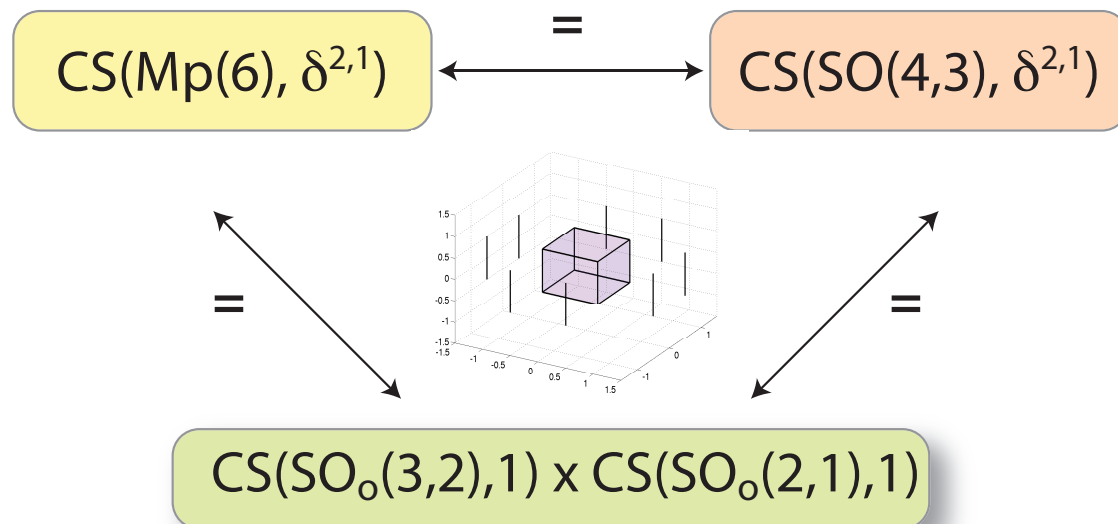


Complementary Series of Split Real Groups

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joint with Annegret Paul and Susana Salamanca-Riba

(some of the techniques used are joint work with D. Barbasch)

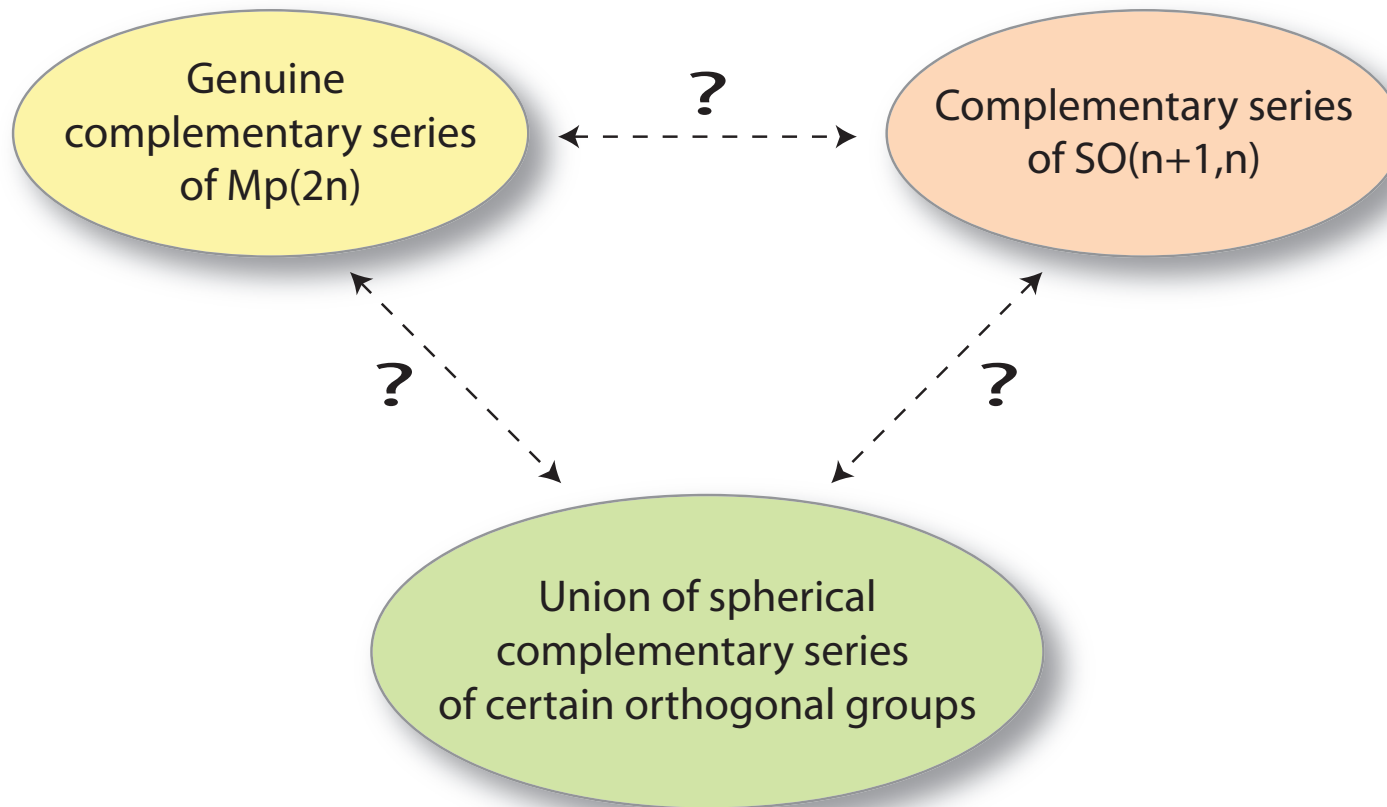


Salt Lake City, July 2009

Introduction

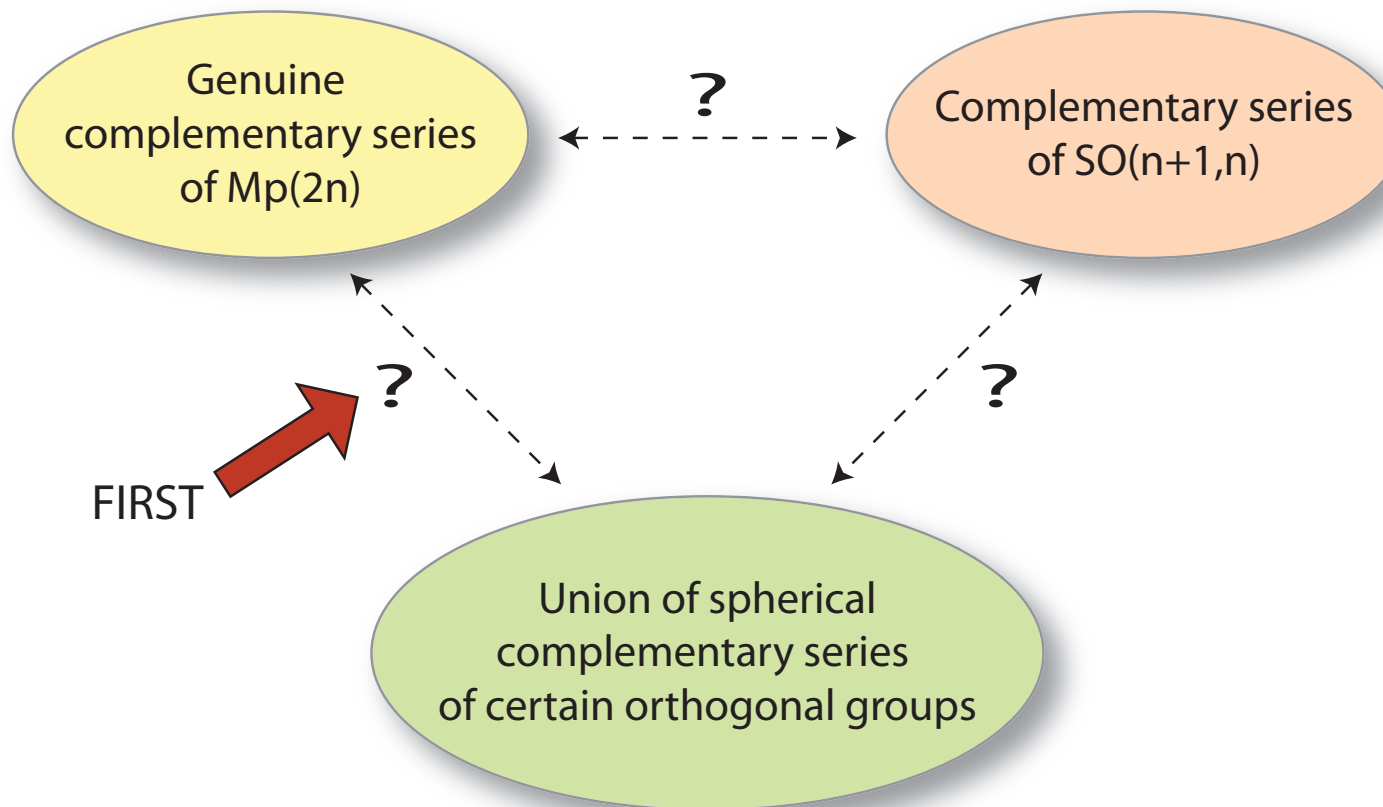
Aim

Discuss the unitarity of minimal principal series of $Mp(2n)$ and $SO(n+1, n)$.



PART 1

Genuine Complementary Series of $Mp(2n)$



NOTATION

- $G := Mp(2n)$ the **connected double cover** of $Sp(2n, \mathbb{R})$
- $K := \tilde{U}(n)$ the **maximal compact** subgroup of G
 $= \{[g, z] \in U(n) \times U(1) : \det(g) = z^2\}$
- $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$
- $\mathfrak{a}_0 :=$ maximal abelian subspace of \mathfrak{p}_0
- $M := Z_K(\mathfrak{a}_0)$
- $\Delta(\mathfrak{g}_0, \mathfrak{a}_0) = \{\pm\epsilon_k \pm \epsilon_l\}_{k,l=1\dots n} \cup \{\pm 2\epsilon_k\}_{k=1\dots n}$ *type C_n*
- $W \simeq S_n \times (\mathbb{Z}/2\mathbb{Z})^n$ *all permutations and sign changes*

The group M and its genuine representations

$M = Z_K(\mathfrak{a}_0)$ subgroup of K generated by the elements

$$m_k = \left[\text{diag}(1, \dots, 1, \underset{k}{-1}, 1, \dots, 1), i \right], \quad k = 1 \dots n \quad (\text{of order } 4)$$

Genuine M -types Irreducible repr.s δ of M s.t. $\delta([I, -1]) \neq +1$.



Subsets $S \subset \{1 \dots n\}$ $m_k^2 = [I, -1] \rightarrow$ each generator m_k acts by $\pm i$
 S keeps track of which generators act by $-i$

$$\delta_S(m_k) = \begin{cases} -i & \text{if } k \in S \\ +i & \text{otherwise} \end{cases}$$

$Mp(6)$	m_1	m_2	m_3
$\delta_{\{2,3\}}$	$+i$	$-i$	$-i$

An action of the Weyl group on genuine M -types

W acts on $\widehat{M} \leftarrow (s_\alpha \cdot \delta)(m) := \delta(\sigma_\alpha^{-1} m \sigma_\alpha) \quad \forall m \in M, \forall \alpha \in \Delta$

The stabilizer of δ in W is $W^\delta := \{w \in W : w \cdot \delta \simeq \delta\}$.

For all $S \subset \{1, \dots, n\}$, set $q = |S|$, $p = |S^c|$.

- $W^{\delta_S} \simeq W(C_p) \times W(C_q) \leftarrow s_{2\epsilon_k} \ \& \ s_{\epsilon_k \pm \epsilon_l}, \ k, l \text{ in } S \text{ or } S^c$
- $W \cdot \delta_S = \{\delta_T : |T| = q, |T^c| = p\}$

W -orbits of genuine M -types \leftrightarrow pairs $(p, q) : p, q \in \mathbb{N}, p + q = n$

Pick representatives $\delta^{p,q} := \delta_{\{p+1, \dots, n\}} \cdot \delta^{p,q}(m_k) = \begin{cases} +i & \text{if } k \leq p \\ -i & \text{if } k > p. \end{cases}$

The group K and its genuine representations

Maximal compact subgroup of G :

$$K = \tilde{U}(n)$$

Genuine K -types

parameterized by highest weight (a_1, \dots, a_n)
with $a_1 \geq a_2 \geq \dots \geq a_n$ and $a_j \in \mathbb{Z} + \frac{1}{2}, \forall j$

fine K -types	highest weight	restriction to M
$\Lambda^p(\mathbb{C}^n) \otimes \det^{-1/2}$	$(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q)$	$W \cdot \delta^{p,q}$

- If we restrict a *fine* K -type to M , we get *one full* W -orbit in \widehat{M}
- Each genuine M -type δ is contained in a *unique fine* K -type μ_δ .

Genuine Complementary Series of $Mp(2n)$

- $MA :=$ Levi factor of a minimal parabolic
- $\delta :=$ genuine irreducible representation of M
- $\nu :=$ real character of A
- $P = MAN :=$ a minimal parabolic making ν weakly dominant.

Minimal Principal Series $I_P(\delta, \nu) := \text{Ind}_P^G (\delta \otimes \nu \otimes 1)$

Langlands Quotient $J(\delta, \nu) :=$ composition factor of $I_P(\delta, \nu) \supseteq \mu_\delta$

δ -Complementary Series $CS(G, \delta) := \{\nu \in \mathfrak{a}_{\mathbb{R}}^* \mid J(\delta, \nu) \text{ is unitary}\}$

Problem: Find $CS(Mp(2n), \delta^{p,q})$

THEOREM 1

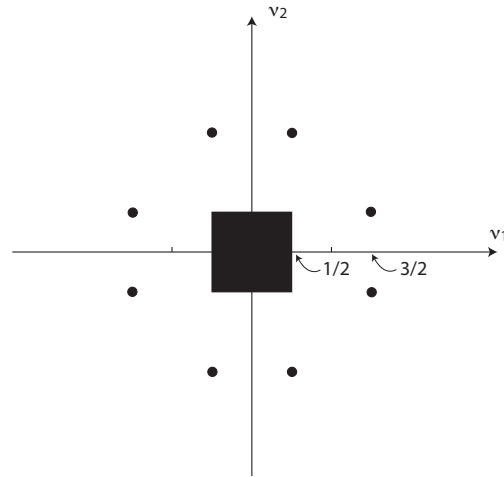
Theorem 1: For all $\nu \in \mathfrak{a}_{\mathbb{R}}^*$, write $\nu := (\nu^p | \nu^q)$. The map:

$$\begin{aligned} CS(Mp(2n), \delta^{p,q}) &\rightarrow CS(SO(p+1, p)_0, 1) \times CS(SO(q+1, q)_0, 1) \\ \nu &\mapsto (\nu^p, \nu^q) \end{aligned}$$

is a well defined injection. (1 denotes the trivial M -type)

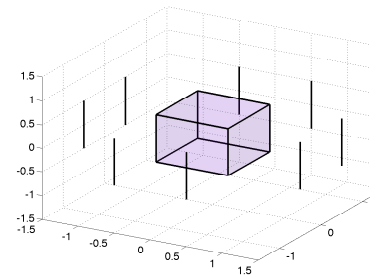
Spherical complementary series of real split orthogonal groups are known (Barbasch). Hence this theorem provides explicit necessary conditions for the unitarity of genuine principal series of $Mp(2n)$.

Example: $CS(Mp(6), \delta^{2,1}) \rightarrow CS(SO(3, 2)_{0, 1}) \times CS(SO(2, 1)_{0, 1})$



$CS(SO(3, 2)_{0, 1})$

$CS(SO(2, 1)_{0, 1})$



$\Rightarrow CS(Mp(6), \delta^{2,1})$ embeds into:

A reformulation of THEOREM 1

For all $p, q \in \mathbb{N}$ s.t. $p + q = n$, set:

$$G^{\delta^{p,q}} \equiv SO(p+1, p)_0 \times SO(q+1, q)_0$$

and note that $W(G^{\delta^{p,q}}) = W^{\delta^{p,q}}$.

$G^{\delta^{p,q}}$:= *connected real split group whose root system is dual to the system of good roots for $\delta^{p,q}$.*

Theorem 1: The $\delta^{p,q}$ -complementary series of $Mp(2n)$ embeds into the spherical complementary series of $G^{\delta^{p,q}}$.

Proof: based on Barbasch's idea to use calculations on petite K-types to compare unitary parameters for different groups.

Comparing unitary parameters for $Mp(2n)$ and $G^{\delta^{p,q}}$

$J(\delta^{p,q}, \nu)$ unitary for $Mp(2n)$



$T(\mu, \delta^{p,q}, \nu)$
 pos. semidefinite
 $\forall \mu \in \widehat{K}$

$J(1, \nu)$ unitary for $G^{\delta^{p,q}}$



$A(\psi, 1, \nu)$
 pos. semidefinite
 $\forall \psi \in \widehat{W^{\delta^{p,q}}}$

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$A(\psi, 1, \nu)$
 pos. semidefinite
 $\forall \psi \in \widehat{W^{\delta^{p,q}}}$ *relevant*

A matching of operators

Key Proposition:

\forall relevant $W^{\delta^{p,q}}$ -type ψ , \exists a “petite” K -type μ s.t.

$$\underbrace{T(\mu, \delta^{p,q}, \nu)}_{\text{operator for } Mp(2n)} = \underbrace{A(\psi, 1, \nu)}_{\text{operator for } G^{\delta^{p,q}}}$$

Sketch of the proof:

- $T(\mu, \delta^{p,q}, \nu)$ is defined on $\text{Hom}_M(\mu, \delta^{p,q})$
- This space carries a representation ψ_μ of $W^{\delta^{p,q}} \leftarrow = W(G^{\delta^{p,q}})$
- Attached to ψ_μ , \exists a spherical operator $A(\psi_\mu, 1, \nu)$ for $G^{\delta^{p,q}}$
- If μ is petite, $T(\mu, \delta^{p,q}, \nu) = A(\psi_\mu, 1, \nu)$
- For all $\psi \in \widehat{W^{\delta^{p,q}}}$ relevant, $\exists \mu \in \widehat{K}$ petite such that $\psi = \psi_\mu$. \square

A matching of relevant $W^{\delta^{p,q}}$ -types with petite K -types

$((p - s) \times (s)) \otimes triv$	$\left(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p-s}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q, \underbrace{-\frac{3}{2}, \dots, -\frac{3}{2}}_s \right)$
$(p - s, s) \otimes triv$	$\left(\underbrace{\frac{3}{2}, \dots, \frac{3}{2}}_s, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p-2s}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q+s} \right)$
$triv \otimes ((q - r) \times (r))$	$\left(\underbrace{\frac{3}{2}, \dots, \frac{3}{2}}_r, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q-r} \right)$
$triv \otimes (q - r, r)$	$\left(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p+r}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q-2r}, \underbrace{-\frac{3}{2}, \dots, -\frac{3}{2}}_r \right)$

$J(\delta^{p,q}, \nu)$ unitary for $Mp(2n)$

\Updownarrow

$T(\mu, \delta^{p,q}, \nu)$
pos. semidefinite
 $\forall \mu \in \widehat{K}$

\Downarrow

$T(\mu, \delta^{p,q}, \nu)$
pos. semidefinite
 $\forall \mu \in \widehat{K}$ *petite*

$J(1, \nu)$ unitary for $G^{\delta^{p,q}}$

\Updownarrow

$A(\psi, 1, \nu)$
pos. semidefinite
 $\forall \psi \in \widehat{W^{\delta^{p,q}}}$

\Updownarrow

$A(\psi, 1, \nu)$
pos. semidefinite
 $\forall \psi \in \widehat{W^{\delta^{p,q}}}$ *relevant*

\Rightarrow

\uparrow

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$\forall \psi \in \widehat{W^{\delta^{p,q}}}$ *relevant*, $\exists \mu \in \widehat{K}$ *petite* s.t. $A(\psi, 1, \nu) = T(\mu, \delta^{p,q}, \nu)$

Non-unitarity certificates

Let $G^{\delta^{p,q}} = SO(p+1, p)_0 \times SO(q+1, q)_0$. For all $\nu = (\nu^p | \nu^q)$:

$J(\delta^{p,q}, \nu)$ unitary for $Mp(2n) \implies J(1, \nu)$ unitary for $G^{\delta^{p,q}}$.

*The spherical unitary dual of split orthogonal groups is known. So we get **non-unitarity certificates** for genuine L.Q.s of $Mp(2n)$.*

Theorem 1': If

- the spherical L.Q. $J(1, \nu^p)$ of $SO(p+1, p)_0$ is *not unitary*, or
- the spherical L.Q. $J(1, \nu^q)$ of $SO(q+1, q)_0$ is *not unitary*

then the genuine L.Q. $J(\delta^{p,q}, (\nu^p | \nu^q))$ of $Mp(2n)$ is also *not unitary*.

An example of non-unitarity certificate

Let $\nu = (\nu_1, \dots, \nu_n)$. We may assume:

$$\nu_1 \geq \dots \geq \nu_p \geq 0 \quad \text{and} \quad \nu_{p+1} \geq \dots \geq \nu_n \geq 0,$$

by $W^{\delta^{p,q}}$ -invariance. (Recall $W^{\delta^{p,q}} = W(C_p) \times W(C_q)$.)

If *any* of the following conditions holds:

- $\nu_p > 1/2$
- $\nu_n > 1/2$
- $\nu_a - \nu_{a+1} > 1$, for some a with $1 \leq a \leq p-1$, or
- $\nu_a - \nu_{a+1} > 1$, for some a with $p+1 \leq a \leq n-1$

then the genuine Langlands quotient $J(\delta^{p,q}, \nu)$ of $Mp(2n)$ is not unitary.

An application

This non-unitarity certificate is a key ingredient in the classification of the ω -regular unitary dual of $Mp(2n)$.

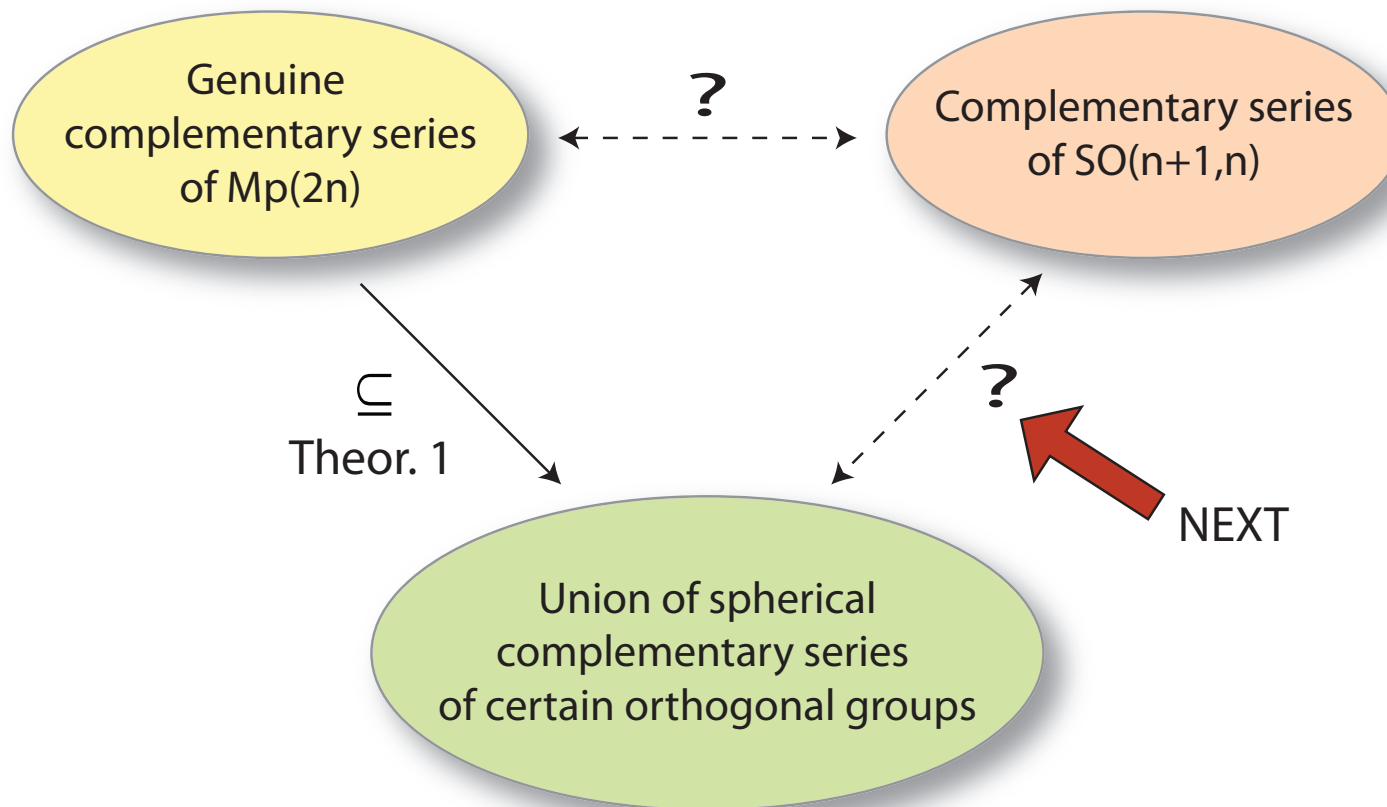
Definition: A representation of $Mp(2n)$ is called ω -regular if its infinitesimal character is at least as regular as the one of the oscillator representation.

Corollary: The only ω -regular complementary series repr.s of $Mp(2n)$ are the two even oscillator representations:

$$J\left(\delta_{0,n}, \left(n - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2}\right)\right) \quad \text{and} \quad J\left(\delta_{n,0}, \left(n - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2}\right)\right).$$

PART 2

Complementary Series of $SO(n+1, n)$



NOTATION

- $G := SO(n + 1, n)$
- $K := S(O(n + 1) \times O(n))$ *maximal compact*
- $\Delta(\mathfrak{g}_0, \mathfrak{a}_0) = \{\pm\epsilon_k \pm \epsilon_l\} \cup \{\pm\epsilon_k\}$ *type B_n* ← dual to previous case
- $W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ ← same Weyl group as before
- $M := Z_K(\mathfrak{a}_0) = \{\text{diag}(1, t_n, \dots, t_1, t_1, \dots, t_n) : t_j = \pm 1, \forall j\}$

M-types

M is generated by the elements

$$m_k = \text{diag}(1, \dots, 1, \underset{n+2-k}{-1}, 1, \dots, 1, \underset{n+1+k}{-1}, 1, \dots, 1)$$

$k = 1 \dots n$ (of order 2).

M -types

\Leftrightarrow

Subsets $S \subset \{1 \dots n\}$

\leftarrow

same parametrization
as before

The set S keeps track of which generators act by -1 :

$$\delta_S(m_k) = \begin{cases} -1 & \text{if } k \in S \\ +1 & \text{otherwise} \end{cases}$$

$SO(4, 3)$	m_1	m_2	m_3
$\delta_{\{2,3\}}$	+1	-1	-1

W-orbits of M-types

Just like before, we look at the action of W on \widehat{M} . Then

- $W^{\delta_S} \simeq W(B_p) \times W(B_q)$, for $q = |S|$, $p = |S^c|$ ←

same as
before

- $W \cdot \delta_S = \{\delta_T : |T| = q, |T^c| = p\}$

- W -orbits of M -types \leftrightarrow pairs $(p, q) : p, q \in \mathbb{N}, p + q = n$

↑

same parametrization as before

Pick representatives $\delta^{p,q} := \delta_{\{p+1, \dots, n\}}$. $\delta^{p,q}(m_k) = \begin{cases} +1 & \text{if } k \leq p \\ -1 & \text{if } k > p. \end{cases}$

K-types (*n even*)

$$K = S(O(n+1) \times O(n)), \quad n \text{ even}$$

$(a_1, \dots, a_{\frac{n}{2}}; b_1, \dots, b_{\frac{n}{2}})$ with $a_j, b_j \in \mathbb{Z}, \forall j$ and

K-types $a_1 \geq \dots \geq a_{\frac{n}{2}} \geq 0; b_1 \geq \dots \geq b_{\frac{n}{2}} \geq 0.$

If $b_{\frac{n}{2}} = 0$, there is also a sign $\epsilon = \pm 1$.

	Fine K-types	realization	res. to M
$q < \frac{n}{2}$	$(0, \dots, 0; \underbrace{1, \dots, 1}_q, 0, \dots, 0; +)$	$triv \otimes \Lambda^q \mathbb{C}^n$	$W \cdot \delta^{p,q}$
$q = \frac{n}{2}$	$(0, \dots, 0; 1, \dots, 1)$	$triv \otimes \Lambda^{\frac{n}{2}} \mathbb{C}^n$	$W \cdot \delta^{p,q}$
$q > \frac{n}{2}$	$(0, \dots, 0; \underbrace{1, \dots, 1}_{n-q}, 0, \dots, 0; -)$	$triv \otimes \Lambda^q \mathbb{C}^n$	$W \cdot \delta^{p,q}$

K-types (*n* odd)

$$K = S(O(n+1) \times O(n)), \quad n \text{ odd}$$

$(a_1, \dots, a_{\frac{n+1}{2}}; b_1, \dots, b_{\frac{n-1}{2}})$ with $a_j, b_j \in \mathbb{Z}, \forall j$ and

K-types $a_1 \geq \dots \geq a_{\frac{n+1}{2}} \geq 0; b_1 \geq \dots \geq b_{\frac{n-1}{2}} \geq 0.$

If $a_{\frac{n+1}{2}} = 0$, there is also a sign $\epsilon = \pm 1$.

	Fine K-types	realization	res. to M
$q < \frac{n}{2}$	$(0, \dots, 0; \underbrace{1, \dots, 1}_q, 0, \dots, 0; +)$	$triv \otimes \Lambda^q \mathbb{C}^n$	$W \cdot \delta^{p,q}$
$q > \frac{n}{2}$	$(0, \dots, 0; \underbrace{1, \dots, 1}_{n-q}, 0, \dots, 0; -)$	$triv \otimes \Lambda^q \mathbb{C}^n$	$W \cdot \delta^{p,q}$

Complementary Series of $SO(n + 1, n)$

- MA : Levi factor of a minimal parabolic
- $\delta \in \widehat{M}$
- $\nu \in \mathfrak{a}_{\mathbb{R}}^*$
- $P = MAN :=$ a minimal parabolic making ν weakly dominant.

Minimal Principal Series $I_P(\delta, \nu)$

Langlands Quotient $J(\delta, \nu)$

δ -Complementary Series $CS(SO(n + 1, n), \delta) = \{\nu \mid J(\delta, \nu) \text{ unitary}\}$

Problem: Find $CS(SO(n + 1, n), \delta^{p,q})$

THEOREM 2

Theorem 2: For all $\nu \in \mathfrak{a}_{\mathbb{R}}^*$, write $\nu := (\nu^p | \nu^q)$. The map:

$$CS(SO(n+1, n), \delta^{p,q}) \rightarrow CS(SO(p+1, p)_0, 1) \times CS(SO(q+1, q)_0, 1)$$
$$\nu \mapsto (\nu^p, \nu^q)$$

is a well defined injection. (1 denotes the trivial M -type.)

↑

same embedding as before

A reformulation of THEOREM 2

Set:

$$G^{\delta^{p,q}} \equiv SO(p+1, p)_0 \times SO(q+1, q)_0$$

←

same as
before

and note that $W(G^{\delta^{p,q}}) = W^{\delta^{p,q}}$.

$$G^{\delta^{p,q}} :=$$

*connected real split group whose root system is
equal to the system of good roots for $\delta^{p,q}$.*

Theorem 2: The $\delta^{p,q}$ -complementary series of $SO(n+1, n)$ embeds into the spherical complementary series of $G^{\delta^{p,q}}$.

Proof: based on a matching of *relevant* W -types for $G^{\delta^{p,q}}$ with *petite* K -types for $SO(n+1, n)$.

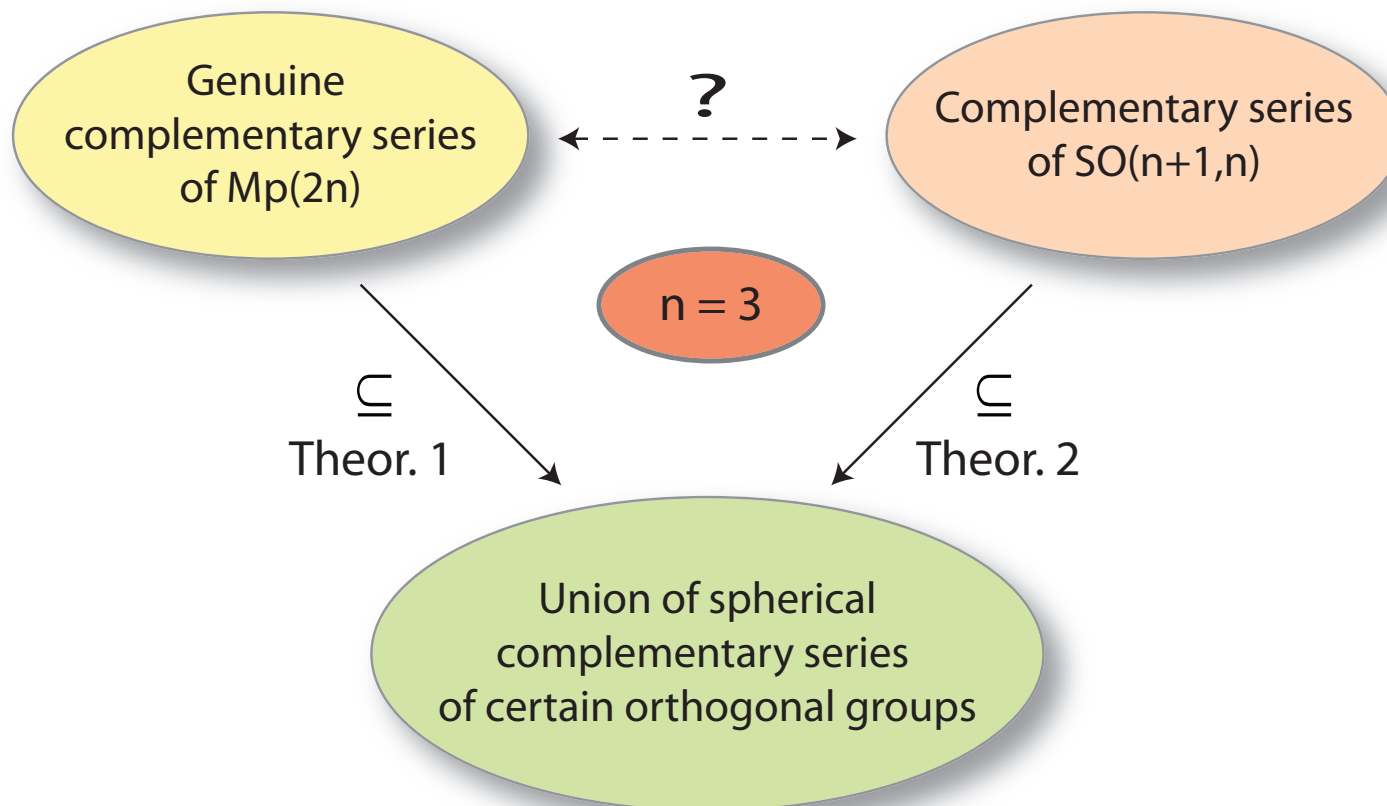
A matching of relevant $W^{\delta^{p,q}}$ -types with petite K -types

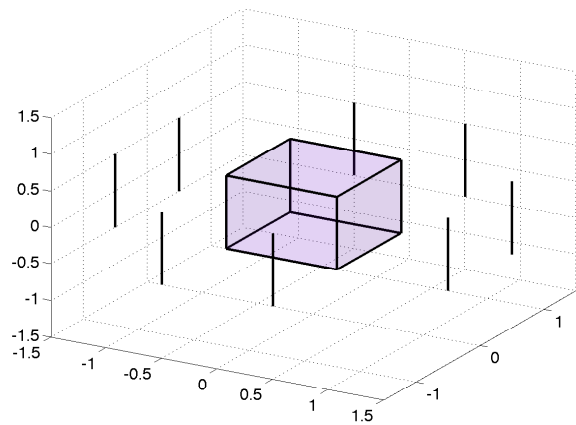
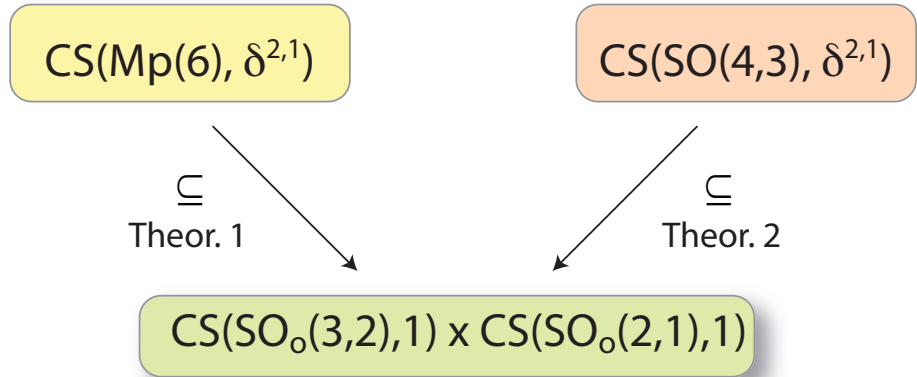
Recall that $W^{\delta^{p,q}} = W(B_p) \times W(B_q)$ and $K = S(O(n+1) \times O(n))$.

$((p-s) \times (s)) \otimes \text{triv}$	$\Lambda^s(\mathbb{C}^{n+1}) \otimes \Lambda^{q+s}(\mathbb{C}^n)$
$(p-s, s) \otimes \text{triv}$	an irreducible submodule of $\text{triv} \otimes [\Lambda^s(\mathbb{C}^n) \otimes \Lambda^{q+s}(\mathbb{C}^n)]$
$\text{triv} \otimes ((q-r) \times (r))$	$\Lambda^r(\mathbb{C}^{n+1}) \otimes \Lambda^{q-r}(\mathbb{C}^n)$
$\text{triv} \otimes (q-r, r)$	an irreducible submodule of $\text{triv} \otimes [\Lambda^r(\mathbb{C}^n) \otimes \Lambda^{q-r}(\mathbb{C}^n)]$

PART 3

An example: $n = 3$





Are these “proper containments” or “equalities”?

Are the L.Q.s $J_{Mp(6)}(\delta^{2,1}, \nu)$ and $J_{SO(4,3)}(\delta^{2,1}, \nu)$ unitary for *all* points ν of the unit cube and *all* points ν of the 8 line segments?

Unitarity of $J_{Mp(6)}(\delta^{2,1}, \nu)$ for ν in the unit cube

Theorem. The Langlands quotient $J(\delta, \nu)$ of $Mp(2n)$ is unitary for all ν in the unit cube $\{\underline{x} \in \mathfrak{a}_{\mathbb{R}}^* \mid 0 \leq |x_j| \leq 1/2, \forall j\}$.

Proof. Note that:

- For $\nu = 0$, all the operators $T(\mu, \delta, \nu)$ are positive definite.
- The signature of $T(\mu, \delta, \nu)$ can only change along the reducibility hyperplanes:

$$\begin{cases} \langle \nu, \beta \rangle \in 2\mathbb{Z} + 1 & \text{for some root } \beta \text{ which is } \textit{good} \text{ for } \delta \\ \langle \nu, \beta \rangle \in 2\mathbb{Z} \setminus \{0\} & \text{for a root } \beta \text{ which is } \textit{bad} \text{ for } \delta. \end{cases}$$

- Away from these hyperplanes, $I(\delta, \nu)$ is irreducible ($= J(\delta, \nu)$), and the operators $T(\mu, \delta, \nu)$ have constant signature. In particular, $J(\delta, \nu)$ is unitary throughout the unit cube. \square

Unitarity of $J_{Mp(6)}(\delta^{2,1}, \nu)$ for $\nu = (\frac{3}{2}, \frac{1}{2}|t)$, $t \in [0, \frac{1}{2}]$

Theorem. The repr. $J(\delta^{p,q}, \nu)$ of $Mp(2n)$ is unitary $\forall \nu = (\nu^p | \nu^q)$ s.t.

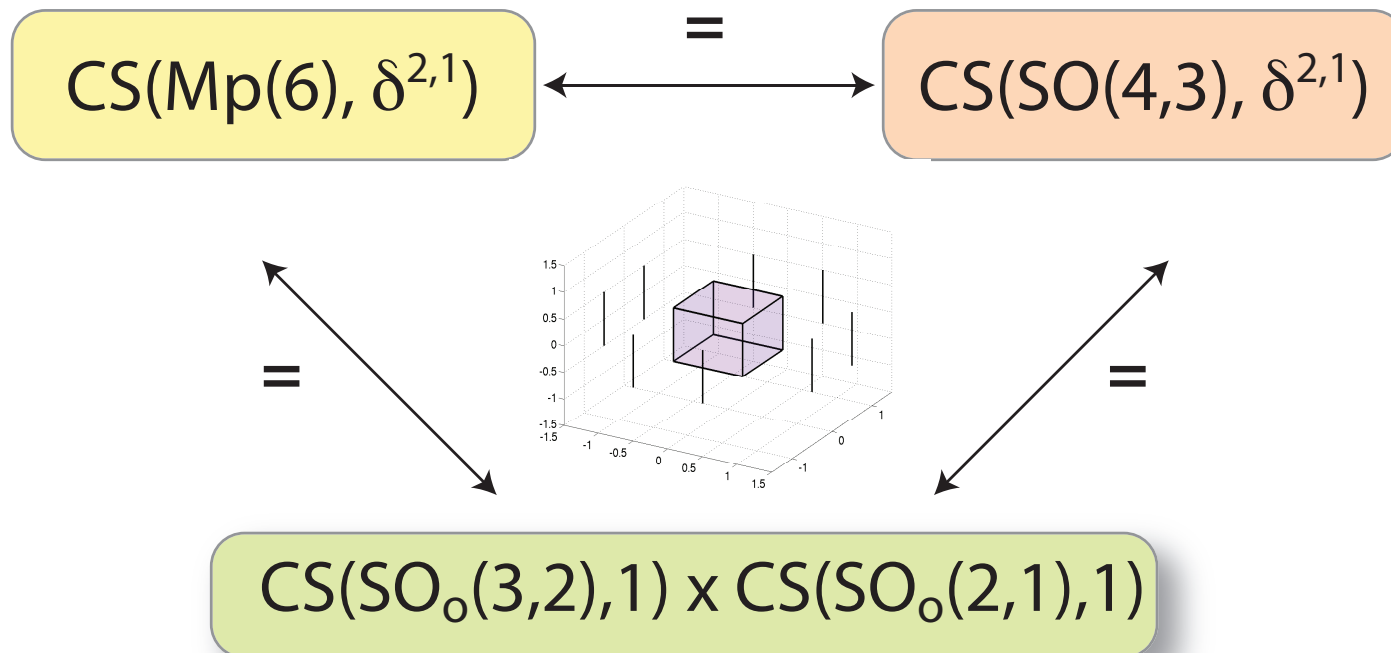
- $\nu^p \in CS(SO(p+1, p)_0, 1)$, with $0 \leq |a_j| \leq 3/2$ or $a_j \in \mathbb{Z} + \frac{1}{2}$
- $\nu^q \in CS(SO(q+1, q)_0, 1)$, with $0 \leq |a_j| \leq \frac{1}{2}$.

Proof. Let P_1 be a parabolic with $M_1 A_1 := Mp(2p) \times (\widetilde{GL}(1, \mathbb{R}))^q$.
By double induction, $J(\delta^{p,q}, \nu)$ is the Langlands quotient of

$$I(\nu^q) := \text{Ind}_{M_1 A_1 N_1}^{Mp(2n)} \left((J(\delta^{p,0}, \nu^p) \otimes \delta^{0,q}) \otimes \nu^q \otimes 1 \right).$$

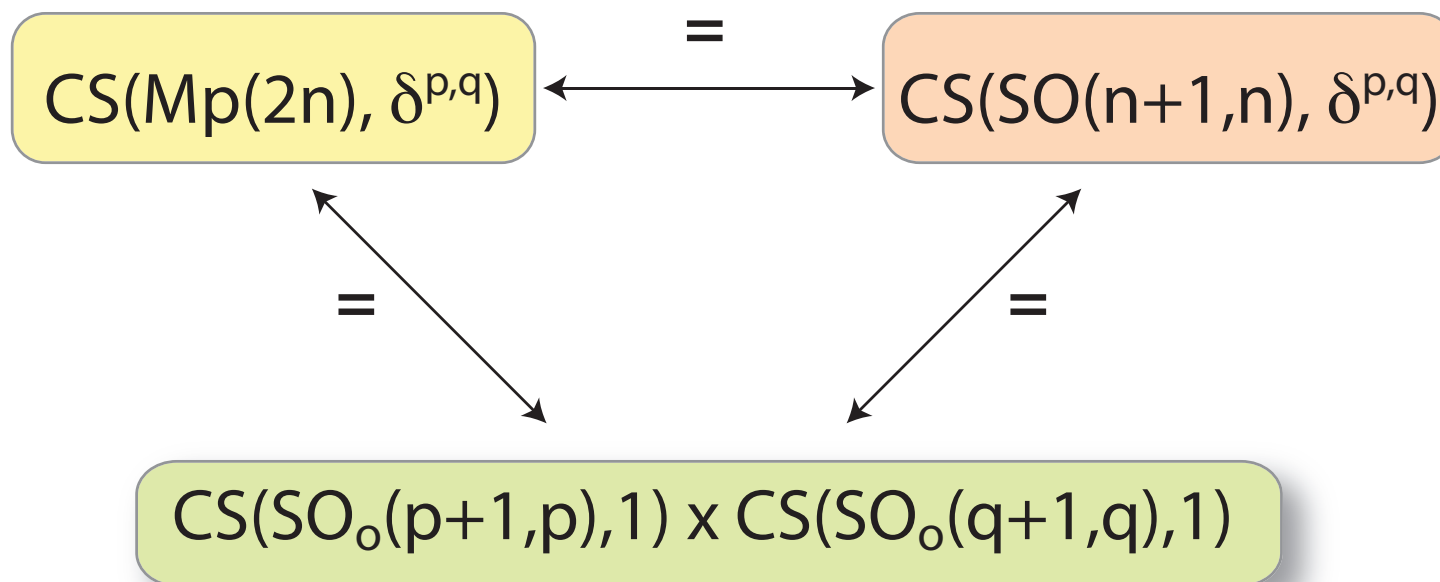
Here $J(\delta^{p,0}, \nu^p)$ is a pseudospherical repr. of $Mp(2p)$. By results of ABPTV, $J(\delta^{p,0}, \nu^p)$ is unitary $\forall \nu^p \in CS(SO(p+1, p)_0, 1)$. Then the repr. $I(\nu^q)$ of $Mp(2n)$ is unitary at $\nu^q=0$ (unitarily induced). For all ν of interest, $I(\nu^q)$ is irreducible, hence it stays unitary by the principle of unitary deformation. \square

Corollary



More generally...

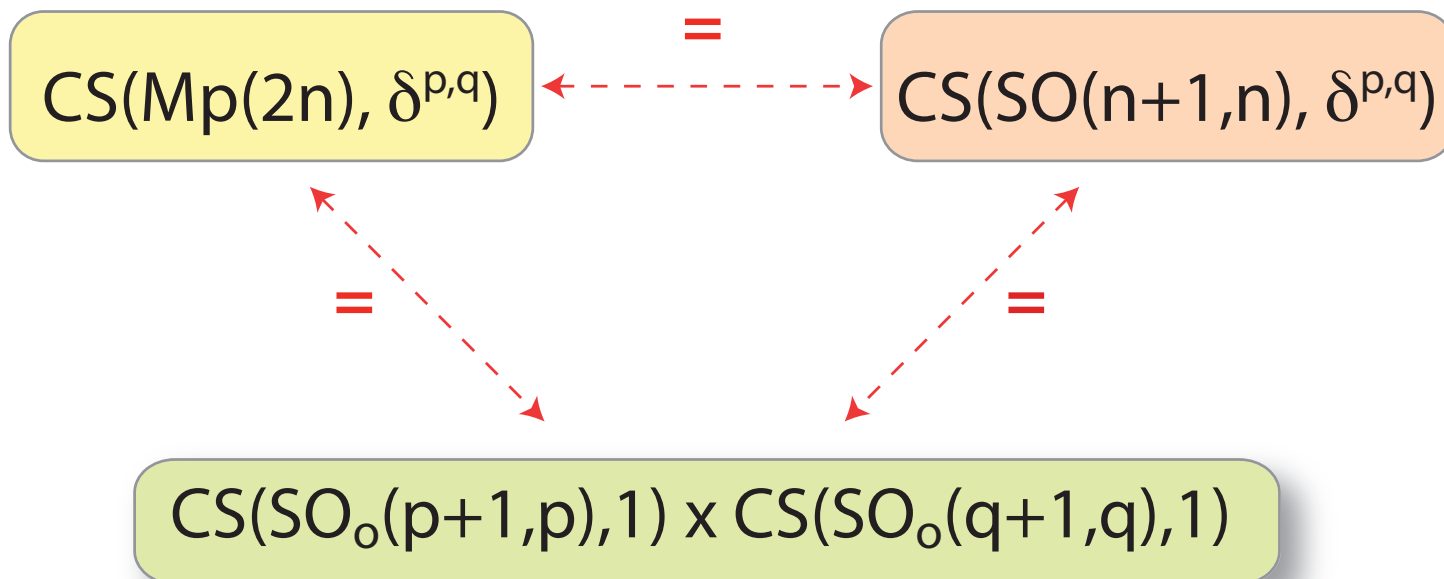
For all $n \leq 4$ and for all $\delta = \delta^{p,q}$, the following equalities hold:



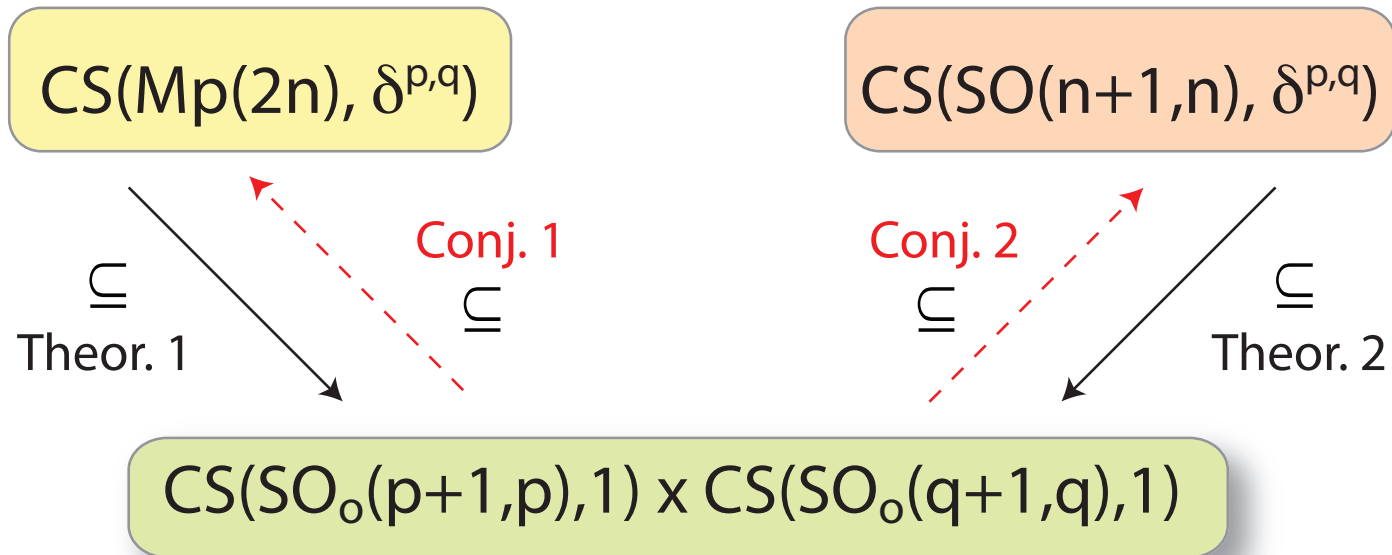
PART 4

A natural conjecture

*Equalities hold for all n
and all choices of $\delta^{p,q}$*



Conjectures 1 and 2



Remark. We may assume $p \geq q$, because

- $J_{Mp(2n)}(\delta^{p,q}, (\nu^p | \nu^q)) = J_{Mp(2n)}(\delta^{q,p}, (\nu^p | \nu^q))^*$
- $J_{SO(n+1,n)}(\delta^{p,q}, (\nu^p | \nu^q)) = J_{SO(n+1,n)}(\delta^{q,p}, (\nu^q | \nu^p)) \otimes \chi$
 ($\chi =$ a unitary character).

(More) evidence for these conjectures

- The case $(p, q) = (n, 0)$

If $(p, q) = (n, 0)$, the conjectures hold for all n . This is the pseudospherical case for $Mp(2n)$ and the spherical case for $SO(n + 1, n)$. (For $Mp(2n)$, the result is due to ABPTV; for $SO(n + 1, n)$, it is an empty statement.)

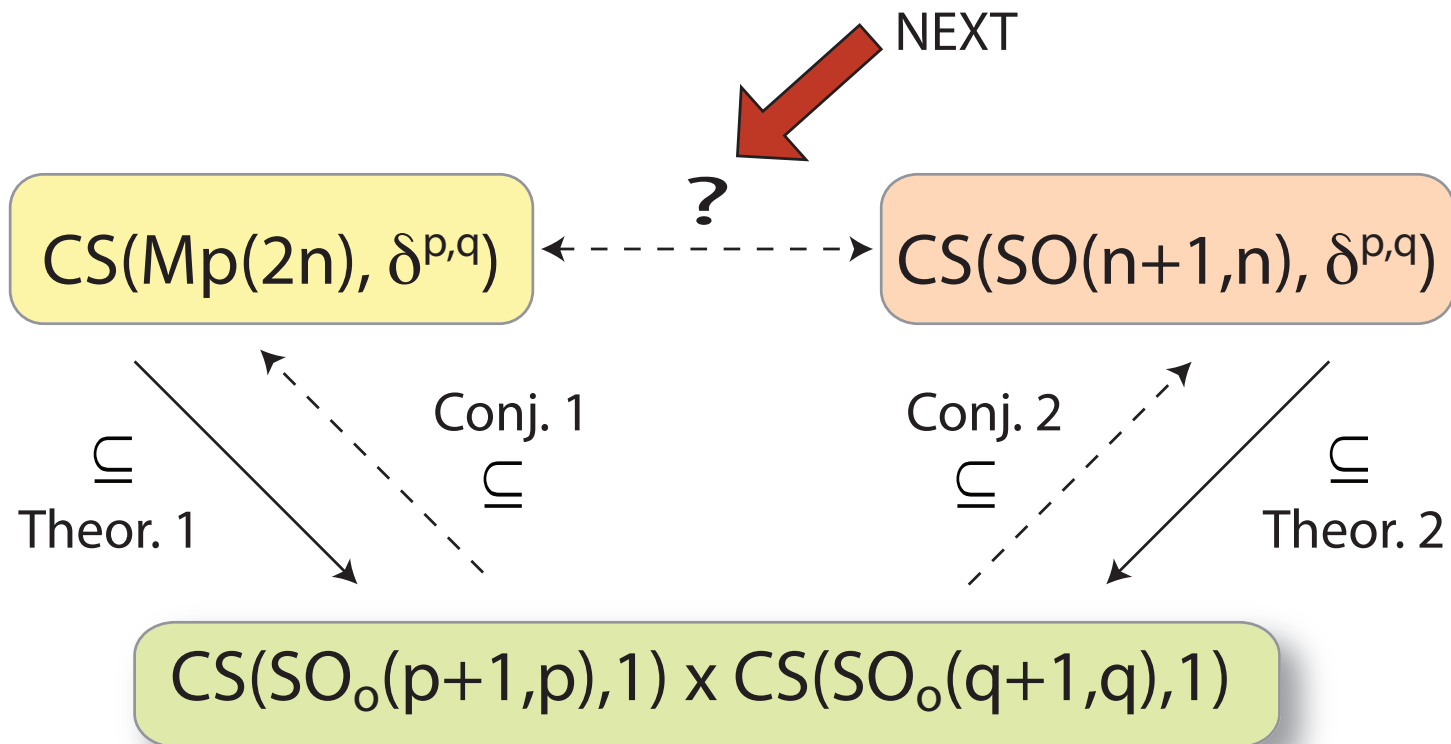
- A large family of examples

Assume $p > q$. The conjectures hold for all $\nu = (\rho^p | \nu^q)$ with

★ $\rho^p = (p - \frac{1}{2}, p - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2})$ = the infinitesimal character of the trivial representation of $SO(p + 1, p)_0$,

★ $\nu^q \in CS(SO(q + 1, q)_0, 1)$.

PART 5

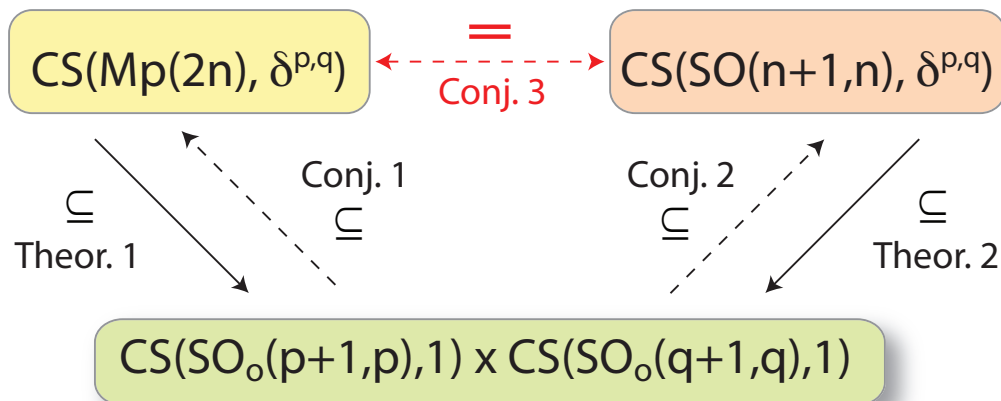


Conjecture 3

Conjecture 3

For all n and all choices of $\delta^{p,q}$:

$$CS(Mp(2n), \delta^{p,q}) = CS(SO(n+1, n), \delta^{p,q}).$$



- Conjecture 3 is true for $n = 2, 3,$ and $4.$

- Conjecture 3 is independent of Conjectures 1 and 2. ←

*new
tools!*

θ -correspondence

Consider $G = Sp(2n, \mathbb{R})$, $G' = O(m+1, m) \subset Sp(2n(m+1), \mathbb{R})$.
Let \tilde{G} and \tilde{G}' be their preimages in $Mp(2n(m+1))$:

$$\tilde{G} = Mp(2n) \quad \tilde{G}' = \tilde{O}(m+1, m) \text{ linear cover.}$$

- (G, G') is a *dual pair* in $Sp(2n(m+1), \mathbb{R})$ (mutual centralizers)
- The θ -correspondence gives a bijection between certain genuine irreducible representations of \tilde{G} and \tilde{G}' .

We can re-interpret this correspondence as a map:

$$\pi \in \widehat{Mp(2n)}_{gen} \leftrightarrow \pi' \in \widehat{SO(m+1, m)}.$$

Some results of Adams, Barbasch and Li

For all $k \geq 0$, let $\rho_k = (k - \frac{1}{2}, \dots, \frac{1}{2})$. The θ -correspondence maps:

$$J_{Mp(2n)}(\delta^{p,q}, \nu) \rightarrow J_{SO(n+k+1, n+k)}(\delta^{p+k,q}, (\rho_k | \nu))$$

$$J_{Mp(2n+2k)}(\delta^{p+k,q}, (\rho_k | \nu)) \leftarrow J_{SO(n+1, n)}(\delta^{p,q}, \nu)$$

for all $p \geq q$.

If $k \geq n + 1$, both arrows preserve unitarity. (Stable Range)

Remark: If $k = 0$, the correspondence

$$J_{Mp(2n)}(\delta^{p,q}, \nu) \leftrightarrow J_{SO(n+1, n)}(\delta^{p,q}, \nu)$$

is not known to preserve unitarity.

Conj.3

$J_{Mp(2n)}(\delta^{p,q}, \nu)$ unitary

\Leftrightarrow

$J_{SO(n+1, n)}(\delta^{p,q}, \nu)$ unitary

THEOREM 3

Theorem 3: Conjecture 3 holds in each of the following cases:

- (i) Conj.s A1 & A2 hold (ii) Conj.s A1 & B1 hold
(iii) Conj.s A2 & B2 hold (iv) Conj.s B1 & B2 hold.

Conjecture A

$$(\rho_{n+2}|\nu) \in CS(Mp(4n+4), \delta^{p+n+2,q})$$

Conj. A1 \uparrow \downarrow **Conj. A2**

$$\nu \in CS(Mp(2n), \delta^{p,q})$$

Conjecture B

$$(\rho_{n+2}|\nu) \in CS(SO(2n+3, 2n+2), \delta^{p+n+2,q})$$

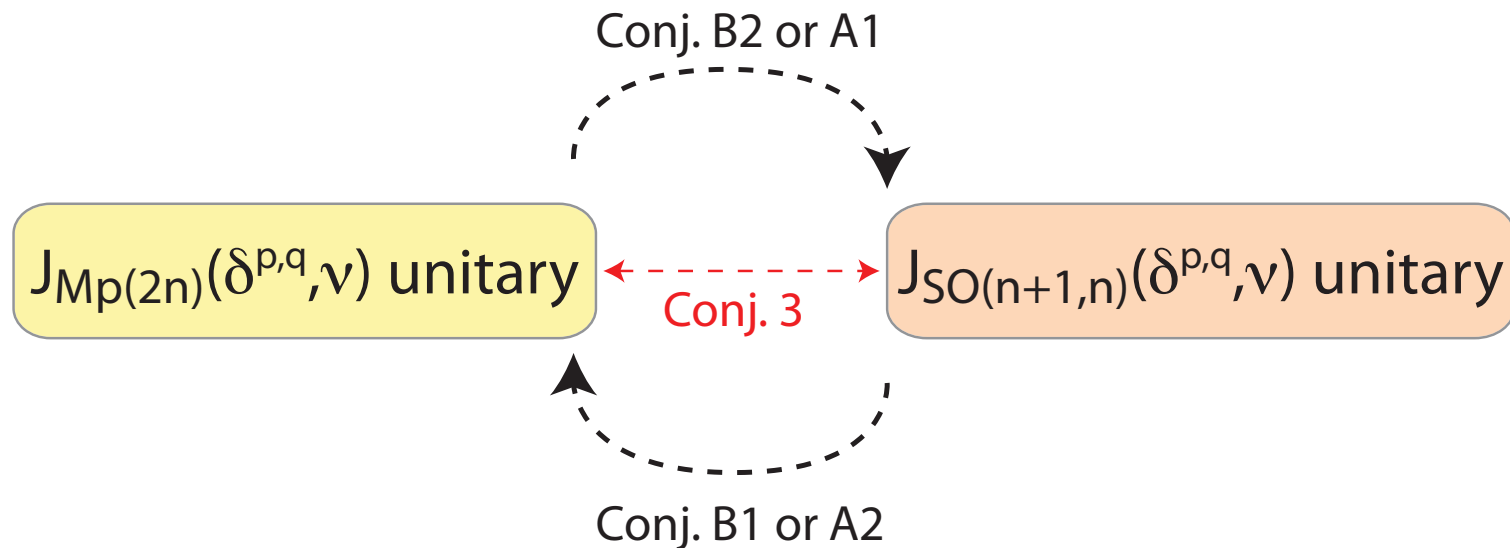
Conj. B1 \uparrow \downarrow **Conj. B2**

$$\nu \in CS(SO(n+1, n), \delta^{p,q})$$

THEOREM 3 (a sketch of the proof)

The idea of the proof is similar to the one in ABPTV.

We show that:



Key ingredients:

- Results on θ -correspondence (Adams, Barbasch, Li, Przebinda).
- Non-unitarity certificates for both $Mp(2n)$ and $SO(n + 1, n)$.

$J_{Mp(2n)}(\delta^{p,q}, \nu)$ unitary

\implies
Conj. B2

$J_{SO(n+1,n)}(\delta^{p,q}, \nu)$ unitary

$J_{SO(2n+3,2n+2)}(\delta^{p+n+2,q}, (\rho^{n+2}|\nu))$ unit.

Stable range

$J_{Mp(2n)}(\delta^{p,q}, \nu)$ unit.

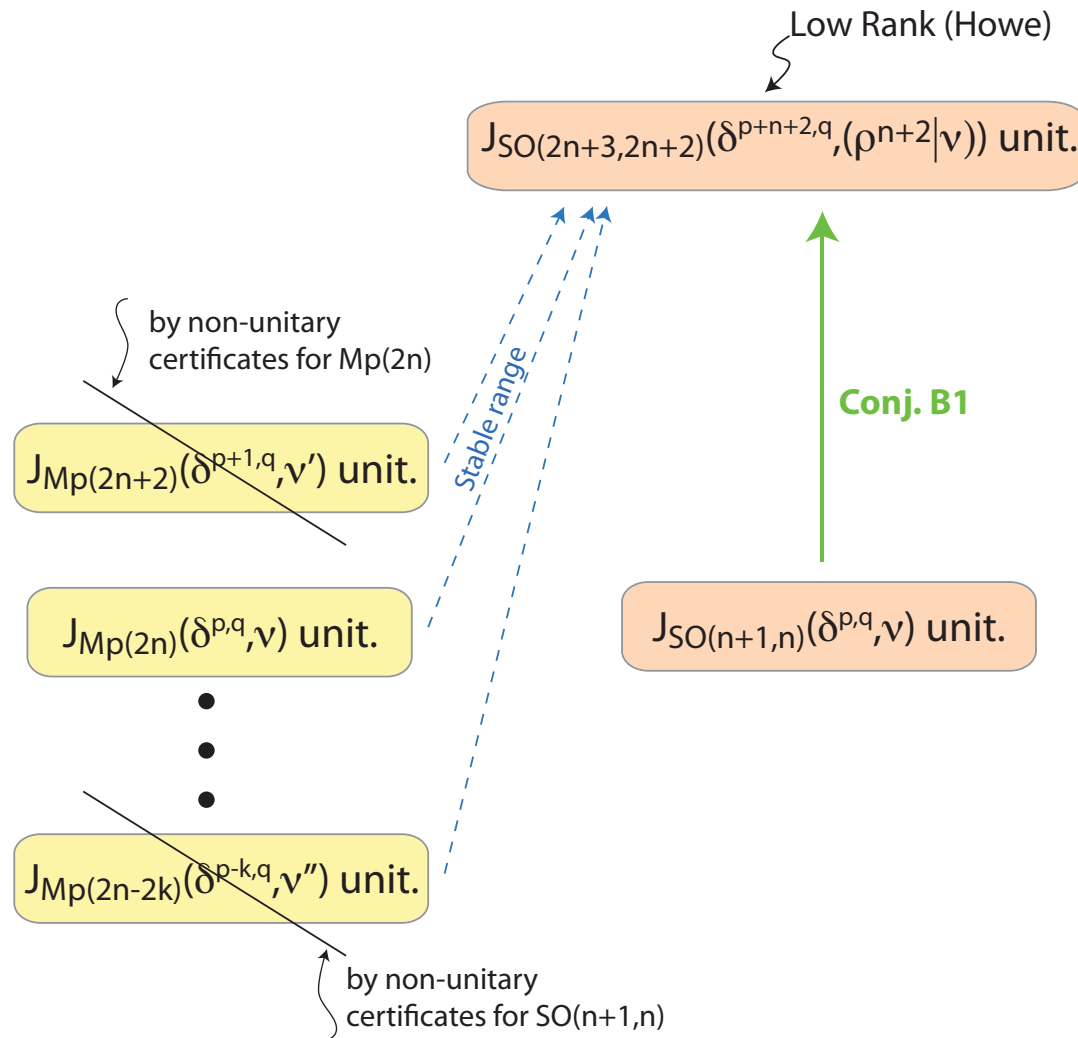
Conj. B2

$J_{SO(n+1,n)}(\delta^{p,q}, \nu)$ unit.

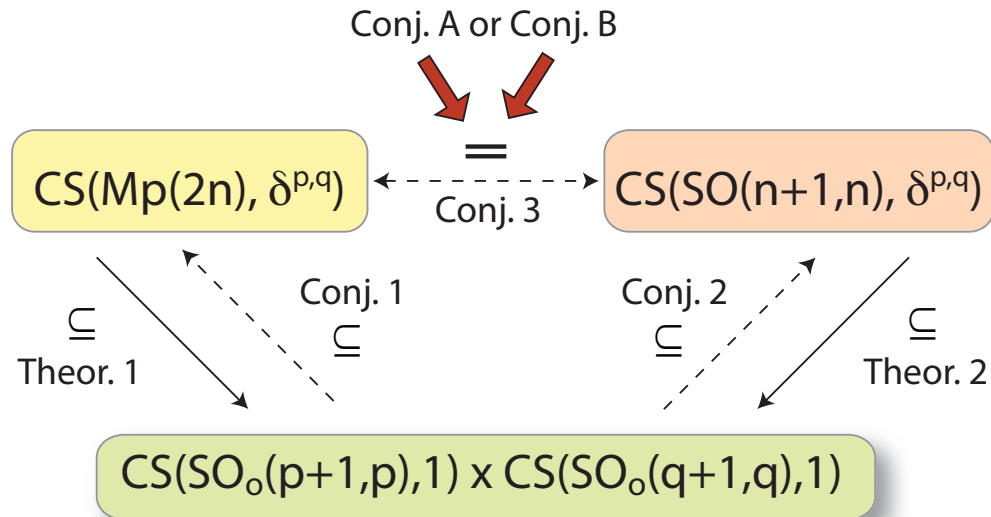
$J_{Mp(2n)}(\delta^{p,q}, \nu)$ unitary

\Leftarrow
Conj. B1

$J_{SO(n+1,n)}(\delta^{p,q}, \nu)$ unitary



Conclusions



- Conj. 1 \Rightarrow Conj. A.
- Conj. 2 \Rightarrow Conj. B.

If either Conj. 1 (alone) or Conj. 2 (alone) holds, then the 3 parameter sets are all equal.