Parameters for representations of real reductive groups, combinatorial aspects

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About parameters

Equivalence classes of representations of real reductive groups can be specified concretely in different ways, all of which are called parameters for these representations

Many (computational) questions, notably those involved in recent work within the Atlas of Lie groups and Representations project to decide **unitarity**, involve establishing precise relations between *several different* representations

They require algorithms that *manipulate* these parameters

Perspective

Focus of this talk: *how* parameters are manipulated

Not (much) discussed will be:

- Background theory of reductive groups, *why* parameters appear in the form they have, what they *mean*
- The design of complete algorithms using parameters, for instance to decide unitarity
- Actual running of these computations

Will give a somewhat *low-level* “gears and pistons” description of how computations function

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Context

Parameters serve to describe equivalence classes of irreducible (infinite dimensional) representations of real reductive groups

They arise as irreducible quotients of standard $(\mathfrak{g}, K)$-modules

Interrelations between these modules (composition series) involve blocks of representations: finite families of parameters

Algebraic structure of block is encoded in a matrix, indexed by block elements, of Kazhdan-Lusztig-Vogan polynomials

Computations for unitarity also involve *continuous deformation* of parameters, with changes to signatures of Hermitian forms occurring at certain discrete points

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Overview

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Basic data

Complex reductive group $G(\mathbb{C})$, Cartan subalgebra $H(\mathbb{C})$

Character lattice $X^* = \mathrm{Hom}(H(\mathbb{C}), \mathbb{C}^\times) \cong \mathbb{Z}^r$, and its dual X_*
Finite set $R \subset X^*$ of roots; corresponding coroots $R^\vee \subset X_*$

Real form σ in $G(\mathbb{C})$ defines $G = G(\mathbb{R}, \sigma) = G(\mathbb{C})^\sigma$, as well as
Cartan involution θ of $G(\mathbb{C})$; assume σ, θ stabilise $H(\mathbb{C})$

(θ arises from comparison of σ with *compact* real form)

This θ induces involution τ of root datum $(X^*, R, X_*, R^\vee)$

This root datum involution determines real Cartan $H = H(\mathbb{C})^\sigma$

G-conjugacy class of real Cartans $\leftrightarrow$ W-conjugacy class of τ

Given real form may admit *several* Cartan classes

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Information associated to an involution

For root datum $(X^*, R, X_*, R^\vee)$ associated to $G(\mathbb{C}), H(\mathbb{C})$, giving an involution τ of gives in particular:

- linear involutions of X^* and of X_*, describing real Cartan H
- an involution of R (and one of $R^\vee$), allowing (co)roots to be classified as imaginary (τ-fixed), real ($-\tau$-fixed), or complex (neither τ-fixed nor $-\tau$-fixed)

Relevant to the first point: an involution in $\mathbf{GL}(n, \mathbb{Z})$ can be brought into block-diagonal form, with each block one of

- $(+1)$ (giving a *compact* factor $U(1)$ of H),
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (giving a *complex* factor $\mathbb{C}^\times$ of H),
- (-1) (giving a *split* factor $\mathbb{R}^\times$ of H)

Information associated to an involution

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Backbone of parameter set: all root datum involutions

Choice of positive roots will matter; therefore *distinguish* W-conjugate root datum involutions τ

For basic root datum involution δ, involutions of form $w\delta$ with $w \in W$ give all root datum involutions for an inner class of real forms

A given real form may allow only *some* of all Cartan classes for its inner class

Example: $\mathrm{Sp}(2)$ has 1 inner class, 4 Cartan classes, 3 real forms, and 6 root datum involutions

E_8			
class	#involn.	fiber	total
0	1	256	256
1	120	64	7680
2	3780	16	60480
3	37800	4	151200
4	113400	1	113400
5	3150	4	12600
6	37800	1	37800
7	3780	1	3780
8	120	1	120
9	1	1	1
Σ	199952		387317

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Fibers over involutions

Real forms in the same inner class can share a real Cartan H

Their involutions θ differ, though same restriction τ to $H(\mathbb{C})$

Extra information, in fiber over given root datum involution τ

A fiber contains elements (essentially) from $\mathrm{Norm}_{G(\mathbb{C})}(H(\mathbb{C}))$, and representing (lifting) the $w \in W$ for which $\tau = w\delta$

$\mathrm{Stab}_W(\tau)$ acts on fiber over τ, each orbit belongs to a real form

Each orbit lives in a coset of elementary 2-group $X_*{}^\tau/(1+\tau)X_*$

For each τ, *one* orbit is chosen in its fiber;
union of these orbits gives one sided parameter set (KGB)

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Information provided by a fiber element

In addition to determining τ (and thus a real Cartan) a fiber element determines a $\mathbb{Z}/2\mathbb{Z}$-grading (compact/non-compact) of the imaginary (τ-fixed) roots at τ (and thus a real form on $G(\mathbb{C})$)

$\mathrm{Stab}_W(\tau)$-action on coset of $X_*{}^\tau/(1+\tau)X_*$, an *affine* space over $\mathbb{Z}/2\mathbb{Z}$, requires knowing this grading for one of its elements

If α imaginary root at τ, then $s_\alpha\tau$ is again root datum involution, with "less compact" real Cartan; whether passage $\tau \to s_\alpha\tau$ (Cayley transform) is possible for a fiber element (and hence for its real form) depends on α being a *non-compact* imaginary root

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Generating the KGB structure

Generating the set KGB requires fixing the real form by choosing one fiber element at the distinguished root datum involution $\tau = \delta$ (an involution giving the most compact Cartan)

From one fiber element, the cross action of W generates all involutions of its Cartan class, and also orbits under $\mathrm{Stab}_W(\tau)$

For each fiber element, its grading *may allow* Cayley transform to reach an involution at a "more split" Cartan class

Stop when no new Cartan class reachable; for all *simple roots* record gradings, its cross actions and Cayley transforms

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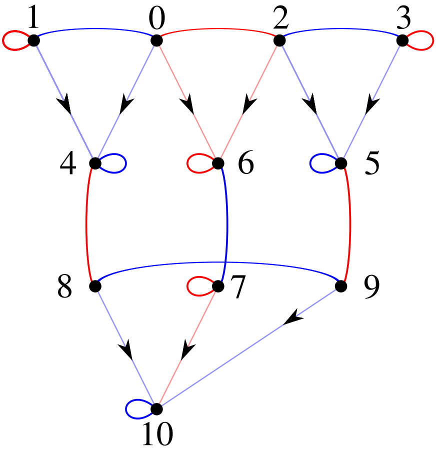
Illustration: KGB for $\mathrm{Sp}(4, \mathbb{R})$



blue: long simple root cross actions and (with arrow) Cayley transforms;

red: same for short simple root

Cartan class 0: $\tau = \delta = \mathrm{id}$, fiber $\{0, 1, 2, 3\}$;
Cartan class 1: $\tau = s_1$, fiber $\{4, 5\}$, and $\tau = s_2 s_1 s_2$, fiber $\{8, 9\}$;
Cartan class 2: $\tau = s_2$, fiber $\{6\}$ and $\tau = s_1 s_2 s_1$, fiber $\{7\}$;
Cartan class 3: $\tau = -\mathrm{id}$, fiber $\{10\}$

Why KGB?

A fiber element ("strong involution" g) determines real form σ and $G = G(\mathbb{R},\sigma) = G(\mathbb{C})^\sigma$, or equivalently Cartan involution θ and complex reductive (maybe disconnected) group $K(\mathbb{C}) = G(\mathbb{C})^\theta$, the complexified maximal compact $K = G^\theta$ of G

Moreover the stabiliser in $G(\mathbb{C})$ of g is precisely $K(\mathbb{C})$, so orbit of g (in "conjugation" action) is in bijection with $G(\mathbb{C}) \,/\, K(\mathbb{C})$

The choice of a set R^+ of positive roots for $H(\mathbb{C}) \subset G(\mathbb{C})$ amounts to the choice of a Borel subgroup $B \supset H(\mathbb{C})$ of $G(\mathbb{C})$, which is its own normaliser in $G(\mathbb{C})$ (so $G(\mathbb{C})/B \cong \{\text{all Borels}\}$)

The *relative position* of $K(\mathbb{C})$ and B corresponds to a double coset in $K(\mathbb{C}) \setminus G(\mathbb{C}) \,/\, B$, abbreviated $K\backslash G/B$ or KGB

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Overview

From KGB to blocks for G

A choice $x \in K\backslash G/B$ forms an important part of parameters

While KGB structure is subtle, it depends *only* on the real form, and can be generated once and for all for it

However, x alone does not distinguish all elements of a block

Additional information needed at involution τ, representable by an element of the elementary 2-group $(X^*)^{-\tau}/(1-\tau)X^*$

This element defines grading (parity condition) of *real* coroots

Abstractly, this information is provided by element y in KGB structure for dual root datum $(X_*, R^\vee, X^*, R)$, but in the fiber at the *negative* (transpose) involution $-\tau$

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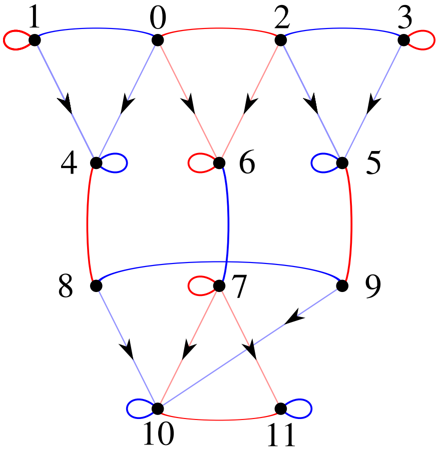

The `atlas` command `block`

Ask for a complex group, an inner class, and a real form, and *ask for* a dual real form (real form for the dual root datum); then

- Check that the two forms have some Cartan class(es) in common (they “go up from most compact” for the real form, and “go down from the most split” for dual real form; these should meet), and therefore involutions in common
- For every common involution τ, take all pairs (x, y) with x in fiber over τ, and y in dual fiber over τ (fibred product)
- Define cross actions and Cayley transforms of (x, y) component-wise (for simple roots); dual Cayley transforms go in the *opposite* direction, and in the end one gets a 2-to-1 (type 1) or 1-to-2 (type 2) relation

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Utility of these blocks

Such a block provides *all structure* necessary for computation of KLV polynomials

Moreover *all possible* blocks occur (up to isomorphism of combinatorial structures) for some inner class, real form, and dual real form

So this method is well suited for *enabling* the computation of tables of KLV polynomials, and *testing* those computations under all possible circumstances

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Limitations of the command `block`

Asking the user to supply a dual real form *circumvents* part of the computation: representations are usually given by data that encode the dual real form only implicitly, and non-obviously

Moreover, one does not get all blocks of representations of G using the root datum and the real form of G *itself*: for representations with "non-integral infinitesimal character" (to be discussed), the block may be associated to a root *sub-datum*, and real and dual real forms defined for this sub-datum

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Overview

Components of parameters

Parameters for G are specified by several components:

- A KGB element x, which determines amongst others a root datum involution τ
- A discrete component $\lambda \in (\rho + X^*)/(1 - \tau)X^*$
- A continuous component $\nu \in (X^*)^{-\tau} \otimes_{\mathbb{Z}} \mathbb{C}$

In actual computations, ν is limited to $(X^*)^{-\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$

The component λ can be further split into two (dependent) parts $\lambda_0 = \frac{1+\tau}{2}\lambda \in (\frac{1}{2}X^*)^{\tau}$ and $\lambda - \lambda_0 \in (\frac{1}{2}X^*)^{-\tau}/(1-\tau)X^*$, the latter being 2-torsion

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The infinitesimal character of the parameter is $\gamma = \lambda_0 + \nu$

The parity condition

A parameter (x, λ, ν) grades the *real coroots* $\alpha^\vee$ for which $\langle \nu, \alpha^\vee \rangle = \langle \gamma, \alpha^\vee \rangle$ is *integral*, by the parity of $\langle \lambda + \nu + \rho_\mathbb{R}, \alpha^\vee \rangle$ where $\rho_\mathbb{R}$ is half the sum of the positive *real* roots

When this parity is odd, $\alpha^\vee$ is said to satisfy the parity condition; the real root α is then called parity, and non-parity otherwise

A reverse Cayley transform is only defined for real parity roots

Since $\langle \lambda_0, \alpha^\vee \rangle = 0$, the parity condition depends on λ only via its (2-)torsion part $\lambda - \lambda_0$

If α is a *simple root* (and real), the parity condition simplifies to: $\langle \lambda + \nu, \alpha^\vee \rangle \in 2\mathbb{Z}$, or alternatively to $\langle \gamma \pm \lambda, \alpha^\vee \rangle \in 2\mathbb{Z}$

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Alternative representation of parameters

Within a block, γ will be an invariant

The following representation of parameters is possible

- Infinitesimal character γ
- KGB element x
- Small vector $y\_bits \in (\mathbb{Z}/2\mathbb{Z})^k$ for 2-torsion part of λ

Concretely y_bits found by first factoring $1 - \tau = UDV$ with $U, V \in \mathbf{GL}(n, \mathbb{Z})$ and D diagonal, entries in $\{0, 1, 2\}$ (Smith); set y_bits to "$D_{i,i} = 2$" coordinates of $U(\lambda - \rho)$ modulo 2: allows determining $\frac{1-\tau}{2}(\lambda - \rho)$, (which is defined) modulo $(1 - \tau)X^*$

One can recover $\nu = \frac{1-\tau}{2}\gamma$, and $\lambda = \frac{1+\tau}{2}(\gamma - \rho) + \frac{1-\tau}{2}(\lambda - \rho)$

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Yet another variation of parameters

During construction of blocks, necessary frequent conversions between λ and *y_bits* would be cumbersome

Therefore, and for another reason discussed later (related to non-integrality), block construction is easier using an more complicated value $y \in \mathbb{Q}^r/((\mathbb{Q}^r)^\tau + (2\mathbb{Z})^r)$ instead of *y_bits*

Define y as image in quotient of $\gamma - \lambda = \nu - (\lambda - \lambda_0)$

This stands for its image $\exp(\pi \mathbf{i} y)$ in “dual torus”

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Behaviour of rational vector y in block construction

The part y behaves well under cross actions for simple α:

- if α is complex, just reflect y by s_α;
- if α is real, reflect and add α (as dual Tits groups dictates)
- if α is imaginary, keep y unchanged

Recall: x changes too, but γ is *invariant*

For Cayley transforms, the following applies:

- For reverse Cayley transform, y can be kept as representative in the coarser quotient
- For forward Cayley transform, it may be necessary to add $\frac{\alpha}{2}$ to y, to ensure parity condition holds afterwards

Recall: parity condition is equivalent to $\langle \gamma - \lambda, \alpha^\vee \rangle \in 2\mathbb{Z}$

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Equivalence of parameters

Recall: the component x of a parameter records relative position of θ and a choice of positive roots $R^+ \subset R$

That relative position is relevant for (γ, x, y) only if γ is *dominant* for this choice of positive roots: $\langle \gamma, \alpha^\vee \rangle \geq 0$ for all $\alpha \in R^+$
In **atlas** encoding, the set R^+ is *always* the same, but θ varies

Assume parameter standard: $\langle \gamma, \alpha^\vee \rangle \geq 0$ for *imaginary* $\alpha \in R^+$

An equivalence relation on standard parameters generated by:

- Simple α is complex: apply (cross action by) s_α to x, λ, ν
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Unlike cross actions in block, this *does change* γ (by s_α)

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Normal form for standard final parameters

Call α singular (for γ) if $\langle\gamma, \alpha^\vee\rangle = 0$, and call a parameter final if there are *no singular* real *parity* roots

For final standard parameters one has a normal form: a unique parameter in the equivalence class, satisfying: γ is dominant, and for every *complex singular* root α, its image $\tau(\alpha)$ is *positive*

Basically, reason is: if equivalence for some *real* simple α leaves γ unchanged, then α is singular, so non-parity (due to "final"), and equivalence does *nothing*; for singular *complex* simple α, use equivalence to diminish set of positive singular roots β with $\tau(\beta)$ negative, until it contains *only real roots*;

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Overview

Integrality of the infinitesimal character

The continuous nature of $\nu = \frac{1-\tau}{2}\gamma$ means that coroots $\alpha^\vee$ *need not satisfy* $\langle\gamma, \alpha^\vee\rangle \in \mathbb{Z}$; this *is* assured though for imaginary α, since then $\langle\gamma, \alpha^\vee\rangle = \langle\lambda_0, \alpha^\vee\rangle = \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z}$

Call $R^\vee_\gamma = \{\, a^\vee \in R^\vee \mid \langle\gamma, \alpha^\vee\rangle \in \mathbb{Z} \,\}$ the integral coroot subsystem, and R_γ the corresponding subset of R

The integral root sub-datum $(X^*, R_\gamma, X_*, R^\vee_\gamma)$ for γ is the relevant root datum for blocks with infinitesimal character γ (dominant)

One takes as positive roots $R^+_\gamma = R_\gamma \cap R^+$, but the set of *simple* roots for R^+_γ is unrelated to the simple roots for R^+

So the *structure* of KGB and of the block *will be different*

But KLV computation is unaffected

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Non-integral KGB structure

The KGB structure used for non-integral blocks could be generated similarly to the KGB structure for G, but using the root *sub-datum*; the resulting elements would then need to be matched with a subset of KGB for G

Simpler to define this subset implicitly: describe cross actions and Cayley transforms in terms of the KGB for G

Write for each simple root β of the sub-datum: $w(\beta) = \alpha$ with $w \in W$ and α a *simple* root for the *full* root datum; then

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Handling y for non-integral blocks

Likewise, one could consider performing the sub-datum cross actions and Cayley transforms directly on the y component

Formulas above apply in principle, as if generating integral block for group G' defined by sub-datum; get valid block for G'

However, taking for y the image of $\gamma - \lambda$ as before *does not* give the correct parity condition; the reason is that this encoding of (the torsion part of) λ is implicitly based on the weight ρ for the root datum, which differs for subsystem

A solution would be to apply a *shift* to the y value when using the sub-datum, but the shift needs to be is τ-dependent in a manner for which we have not (yet) found a formula

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How `atlas` generates possibly non-integral blocks

Prepare: compute dominant value of γ and its integral sub-datum, and for each simple root β of the sub-datum an element $w_\beta \in W$ mapping it to a simple root α of the full datum

The encoding of y is as vector in $(\mathbb{Q}/2\mathbb{Z})^r$.

To perform cross actions or (reverse) Cayley transform for β, do

1. Apply cross actions for w_β to x; modify y accordingly
2. Apply (reverse) Cayley or cross action for α to x and y
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Reduce y modulo $(Q^r)^\tau$ when testing equality of pairs (x, y)

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Overview

Deformation of continuous part ν towards 0

KLV polynomials, and therefore blocks, serve two purposes

- Expressing irreducible module in terms of standard modules (character formula)
- Describing decomposition of the standard module at special values of ν, to keep track of signature changes

The latter allows signature computation by deformations of continuous part ν of γ to 0

The special values of ν are those where the parameter does not occur at the "bottom" of its block, as happens generically

Recall: sub-datum for block has coroots integral on $\gamma = \nu + \lambda_0$

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Some observations

- Deformation applied recursively gives large collection of parameters (its nature is not so clear for now...)
- Large number of blocks occur, small on average
- Often parameter used is near bottom of its block: it is useful to generate only the Bruhat interval below this parameter, and to limit the KLV computation to this interval
- When reaching $\nu = 0$, parameter may become non-final (real parity roots become singular), in which case rewrite it
- Signature calculations are easy at $\nu = 0$; this is the base case for the computation