Computing twisted KLV polynomials

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This is an attempt to convert [5] to the atlas setting, and write down explicit recursion formulas for the Kazhdan-Lusztig-Vogan polynomials which arise.

1 The Setup

{s:setup}

The starting point is: a group G, a Cartan involution θ, and another involution σ of finite order, commuting with θ.

It is natural to consider the coset $\sigma K = \{\sigma \circ \mathrm{int}(k) \mid k \in K\} \subset \mathrm{Aut}(G)$. Every element of this coset commutes with θ.

We're mainly interested when σ is an involution, especially the case $\sigma = \theta$.

Now fix a pinning $\mathcal{P} = (H, B, \{X_\alpha\})$, and write $\delta, \epsilon \in \mathrm{Aut}(G)$ for the images via the embedding $\mathrm{Out}(G) \hookrightarrow \mathrm{Aut}(G)$ (the image consists of $\mathcal{P}$-distinguished automorphisms). Then $\delta, \epsilon \in \mathrm{Aut}(G)$ commute.

We now introduce the usual atlas structure. See [1] for details. Let ${}^\delta G = G \rtimes \langle\delta\rangle$ be the usual extended group; σ acts on it, trivially on δ.

Recall $\mathcal{X} = \{\xi \in \mathrm{Norm}_{G\delta}(H) \mid \xi^2 \in Z(G)\}/H$, and $\widetilde{\mathcal{X}}$ is the numerator. We'll write x for elements of $\mathcal{X}$, ξ for elements of $\widetilde{\mathcal{X}}$, and $p : \widetilde{\mathcal{X}} \to \mathcal{X}$. It is important to distinguish between elements of $\mathcal{X}$ and $\widetilde{\mathcal{X}}$.

For $\xi \in \widetilde{\mathcal{X}}$, let $\theta_\xi = \mathrm{int}(\xi) \in \mathrm{Aut}(G)$, $K_\xi = G^{\theta_\xi}$.The restriction of θ_ξ to H only depends on $p(\xi) \in \mathcal{X}$, and is denoted θ_x. It is important to remember θ_x is only an involution of H, not of G, and K_x is not well defined.

It is immediate that $\sigma(\mathcal{X}) = \mathcal{X}$.

After conjugating we can assume $\theta = \mathrm{int}(\xi_0)$ for some $\xi_0 \in \widetilde{\mathcal{X}}$. Let $x_0 = p(\xi_0)$, and define $\mathcal{X}_\theta = \{x \in \mathcal{X} \mid x \text{ is } G\text{-conjugate to } x_0\}$ (G-conjugacy of elements of $\mathcal{X}$ is well defined). Let $K = G^\theta = K_{\xi_0}$. Then there is a canonical bijection $\mathcal{X}_\theta \leftrightarrow K\backslash G/B$.

Since $\{\sigma, \theta\} = 1$, σ acts on $K\backslash G/B$. Write $x \to \sigma^\dagger(x)$ for the automorphism of X_θ, corresponding to the action of σ on $K\backslash G/B$. We need to compute $\sigma^\dagger$.

The condition $\{\sigma, \theta\} = 1$ holds if and only if $\sigma(\xi_0) \in \xi_0 Z$. It is convenient to define $z_0 = \sigma(\xi_0^{-1})\xi_0 \in Z^{-\sigma}$, so

$$\sigma(\xi_0) = \xi_0 z_0^{-1}.$$

(The choice of inverse on z_0 is so that it goes away later.) It makes sense to write:

$$\sigma(x_0) = x_0 z_0^{-1}. \tag{1.1}$$

(both sides being defined up to conjugacy by H).

{p:kgbaction}

Proposition 1.2 *After replacing σ by another element of σK, we may assume σ normalizes H. Define $v \in \mathrm{Norm}_G(H)$ by $\sigma(B) = vBv^{-1}$. Then*

$$\sigma^\dagger(x) = v^{-1}\sigma(x)vz_0 \quad (x \in X_\theta). \tag{1.3(a)}$$

If sigma preserves the base orbit, i.e. $\sigma(K \cdot B) = K \cdot B$, then after replacing σ with another element of σK we may assume σ normalizes (B, H). In this case

$$\sigma^\dagger(x) = \sigma(x)z_0. \tag{1.3(b)}$$

The coset vH of v in $W = \mathrm{Norm}_G(H)/H$ is well defined, and it makes no difference if we view v as an element of $\mathrm{Norm}_G(H)$ or W.

Remark 1.4 *After replacing σ with another element of σK we may assume $\sigma(H) = H$, and $\sigma(B \cap K) = B \cap K$. Having done this, we conclude v normalizes $B \cap K$. The normalizer in W of $B \cap K$, equivalently ρ_K, is very small; in particular it is a product of A_1 factors.*

Assume $\sigma(K \cdot B) = K \cdot B$, i.e. $v = 1$. Then we are in the following setting. We have an involution θ and an automorphism σ, preserving (B, H), and commuting with θ. The induced automorphism of (W, S) also written σ, has finite order (S is the set of simple reflections). Finally the corresponding automorphism $\sigma^\dagger$ of X_θ is $\sigma^\dagger(x) = \sigma(x)z_0$.

The main case of interest is:

{c:kgbaction}

Corollary 1.5 *Suppose $\sigma=\theta$. After replacing θ with a G-conjugate, we have $\sigma^\dagger(x)=\delta(x)$, and θ,δ commute.*

Proof. After replacing θ with a G-conjugate we may assume $\xi_0\in H\delta$. Then $\sigma(B)=\theta(B)=B$, also $z_0=1$, so $\sigma^\dagger(x)=\sigma(x)=\theta(x)$. But $\theta=\text{int}(\xi_0)$ and δ differ by an element of H, so $\theta(x)=\delta(x)$ (since X has conjugation by H built in). $\square$

Proof of the Proposition.

We recall a few details of the bijection $X_\theta\leftrightarrow K\backslash G/B$ [1, Sections 8 and 9].

Let $\widehat{P}^\sigma=\{(x,B')\mid x\in X_\theta, B'\in\mathcal{B}\}/G$. There are bijections:

$$\begin{aligned}
X_\theta \longleftrightarrow \quad & \widehat{P}^\sigma \longleftrightarrow K\backslash\mathcal{B}\\
x\longrightarrow \quad & (x,B)\\
& (x_0,B')\longleftarrow K\cdot B'
\end{aligned}\tag{1.6}$$

Since H is a fundamental Cartan subgroup with respect to θ, and all such Cartan subgroups are K-conjugate, we can modify σ by k so that σ normalizes H.

Here is the computation.

$$\begin{aligned}
X_\theta\ni x &\to (x,B)\in\widehat{P}^\sigma\\
&=(gx_0g^{-1},B)\quad (x=gx_0g^{-1})\\
&=(x_0,g^{-1}Bg)\\
&\to g^{-1}Bg\in K\backslash\mathcal{B}\\
&\to \sigma(g^{-1}Bg)\text{ by the action of }\sigma\text{ on }K\backslash\mathcal{B}\\
&=\sigma(g^{-1})vBv^{-1}\sigma(g)\text{ where }\sigma(B)=vBv^{-1}\\
&\to (x_0,\sigma(g^{-1})vBv^{-1}\sigma(g))\in\widehat{P}^\sigma\\
&=(v^{-1}\sigma(g)x_0\sigma(g^{-1})v,B)\\
&\to (v^{-1}\sigma(g)x_0\sigma(g^{-1})v\in X_\theta\\
&=v^{-1}\sigma(g\sigma(x_0)g^{-1})v,\\
&=v^{-1}\sigma(gx_0z_0^{-1}g^{-1})v\quad\text{(by (1.1))}\\
&=v^{-1}\sigma(gx_0g^{-1})\sigma(z_0^{-1})v\\
&=v^{-1}\sigma(x)\sigma(z_0^{-1})v\quad (gx_0g^{-1}=x)\\
&=v^{-1}\sigma(x)vz_0\quad (\sigma(z_0^{-1})=z_0)
\end{aligned}$$

□

2 The group W^σ and the twisted Hecke algebra $\mathbf{H}$

{s:hecke}

We continue in the setting of Section 1, and we now assume σ is an involution. Let $K = G^\theta$.

For simplicity let's assume $\sigma(K\cdot B) = K\cdot B$, so $v = 1$ (see Proposition 1.2). Then, after replacing σ with an element of σK, we may assume σ commutes with θ, and satisfies $\sigma(B,H) = (B,H)$. Although σ may not have finite order, $\sigma^2 = \mathrm{int}(h)$ for some $h \in H$, so σ induces an involution, also denoted σ, of (W,S). Let $\overline{S}$ be the set of orbits of the action of σ on S.

We are primarily interested in the case $\sigma = \theta$. In this case, after conjugating by G we may assume the induced automorphism of (W,S) is δ (Corollary 1.5), and $\{\theta,\delta\} = 1$.

If $\kappa \in \overline{S}$ let $W(\kappa)$ be the subgroup of G generated by κ. Write $\kappa = \{s_\alpha\}$ or $\{s_\alpha, s_\beta\}$, with α,β simple. In each case there is a unique long element $w_\kappa \in W(\kappa)$. Define $\ell(\kappa) = \ell(w_\kappa)$.

$$W(\kappa) = \begin{cases} \ell(\kappa) = 1 & S_2 & \sigma(s_\alpha) = s_\alpha \\ \ell(\kappa) = 2 & S_2\times S_2 & \sigma(s_\alpha) = s_\beta, \langle\alpha,\beta^\vee\rangle = 0 \\ \ell(\kappa) = 3 & S_3 & \sigma(s_\alpha) = s_\beta, \langle\alpha,\beta^\vee\rangle = -1 \end{cases} \tag{2.1}$$

Lusztig and Vogan define a Hecke algebra $\mathbf{H}$ over $\mathbb{Z}[u,u^{-1}]$ (u is an indeterminate). See the end of [5, Section 3.1]. It has generators T_w ($w \in W^\sigma$) and relations

$$\begin{aligned} &T_wT_{w'} = T_{ww'} \quad w,w' \in W^\sigma, \ell(ww') = \ell(w)+\ell(w') \\ &(T_{w_\kappa}+1)(T_{w_\kappa} - u^{\ell(w_\kappa)}) = 0 \quad (\kappa \in \overline{S}) \end{aligned} \tag{2.2}$$

The quotient of the root system by σ is itself a root system (nonreduced if length 3 occurs), with simple roots parametrized by $\overline{S}$, and W^σ is the Weyl group of this root system. In particular W^σ is generated by $\{w_\kappa \mid \kappa \in \overline{S}\}$, and $\mathbf{H}$ is generated by $\{T_{w_\kappa} \mid \kappa \in \overline{S}\}$. So in fact $\mathbf{H}$ has generators and relations

$$(2.3)\quad \begin{aligned} &T_{w_\kappa}T_{w'} = T_{w_\kappa w'} \quad \kappa\in\overline{S}, \quad \ell(w_\kappa w') = \ell(w_\kappa)+\ell(w') \\ &(T_{w_\kappa}+1)(T_{w_\kappa}-u^{\ell(w_\kappa)}) = 0 \quad (\kappa\in\overline{S}) \end{aligned}$$

This makes **H** a quasisplit Hecke algebra [5, §4.7].

Let $\mathcal{D}=\mathcal{Z}[x]$ be the subset of $\mathcal{Z}$ having to do with x. That is $\mathcal{Z}[x]\subset\mathcal{X}[x]\times\mathcal{X}^\vee$ ($\mathcal{X}^\vee$ is the dual KGB space), where $\mathcal{Z}[x]\simeq K_\xi\backslash G/B$, and $\mathcal{D}$ is parametrized by the K_ξ-invariant local systems on $K_\xi\backslash G/B$. Then that σ acts on $\mathcal{D}$, and let $\mathcal{D}^\sigma$ be the fixed points.

Lusztig and Vogan define a **H** module M, with basis $\{a_\gamma \mid \gamma\in\mathcal{D}^\sigma\}$. We are going to write down formulas for the action of **H** on M.

3 Extended Cartans and Parameters

{s:extended}

If $\gamma\in\mathcal{D}^\sigma$ there is an isomorphism between the representations parametrized by γ and $\sigma(\gamma)$. It is possible to normalize this isomorphism to have "square 1", i.e. in such a way that there are two choices, $\pm\alpha_\gamma$. This leads to *extended parameters*: for each $\gamma\in\mathcal{D}^\sigma$ there are two extended parameters corresponding to the two choices of α_γ. Write $\widehat{\gamma}$ for an extended parameter corresponding to γ.

Strictly speaking, the module M is spanned by vectors $a_{\widehat{\gamma}}$ as $\widehat{\gamma}$ runs over extended parameters, and the Hecke algebra action is naturally defined in these terms. If $\widehat{\gamma}^\pm$ are the two choices of extension, in the module M we have $a_{\widehat{\gamma}^-}=-a_{\widehat{\gamma}^+}$, and the dimension of M is $|\mathcal{D}^\sigma|$.

{desideratum}

Desideratum 3.1 *For each parameter $\gamma\in\mathcal{D}^\sigma$, it is possible to choose one extended parameter, denoted $\widehat{\gamma}_+$ so that the formulas of* [5] *hold with $\widehat{\gamma}_+$ and $a_{\widehat{\gamma}_+}$ everywhere.*

3.1 Extended Cartans

We probably don't need this subsection and the next one. They are vestiges of a version in which we worked in terms of extended parameters. But it might be helpful to include a few basic facts.

In this section and the next we assume $\sigma=\delta$. Probably this isn't serious, but in any event in the rest of the paper we only assume σ is an involution.

We are interested in KGB elements $x \in \mathcal{X}$ which are fixed by δ. A key point is that if $\delta(x) = x$, and $\xi \in p^{-1}(x) \in \widetilde{\mathcal{X}}$, then $\delta(\xi) = h\xi h^{-1}$ for some $h \in H$. We cannot assume we can choose ξ so that $\delta(\xi) = \xi$.

{l:onlyfixedx}

Lemma 3.1.1 *Suppose $\xi \in \widetilde{\mathcal{X}}$, and let $x = p(\xi) \in \mathcal{X}$. The following conditions are equivalent.*

(1) θ_ξ normalizes ${}^\delta H$, and $({}^\delta H)^{\theta_\xi}$ meets both components of ${}^\delta H$;

(2) $\delta(x) = x$.

Proof. Suppose (1) holds. The second part of (1) says that $\xi(t\delta)\xi^{-1} = t\delta$ for some $t \in H$, i.e.

(3.1.2)(a) $$\theta_x(t)(\xi\delta\xi^{-1}) = t\delta.$$

The first part of (1) says $\xi\delta\xi^{-1} = h\delta$ for some $h \in H$, i.e.

(3.1.2)(b) $$\delta(\xi) = h^{-1}\xi$$

Plug in $\xi\delta\xi^{-1} = h\delta$ to (a): $\theta_x(t)h\delta = t\delta$, so $h^{-1} = t^{-1}\theta_x(t)$. Then by (b):

(3.1.2)(c) $$\delta(\xi) = h^{-1}\xi = t^{-1}\theta_x(t)\xi = t^{-1}\xi t.$$

Projecting to $\mathcal{X}$ this says $\delta(x) = x$.

Conversely, suppose $\delta(x) = x$. By definition this means $\delta(\xi) = h^{-1}\xi h$ for some $h \in H$. Note that

(3.1.3) $$\delta(\xi) = h^{-1}\xi h \Leftrightarrow \xi(h\delta)\xi^{-1} = h\delta.$$

In other words the second condition of (1) holds. Also the first condition holds: the right hand side gives $\theta_x(h)\xi\delta\xi^{-1} = h\delta$, i.e. $\xi\delta\xi^{-1} \in H\delta$. □

From now on we will usually assume $\delta(x) = x$.

Definition 3.1.4 *Suppose $\xi \in \widetilde{\mathcal{X}}$. Let $x = p(\xi) \in \mathcal{X}$, and assume $\delta(x) = x$. The extended Cartan defined by ξ is ${}^1H_\xi = ({}^\delta H)^{\theta_\xi}$. It contains H^{θ_x} as a subgroup of index 2, and meets both components of ${}^\delta H$.*

In other words

(3.1.5) $${}^1H_\xi = \langle H^{\theta_x}, h\delta\rangle$$

where $\xi(h\delta)\xi^{-1} = h\delta$, equivalently $h^{-1}\xi h = \delta(\xi)$.

This is related to [2, Definition 13.5]. Note that $h\delta$ normalizes Δ^+.

3.2 Extended Parameters

{s:extended}

We work only at a fixed regular infinitesimal character, so we fix $\lambda \in \mathfrak{h}^*$, dominant for Δ^+.

By a *character* we mean a pair (x, Λ), where Λ is an $(\mathfrak{h}, H^{\theta_x})$-module. We're ignoring the ρ-cover; this isn't hard to fix. The differential of Λ (the $\mathfrak{h}$ part) is $\lambda \in \mathfrak{h}^*$, so we usually identify Λ with a character of H^{θ_x}.

Definition 3.2.1 *Given $\xi \in \widetilde{\mathcal{X}}$, an extended character is a pair $(\xi, \ {}^1\Lambda)$ where ${}^1\Lambda$ is an $(\mathfrak{h}, {}^1H_\xi)$ module. Equivalence of extended characters is by conjugation by H.*

Recall given a parameter (x, y) as usual, compatible with λ we obtain a character (x, Λ), as above we think of Λ as a character of H^{θ_x}.

Definition 3.2.2 *An extended parameter, at infinitesimal character λ, is a quadruple $(\xi, y, h\delta, z)$ satisfying the following conditions. Set $x = p(\xi) \in \mathcal{X}$, and we assume $\delta(x) = x$.*

(1) (x, y) is a parameter at λ, defining a character Λ of H^{θ_x}.

(2) $\Lambda^\delta = \Lambda$ (i.e. Λ is fixed by δ),

(3) $h\delta$ is in the extended group $({}^\delta H)^{\theta_\xi}$, i.e. $h\delta$ commutes with ξ,

(4) $z \in \mathbb{C}^*, z^2 = \Lambda(h\delta(h))$

Equivalence of extended parameters is generated by conjugation by H, and

$$(\xi, y, h\delta, z) \simeq (\xi, y, th\delta, \Lambda(t)z) \quad (t \in H^{\theta_x}). \tag{3.2.3}$$

Remark 3.2.4 Condition (2) implies (but is not equivalent to): $\delta^t(y) = y$. So we may as well assume this holds as well as $\delta(x) = x$.

Proposition 3.2.5 *There is a bijection between equivalence classes of extended characters and equivalence classes of extended parameters.*

The bijection is

$$(\xi, y, h\delta, z) \leftrightarrow (\xi, \ {}^1\Lambda) \tag{3.2.6}$$

From left to right, take ${}^1\Lambda|_{H^{\theta_x}}$ to be the character defined by $(x = p(\xi), y)$, and ${}^1\Lambda(h\delta) = z$. Conversely, given ${}^1\Lambda$, choose y so that (x, y) corresponds to ${}^1\Lambda|_{H^{\theta_x}}$. Choose any $h\delta \in ({}^\delta H)^{\theta_\xi}$, and let $z = {}^1\Lambda(h\delta)$. There are a few straightforward checks that this works. One of the main points is that if we choose $h_i\delta \in ({}^\delta H)^{\theta_\xi}$ $(i = 1, 2)$, then $h_2 = h_1 t$ for $t \in H^{\theta_x}$, which is taken care of by the equivalence.

4 Cayley transforms and cross actions

{s:cayley}

Cayley transforms and cross actions can naturally be defined in terms of extended parameters. (This was done in an earlier version of these notes.) As discussed at the beginning of Section 3.2, implicit in [5] is the assertion that, for each parameter γ, there is a choice of extended parameter, which we'll label $\widehat{\gamma}^+$, so that the following formulas hold with $\widehat{\gamma}^+$ in place of γ everywhere *except* in cases `2i12/2r21`. For these see Section 5.

Some of these new "Cayley transforms" are iterated Cayley transforms, but some involve a combination of cross actions and Cayley transforms.

4.1 Length 1

In length 1, these are essentially the usual definitions, except in the `1i2s` case, when the Cayley transform is not defined.

Suppose $\ell(\kappa) = 1$, so $\kappa = \{s_\alpha\}$ and $w_\kappa = s_\alpha$, where $\sigma(\alpha) = \alpha$.

In the classical case α has type `C+, C-, i, i2, ic, r, r2` or `rn`. We write these `1C+,...,1rn` to emphasize the length of κ.

Suppose α is of type `1i2`, so the Cayley transform is double valued: $\gamma_1^\alpha, \gamma_2^\alpha$. Then $\sigma(\alpha) = \alpha$ implies σ preserves the set $\{\gamma_1^\alpha, \gamma_2^\alpha\}$. This yields two subcases in the new setting: denote these `1i2f` ("fixed") or `1i2s` ("switched"), depending on whether σ acts trivially on this set, or interchanges the two members.

Type `1r1` is similar; the double-valued Cayley transform is written $\{\gamma_\alpha^1, \gamma_\alpha^2\}$.

type	definition	Cayley transform
`1C+`	α complex, $\theta\alpha > 0$	
`1C-`	α complex, $\theta\alpha < 0$	
`1i1`	α imaginary, noncompact, type 1	$\gamma^\kappa = \gamma^\alpha$
`1i2f`	α imaginary, noncompact, type 2 σ fixes both terms of γ^α	$\gamma^\kappa = \gamma^\alpha = \{\gamma_1^\kappa, \gamma_2^\kappa\}$
`1i2s`	α imaginary, noncompact, type 2 σ switches the two terms of γ^α	
`1ic`	α compact imaginary	
`1r1f`	α real, parity, type 1 σ switches the two terms of γ^α	$\gamma_\kappa = \gamma_\alpha = \{\gamma_\kappa^1, \gamma_\kappa^2\}$
`1r1s`	α real, parity, type 1 σ switches the two terms of γ_α	
`1r2`	α real, parity, type 2	$\gamma_\kappa = \gamma_\alpha$
`1rn`	α real, non-parity	

4.2 Length 2

Suppose $\alpha \in S, \beta = \sigma(\alpha) \in S$, and $\langle \alpha, \beta^\vee \rangle = 0$. Let $\kappa = \{s_\alpha, s_\beta\}$, so $w_\kappa = s_\alpha s_\beta \in W^\sigma$. It is easy to see that α, β have the same type with respect to θ. Here are the twelve cases as listed in [5, Section 7.5].

In the length 2 and 3 cases we include the terminology from [5] in a separate column.

type	LV terminology	definition	Cayley transform
2C+	two-complex ascent	α,β complex $\theta\alpha>0$ $\theta\alpha\ne\beta$	
2C-	two-complex ascent	α,β complex $\theta\alpha<0$ $\theta\alpha\ne\beta$	
2Ci *	two-semiimaginary ascent	α,β complex, $\theta\alpha=\beta$	$\gamma^\kappa=s_\alpha\times\gamma=s_\beta\times\gamma$
2Cr *	two-semireal descent	α,β complex,$\theta\alpha=-\beta$	$\gamma_\kappa=s_\alpha\times\gamma=s_\beta\times\gamma$
2i11	two-imaginary noncpt type I-I ascent	α,β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ single valued	$\gamma^\kappa=(\gamma^\alpha)^\beta$
2i12 †	two-imaginary noncpt type I-II ascent	α,β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ double valued	$\gamma^\kappa=\{\gamma_1^\kappa,\gamma_2^\kappa\}=(\gamma^\alpha)^\beta$
2i22	two-imaginary noncpt type II-II ascent	α,β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ has 4 values	$\gamma^\kappa=\{\gamma_1^\kappa,\gamma_2^\kappa\}=\{\gamma^{\alpha,\beta}\}^\sigma$
2r22	two-real type II-II descent	α,β real, parity, type 2 $(\gamma_\alpha)_\beta$ single valued	$\gamma_\kappa=(\gamma_\alpha)_\beta$
2r21	two-real type II-I descent	α,β real, parity, type 2 $(\gamma_\alpha)_\beta$ double valued	$\gamma_\kappa=\{\gamma_\kappa^1,\gamma_\kappa^2\}=(\gamma_\alpha)_\beta$
2r11	two-real type I-I descent	α,β real, parity, type 2 $(\gamma_\alpha)_\beta$ has 4 values	$\gamma_\kappa=\{\gamma_\kappa^1,\gamma_\kappa^2\}=\{(\gamma_\alpha)_\beta\}^\sigma$
2rn	two-real nonparity ascent	α,β real, nonparity	
2ic	two-imaginary compact descent	α,β compact imaginary	

∗: defect=1 (see Definition 9.1.4).
†: See Section 5

4.3 Length 3

Suppose $\alpha\in S,\beta=\sigma(\alpha)\in S$, and $\langle\alpha,\beta^\vee\rangle\ne 0$ (equivalently ±1). In this case $w_\kappa=s_\alpha s_\beta s_\alpha=s_\beta s_\alpha s_\beta\in W^\sigma$.

Again it is easy to see that α,β have the same type with respect to θ. Here are the cases.

Length 3

type	LV terminology	definition	Cayley transform
3C+	three-complex ascent	α, β complex $\theta\alpha > 0$, $\theta\alpha \neq \beta$	
3C-	three-complex descent	α, β complex $\theta\alpha < 0$, $\theta\alpha \neq \beta$	
3Ci *	three-semiimaginary ascent	α, β complex, $\theta\alpha = \beta$	$\gamma^\kappa = (s_\alpha \times \gamma)^\beta \cap (s_\beta \times \gamma)^\alpha$
3Cr *	three-semireal descent	α, β complex, $\theta\alpha = -\beta$	$\gamma_\kappa = (s_\alpha \times \gamma)_\beta \cap (s_\beta \times \gamma)_\alpha$
3i *	three imaginary noncompact ascent	α, β noncpt imaginary, type 1	$\gamma^\kappa = s_\alpha \times \gamma^\beta = s_\beta \times \gamma^\alpha$
3r *	three-real descent	α, β real, parity, type 2	$\gamma_\kappa = s_\alpha \times \gamma_\beta = s_\beta \times \gamma_\alpha$
3rn	three-real non-parity ascent	α, β real, nonparity	
3ic	three-imaginary compact descent	α, β noncompact imaginary	

$*$: defect=1 (see Definition 9.1.4).

The type of κ depends on a parameter $\gamma \in \mathcal{D}^\sigma$. We say κ is of a given type with respect to γ.

{d:type}

Definition 4.3.1 *If $\kappa \in \overline{S}$ and $\gamma \in \mathcal{D}^\sigma$ write $t_\gamma(\kappa)$ for the type of κ with respect to γ.*

For the notion of ascent/descent in these tables see Lemma 9.3.1.

{d:tau}

Definition 4.3.2 *The τ-invariant of $\gamma \in \mathcal{D}^\sigma$ is*

$$\tau(\gamma) = \{\kappa \in \overline{S} \mid \kappa \textit{ is a descent for } \gamma\}.$$

Here is a list of the $10 + 12 + 8 = 30$ types:

Table 4.3.3

$\ell(\kappa)$	ascent ($\kappa \notin \tau(\gamma)$)	descent ($\kappa \in \tau(\gamma)$)
1	1C+, 1i1, 1i2f, 1i2s, 1rn	1C-, 1r1f, 1r1s, 1r2, 1ic
2	2C+, 2Ci, 2i11, 2i12, 2i22, 2rn	2C-, 2Cr, 2r11, 2r21, 2r22, 2ic
3	3C+, 3Ci, 3i, 3rn	3C–, 3Cr, 3r, 3ic

{table:types}

$*$: defect=1 (see Definition 9.1.4).

5 Cases 2i12 and 2r21

{s:2i12}

Recall we need to address the issue, discussed at the beginning of Section 4, of types 2i12/2r21.

Suppose $\gamma \in \mathcal{D}^\sigma$, and $\kappa = \{\alpha, \beta\}$ is of type 2i12 with respect to γ. Then $s_\alpha \times \gamma = s_\beta \times \gamma$ is also of type 2i12. Label these two parameters $\{\gamma_1, \gamma_2\}$. Then, on the level of non-extended parameters, $(\gamma_1)^\kappa = (\gamma_2)^\kappa$ is double-valued, label these two parameters λ_1, λ_2.

Thus we are given two *unordered* pairs $\{\gamma_1, \gamma_2\}$ and $\{\lambda_1, \lambda_2\}$; κ is of type 2i12 and 2r21 respectively.

We want to define the extensions inductively, starting on the fundamental Cartan. Assume that we have already chosen an extension $\widehat{\gamma}_1^+$ of γ_1.

The Cayley transform of $\widehat{\gamma}_1^+$ by κ is a well defined pair of extended parameters (see the old version of these notes). Use this to define the + labelling on the extended parameters for λ_i:

(5.1)(a) $$(\widehat{\gamma}_1^+)^\kappa = \{\widehat{\lambda}_1^+, \widehat{\lambda}_2^+\}$$

Now fix an extension $\widehat{\gamma}_2^+$ of γ_2. Then $(\widehat{\gamma}_2^+)^\kappa$ is either $\{\widehat{\lambda}_1^+, \widehat{\lambda}_2^-\}$ or $\{\widehat{\lambda}_1^-, \widehat{\lambda}_2^+\}$. (This computation was done in the old notes, in $SL(4,\mathbb{R})$). After switching λ_1, λ_2 if necessary, we can assume

(5.1)(b) $$(\widehat{\gamma}_2^+)^\kappa = \{\widehat{\lambda}_1^+, \widehat{\lambda}_2^-\}$$

Alternatively, define $\widehat{\gamma}_2^+$ by the requirement: $(\widehat{\gamma}_2^+)^\kappa = \{\widehat{\lambda}_1^+, \widehat{\lambda}_2^-\}$ (then $(\widehat{\gamma}_2^-)^\kappa = \{\widehat{\lambda}_1^-, \widehat{\lambda}_2^+\}$).

Clearly the chosen extensions of λ_1, λ_2 depend on the extensions of γ_1, γ_2, and also the fact that we've chosen an order of each pair (γ_1, γ_2) and (λ_1, λ_2). For example, suppose we switch γ_1, γ_2, but keep the same extensions of these two parameters. This would induce new definitions of $\widehat{\lambda}_1^+, \widehat{\lambda}_2^+$: $\widehat{\lambda}_1^+$ wouldn't change, but what we labelled $\widehat{\lambda}_2^+$ before would now be labelled $\widehat{\lambda}_2^-$. See the table at the end of this section.

Conclusion: some additional information is needed to determine unique preferred extensions for λ_1, λ_2.

What was here before was incorrect, and I don't know how to fix it at the moment. So I'm leaving this as a conjecture.

{c:distinguish}

Conjecture 5.2 *Assume $\kappa = \{\alpha, \beta\}$, where α, β are orthogonal and interchanged by σ. Suppose κ is of type* 2i12 *or* 2r21 *for parameters γ and*

$\gamma' = s_\alpha \times \gamma = s_\beta \times \gamma$. *There is a canonical way to distinguish* γ, γ'*, and so write them as an ordered pair* (γ_1, γ_2).

Assuming this, start with the ordered pair (γ_1, γ_2), and assume we have chosen $\widehat{\gamma}_1^+$. Then the Cayley transform $(\gamma_1)^\kappa = (\gamma_2)^\kappa$ is an ordered pair (λ_1, λ_2). Define $\widehat{\lambda}_1^+, \widehat{\lambda}_2^+$ by (a): $(\widehat{\gamma}_1^+)^\kappa = \{\widehat{\lambda}_1^+, \lambda_2^+\}$. Furthermore define $\widehat{\gamma}_2^+$ by the requirement: $(\widehat{\gamma}_2^+)^\kappa = \{\widehat{\lambda}_1^+, \widehat{\lambda}_2^-\}$ (exactly one of the two extensions of $\widehat{\gamma}_2$ satisfy this).

Clearly the choice of these extensions depends on the fact that (γ_1, γ_2) and (λ_1, λ_2) are ordered pairs.

There might be an issue of consistency here: if we've already chosen $\widehat{\gamma}_2^+$, it may conflict with the one just made. (Similar issues possibly could arise elsewhere.) Let's ignore this issue for now, and hope it works. If so, we have chosen a preferred extension of each parameter, and all formulas are in terms of this extension. See the Desideratum 3.1.

5.1 A Table

It is possible that the ordering we've chosen in the previous section, while natural, isn't the right one. Hopefully this table will never be needed, but it shows the affect of different choices.

Assume we've decided on an ordering of γ_1, γ_2, and extensions of these two parameters, labelled $+$. This uniquely determines an ordering of λ_1, λ_2, and extensions of these. This is the first row of the table.

The subsequent rows show the affect of the choices. For example, suppose we keep the same order of γ_1, γ_2, but choose the other extension of γ_2. Then since

$$\begin{aligned}\widehat{\gamma}_1^+ &\to \widehat{\lambda}_1^+, \widehat{\lambda}_2^+\\ \widehat{\gamma}_2^+ &\to \widehat{\lambda}_1^+, \widehat{\lambda}_2^-\end{aligned}$$

with our new choices we have

$$\begin{aligned}\widehat{\gamma}_1^+ &\to \widehat{\lambda}_1^+, \widehat{\lambda}_2^+\\ \widehat{\gamma}_2^- &\to \widehat{\lambda}_1^-, \widehat{\lambda}_2^+\end{aligned}$$

meaning the sign has changed on the first member of the target pair. So we should change their order:

$$\begin{aligned}\widehat{\gamma}_1^+ &\to \widehat{\lambda}_2^+, \widehat{\lambda}_1^+\\ \widehat{\gamma}_2^- &\to \widehat{\lambda}_2^+, \widehat{\lambda}_1^-\end{aligned}$$

This amounts to switching λ_1, λ_2, giving the second row of the table.

$\widehat{\gamma}_1^+$	$\widehat{\gamma}_2^+$	$\widehat{\lambda}_1^+$	$\widehat{\lambda}_2^+$
$\widehat{\gamma}_1^+$	$\widehat{\gamma}_2^-$	$\widehat{\lambda}_2^+$	$\widehat{\lambda}_1^+$
$\widehat{\gamma}_1^-$	$\widehat{\gamma}_2^+$	$\widehat{\lambda}_2^-$	$\widehat{\lambda}_1^-$
$\widehat{\gamma}_1^-$	$\widehat{\gamma}_2^-$	$\widehat{\lambda}_1^-$	$\widehat{\lambda}_2^-$
$\widehat{\gamma}_2^+$	$\widehat{\gamma}_1^+$	$\widehat{\lambda}_1^+$	$\widehat{\lambda}_2^-$
$\widehat{\gamma}_2^+$	$\widehat{\gamma}_1^-$	$\widehat{\lambda}_2^-$	$\widehat{\lambda}_1^+$
$\widehat{\gamma}_2^-$	$\widehat{\gamma}_1^+$	$\widehat{\lambda}_2^+$	$\widehat{\lambda}_1^-$
$\widehat{\gamma}_2^-$	$\widehat{\gamma}_1^-$	$\widehat{\lambda}_1^-$	$\widehat{\lambda}_2^+$

5.2 The sign $\epsilon(\gamma, \lambda)$

The sign which arises in the `2i12/2r21` cases appears frequently, so we need some notation for it.

{`d:epsilon`}

Definition 5.2.1 *Suppose $\gamma, \lambda \in \mathcal{D}^\sigma$, $\kappa \in \overline{S}$, and $\gamma \xrightarrow{\kappa} \lambda$.*

If $t_\gamma(\kappa) \neq$`2r21` *define $\epsilon(\gamma, \lambda) = 1$.*

Assume $t_\gamma(\kappa) =$`2r21`*, so $t_\lambda(\kappa) =$*`2i12`*. By Lemma 5.2, γ and λ are members of ordered pairs (γ_1, γ_2) and (λ_1, λ_2), respectively. Define:*

$$\epsilon(\gamma_i, \lambda_j) = \begin{cases} -1 & i = j = 2 \\ 1 & \text{otherwise} \end{cases} \tag{5.2.2}$$

6 Formulas for the H action on M

{`s:formulas`}

Implicit in the following formulas is the fact that we have chosen an extension of each parameter as discussed in Section 3.2. For each $\gamma \in \mathcal{D}^\sigma$ we have chosen an extension $\widehat{\gamma}^+$; in the following formulas each a_γ is really $a_{\widehat{\gamma}^+}$.

6.1 Length 1

Suppose $\sigma(\alpha) = \alpha$, and $\gamma \in \mathcal{D}^\sigma$. Then $T_{w_\kappa}(\gamma)$ is given by the usual formulas, taking the quotients in types `1i2s,1r1s` into account. The first column is $t_\gamma(\kappa)$, the type of κ with respect to γ.

`1C+`: $T_{w_\kappa}(a_\gamma) = a_{w_\kappa \times \gamma}$

`1C-`: $T_{w_\kappa}(a_\gamma) = (u-1)a_\gamma + ua_{w_\kappa \times \gamma}$

`1i1`: $T_{w_\kappa}(a_\gamma) = a_{w_\kappa \times \gamma} + a_{\gamma^\kappa}$

`1i2f`: $T_{w_\kappa}(a_\gamma) = a_\gamma + (a_{\gamma_1^\kappa} + a_{\gamma_2^\kappa})$

`1i2s`: $T_{w_\kappa}(a_\gamma) = a_\gamma$

`1ic`: $T_{w_\kappa}(a_\gamma) = ua_\gamma$

`1r1f`: $T_{w_\kappa}(a_\gamma) = (u-2)a_\gamma + (u-1)(a_{\gamma_\kappa^1} + a_{\gamma_\kappa^2})$

`1r1s`: $T_{w_\kappa}(a_\gamma) = (u-2)a_\gamma$

`1r2`: $T_{w_\kappa}(a_\gamma) = (u-1)a_\gamma - a_{w_\kappa \times \gamma} + (u-1)a_{\gamma_\kappa}$

`1rn`: $T_{w_\kappa}(a_\gamma) = -a_\gamma$

6.2 Length 2

`2C+`: $T_{w_\kappa}(a_\gamma) = a_{w_\kappa \times \gamma}$

`2C-`: $T_{w_\kappa}(a_\gamma) = (u^2-1)a_\gamma + u^2 a_{w_\kappa \times \gamma}$

`2Ci`: $T_{w_\kappa}(a_\gamma) = ua_\gamma + (u+1)a_{\gamma^\kappa}$

`2Cr`: $T_{w_\kappa}(a_\gamma) = (u^2-u-1)a_\gamma + (u^2-u)a_{\gamma_\kappa}$

`2i11`: $T_{w_\kappa}(a_\gamma) = a_{w_\kappa \times \gamma} + a_{\gamma^\kappa}$

`2i12`: $T_{w_\kappa}(a_\gamma) = a_\gamma + \sum_{\lambda | \lambda \xrightarrow{\kappa} \gamma} \epsilon(\lambda, \gamma) a_\lambda$

`2i22`: $T_{w_\kappa}(a_\gamma) = a_\gamma + (a_{\gamma_1^\kappa} + a_{\gamma_2^\kappa})$

`2r22`: $T_{w_\kappa}(a_\gamma) = (u^2-1)a_\gamma - a_{w_\kappa \times \gamma} + (u^2-1)a_{\gamma_\kappa}$

`2r21`: $T_{w_\kappa}(a_\gamma) = (u^2-2)a_\gamma + (u^2-1) \sum_{\lambda | \gamma \xrightarrow{\kappa} \lambda} \epsilon(\gamma, \lambda) a_\lambda$

`2r11`: $T_{w_\kappa}(a_\gamma) = (u^2-2)a_\gamma + (u^2-1)(a_{\gamma_\kappa^1} + a_{\gamma_\kappa^2})$

`2rn`: $T_{w_\kappa}(a_\gamma) = -a_\gamma$

`2ic`: $T_{w_\kappa}(a_\gamma) = u^2 a_\gamma$

Remark 6.2.1 In the `2i12` case, if $\lambda \overset{\kappa}{\to} \gamma$, recall λ, γ occur in ordered pairs (λ_1, λ_2) and (γ_1, γ_2) (Section 5). With $\epsilon(\lambda, \gamma)$ given by Definition 5.2.1, the stated formula in this case is shorthand for:

$$T_{w_\kappa}(a_{\gamma_1}) = a_{\gamma_1} + (a_{\lambda_1} + a_{\lambda_2})$$

$$T_{w_\kappa}(a_{\gamma_2}) = a_{\gamma_2} + (a_{\lambda_1} - a_{\lambda_2}),$$

We could state the other formulas using $\epsilon(\gamma, \lambda)$ as well, for example if $t_\gamma(\kappa) =$`2r22`. But this doesn't seem worth it.

6.3 Length 3

`3C+`: $T_{w_\kappa}(a_\gamma) = w_\kappa \times a_\gamma$

`3C-`: $T_{w_\kappa}(a_\gamma) = (u^3 - 1)a_\gamma + u^3(a_{w_\kappa \times a_\gamma})$

`3Ci`: $T_{w_\kappa}(a_\gamma) = u a_\gamma + (u + 1)a_{\gamma^\kappa}$

`3Cr`: $T_{w_\kappa}(a_\gamma) = (u^3 - u - 1)a_\gamma + (u^3 - u)a_{\gamma_\kappa}$

`3i`: $T_{w_\kappa}(a_\gamma) = u a_\gamma + (u + 1)a_{\gamma^\kappa}$

`3r`: $T_{w_\kappa}(a_\gamma) = (u^3 - u - 1)a_\gamma + (u^3 - u)a_{\gamma_\kappa}$

`3rn`: $T_{w_\kappa}(a_\gamma) = -a_\gamma$

`3ic`: $T_{w_\kappa}(a_\gamma) = u^3 a_\gamma$

Remark 6.3.1 In terms of extended parameters, the `2i12/2r21` cases are simpler. Suppose κ is of type `2i12` with respect to an ordinary parameter γ.

Suppose $\widehat{\gamma}$ is an extension of γ. On the level of extended parameters $\widehat{\gamma}$ has a well defined Cayley transform $(\widehat{\gamma})^\kappa$, which is an unordered pair of extended parameters. Then:

$$T_{w_\kappa}(a_{\widehat{\gamma}}) = a_{\widehat{\gamma}} + \sum_{\widehat{\lambda} \in (\widehat{\gamma})^\kappa} a_{\widehat{\lambda}}$$

If we write $-\widehat{\gamma}$ for the opposite extension then $(-\widehat{\gamma})^\kappa$ is the same set of two elements, with the sign changed on one of them. There are no choices involved here.

We have the usual notion of the W-graph associated to this Hecke algebra action.

Definition 6.3.2 *Suppose $\kappa \in \overline{S}, \gamma, \lambda \in \mathcal{D}^\sigma$, and $\kappa \in \tau(\gamma)$. Then we say $\gamma \overset{\kappa}{\to} \lambda$ if $\kappa \notin \tau(\lambda)$, and a_λ appears in $T_{w_\kappa}(a_\gamma)$.*

7 Kazhdan-Lusztig-Vogan algorithm

{s:klv}

We use a hybrid notation combining Fokko's notes *Implementation of Kazhdan-Lusztig Algorithm*, and [5].

Recall $\mathbf{H}$ is an algebra over $\mathbb{Z}[u, u^{-1}]$, with generators parametrized by $\overline{S}$. Also M is a $\mathbf{H}$-module, with $\mathbb{Z}[u, u^{-1}]$-basis $\{a_\gamma \mid \gamma \in \mathcal{D}^\sigma\}$.

Write $\mathbf{D}$ for the canonical involution of M. It satisfies

(7.1)
$$\mathbf{D}(um) = u^{-1}\mathbf{D}(m).$$

The order on $\mathcal{D}^\sigma$ is defined in [5, 5.1], and length $\ell(\gamma)$ is inherited from $\mathcal{D}$.

{t:Cdelta}

Theorem 7.2 ([5], Theorem 5.2) *There is a unique basis $\{C_\delta \mid \delta \in \mathcal{D}^\sigma\}$ of M satisfying the following conditions. There are polynomials $P^\sigma(\gamma, \delta) \in \mathbb{Z}[u]$ such that*

(7.3)
$$C_\delta = \sum_\gamma P^\sigma(\gamma, \delta) a_\gamma,$$

and:

(1) $\mathbf{D}(C_\gamma) = u^{-\ell(\gamma)} C_\gamma$*;*

(2) $P^\sigma(\gamma, \delta) \neq 0$ *implies* $\gamma \leq \delta$*;*

(3) $P^\sigma(\gamma, \gamma) = 1$

(4) $\deg(P^\sigma(\gamma, \delta)) \leq \frac{1}{2}(\ell(\delta) - \ell(\gamma) - 1)$.

Introduce a new variable v satisfying $v^2 = u$, and tensor everything with $\mathbb{Z}[v, v^{-1}]$, so $\mathbf{H}$ becomes an algebra over $\mathbb{Z}[v, v^{-1}]$. Define

(7.4)(a)
$$\widehat{a}_\gamma = v^{-\ell(\gamma)} a_\gamma$$

and

(7.4)(b) $$\widehat{C}_\delta = v^{-\ell(\delta)}C_\delta$$

With this notation Theorem 7.2 can be written as

{t:Chatdelta}

Theorem 7.5 *There is a unique basis* $\{\widehat{C}_\delta \mid \delta \in \mathcal{D}^\sigma\}$ *of* M *satisfying the following conditions. There are polynomials* $\widehat{P}^\sigma(\gamma,\delta)(v) \in \mathbb{Z}[v^{-1}]$ *such that*

(7.6) $$\widehat{C}_\delta = \sum_\gamma \widehat{P}^\sigma(\gamma,\delta)(v)\widehat{a}_\gamma,$$

and:

(1) $\mathbf{D}(\widehat{C}_\gamma) = \widehat{C}_\gamma$*;*

(2) $\widehat{P}^\sigma(\gamma,\delta) \neq 0$ *implies* $\gamma \le \delta$*;*

(3) $\widehat{P}^\sigma(\gamma,\gamma) = 1$*;*

(4) if $\gamma \neq \delta$*, then* $\widehat{P}^\sigma(\gamma,\delta) \in v^{-1}\mathbb{Z}[v^{-1}]$*; and*

(5) $\deg(\widehat{P}^\sigma(\gamma,\delta))(v^{-1}) \le \ell(\gamma)-\ell(\delta)$

It is easy to see that

(7.7) $$\widehat{P}^\sigma(\gamma,\delta)(v) = v^{\ell(\gamma)-\ell(\delta)}P^\sigma(\gamma,\delta)(v^2).$$

Fix $\gamma < \delta$, and suppose

(7.8)(a) $$P^\sigma(\gamma,\delta) = c_0 + c_1u + \cdots + c_nu^n$$

with

(7.8)(b) $$n = \begin{cases} (\ell(\delta)-\ell(\gamma)-1)/2 & \ell(\delta)-\ell(\gamma) \text{ odd} \\ (\ell(\delta)-\ell(\gamma)-2)/2 & \ell(\delta)-\ell(\gamma) \text{ even} \end{cases}$$

Then

(7.9)

$$\widehat{P}^\sigma(\gamma,\delta) = \begin{cases} c_nv^{-1} + c_{n-1}v^{-3} + \cdots + c_0v^{\ell(\gamma)-\ell(\delta)=-(2n+1)} & \ell(\delta)-\ell(\gamma) \text{ odd} \\ c_nv^{-2} + c_{n-1}v^{-4} + \cdots + c_0v^{\ell(\gamma)-\ell(\delta)=-(2n+2)} & \ell(\delta)-\ell(\gamma) \text{ even} \end{cases}$$

or alternatively

$$\widehat{P}^\sigma(\gamma,\delta) = \begin{cases} v^{-1}[c_n + c_{n-1}v^{-2} + \cdots + c_0v^{\ell(\gamma)-\ell(\delta)+1}] & \ell(\gamma)-\ell(\delta) \text{ odd} \\ v^{-1}[c_nv^{-1} + c_{n-1}v^{-3} + \cdots + c_0v^{\ell(\gamma)-\ell(\delta)+1}] & \ell(\gamma)-\ell(\delta) \text{ even} \end{cases}$$

8 Action of $T_{w_\kappa} + 1$

An important role is played by the operator $T_{w_\kappa} + 1$, which we renormalize. For $\kappa \in \overline{S}$ define

$$\widehat{T}_\kappa = v^{-\ell(\kappa)}(T_{w_\kappa} + 1) \tag{8.1}$$

Note that Fokko has both T_s and t_s. One can deduce they are related by $t_s = v^{-1}T_s$. These correspond to our T_{w_κ} and $v^{-\ell(w_\kappa)}T_{w_\kappa}$, respectively. Also Fokko has an operator c_s, which can be seen to be $t_s + v^{-1} = v^{-1}(T_s + 1)$, which is our $\widehat{T}_\kappa$.

8.1 Formulas for $\widehat{T}_\kappa$

{s:formulasfortkap

Here are 30 formulas, for $\widehat{T}_\kappa(\hat{a}_\gamma)$, depending on the type of κ with respect to γ.

Type 1: $\widehat{T}_\kappa = v^{-1}(T_{w_\kappa} + 1) = v^{-1}(T_{s_\alpha} + 1)$

These are copied from [3] Section 1.

$t_\gamma(\kappa)$	$\widehat{T}_\kappa(\hat{a}_\gamma)$
1C+	$v^{-1}\hat{a}_\gamma + \hat{a}_{w_\kappa \times \gamma}$
1C-	$v\hat{a}_\gamma + \hat{a}_{w_\kappa \times \gamma}$
1i1	$v^{-1}(\hat{a}_\gamma + \hat{a}_{w_\kappa \times \gamma}) + \hat{a}_{\gamma^\kappa}$
1i2f	$2v^{-1}\hat{a}_\gamma + (\hat{a}_{\gamma_1^\kappa} + \hat{a}_{\gamma_2^\kappa})$
1i2s	0
1r1f	$(v - v^{-1})\hat{a}_\gamma + (1 - v^{-2})(\hat{a}_{\gamma_\kappa^1} + \hat{a}_{\gamma_\kappa^2})$
1r1s	$(v - v^{-1})\hat{a}_\gamma$
1r2	$v\hat{a}_\gamma - v^{-1}\hat{a}_{w_\kappa \times \gamma} + (1 - v^{-2})\hat{a}_{\gamma_\kappa}$
1rn	0
1ic	$(v + v^{-1})\hat{a}_\gamma$

Type 2: $\widehat{T}_\kappa = v^{-2}(T_{w_\kappa} + 1) = v^{-2}(T_{s_\alpha s_\beta} + 1)$

$t_\gamma(\kappa)$	$\widehat{T}_\kappa(\hat{a}_\gamma)$
2C+	$v^{-2}\hat{a}_\gamma + \hat{a}_{w_\kappa \times \gamma}$
2C-	$v^{2}\hat{a}_\gamma + \hat{a}_{w_\kappa \times \gamma}$
2Ci	$(v+v^{-1})[v^{-1}\hat{a}_\gamma + \hat{a}_{\gamma^\kappa}]$
2Cr	$(v^2-1)\hat{a}_\gamma + (v-v^{-1})\hat{a}_{\gamma_\kappa}$
2i11	$v^{-2}(\hat{a}_\gamma + \hat{a}_{w_\kappa \times \gamma}) + \hat{a}_{\gamma^\kappa}$
2i12	$2v^{-2}\hat{a}_\gamma + \sum_{\gamma' \mid \gamma' \xrightarrow{\kappa} \gamma} \epsilon(\gamma', \gamma)\hat{a}_{\gamma'}$
2i22	$2v^{-2}\hat{a}_\gamma + \hat{a}_{\gamma_1^\kappa} + \hat{a}_{\gamma_2^\kappa}$
2r22	$v^{2}\hat{a}_\gamma - v^{-2}\hat{a}_{w \times \gamma} + (1-v^{-4})\hat{a}_{\gamma_\kappa}$
2r21	$(v^2-v^{-2})\hat{a}_\gamma + (1-v^4)\sum_{\gamma' \mid \gamma \xrightarrow{\kappa} \gamma'} \epsilon(\gamma, \gamma')\hat{a}_{\gamma'}$
2r11	$(v^2-v^{-2})\hat{a}_\gamma + (1-v^{-4})(\hat{a}_{\gamma_\kappa^1} + a_{\gamma_\kappa^2})$
2rn	0
2ic	$(v^2+v^{-2})\hat{a}_\gamma$

Type 3: $\widehat{T}_\kappa = v^{-3}(T_{w_\kappa}+1) = v^{-3}(T_{s_\alpha s_\beta s_\alpha}+1)$

$t_\gamma(\kappa)$	$\widehat{T}_\kappa(\hat{a}_\gamma)$
3C+	$v^{-3}\hat{a}_\gamma + \hat{a}_{w_\kappa \times \gamma}$
3C-	$v^{3}\hat{a}_\gamma + \hat{a}_{w_\kappa \times \gamma}$
3Ci	$(v+v^{-1})v^{-2}\hat{a}_\gamma + (v+v^{-1})\hat{a}_{\gamma^\kappa}$
3Cr	$(v^2-v^{-2})v\hat{a}_\gamma + (v^2-v^{-2})v^{-1}\hat{a}_{\gamma_\kappa}$
3i	$(v+v^{-1})v^{-2}\hat{a}_\gamma + (v+v^{-1})\hat{a}_{\gamma\kappa}$
3r	$(v^2-v^{-2})v\hat{a}_\gamma + (v^2-v^{-2})v^{-1}\hat{a}_{\gamma\kappa}$
3rn	0
3ic	$(v^3+v^{-3})\hat{a}_\gamma$

8.2 Summary

{s:summary}

We write some of these formulas in a slightly different form in the following table of all $\widehat{T}_\kappa\hat{a}_\gamma$.

Table 8.2.1

$t_\gamma(\kappa)$	$\widehat{T}_\kappa \hat{a}_\gamma$	$t_\gamma(\kappa)$	$T_\kappa \hat{a}_\gamma$
`1C+`	$[\hat{a}_{w_\kappa\times\gamma} + v^{-1}a_\gamma]$	`1C-`	$v[\hat{a}_\gamma + v^{-1}\hat{a}_{w_\kappa\times\gamma}]$
`1i1`	$[\hat{a}_{\gamma^\kappa} + v^{-1}(\hat{a}_\gamma + \hat{a}_{w_\kappa\times\gamma})]$	`1r1f`	$(v-v^{-1})[\hat{a}_\gamma + v^{-1}(\hat{a}_{\gamma^1_\kappa} + \hat{a}_{\gamma^2_\kappa})]$
`1i2f`	$[\hat{a}_{\gamma^\kappa_1} + v^{-1}\hat{a}_\gamma] + [\hat{a}_{\gamma^\kappa_2} + v^{-1}\hat{a}_\gamma]$	`1r2`	$v[\hat{a}_\gamma + v^{-1}\hat{a}_{\gamma_\kappa}]$ $-v^{-1}[\hat{a}_{w_\kappa\times\gamma} + v^{-1}\hat{a}_{\gamma_\kappa}]$
`1i2s`	0	`1r1s`	$(v-v^{-1})[\hat{a}_\gamma]$
`1rn`	0	`1ic`	$(v+v^{-1})[\hat{a}_\gamma]$
`2C+`	$[\hat{a}_{w_\kappa\times\gamma} + v^{-2}\hat{a}_\gamma]$	`2C-`	$v^2[\hat{a}_\gamma + v^{-2}\hat{a}_{w_\kappa\times\gamma}]$
`2Ci`	$(v+v^{-1})[\hat{a}_{\gamma^\kappa} + v^{-1}\hat{a}_\gamma]$	`2Cr`	$v(v-v^{-1})[\hat{a}_\gamma + v^{-1}\hat{a}_{\gamma_\kappa}]$
`2i11`	$[\hat{a}_{\gamma^\kappa} + v^{-2}(\hat{a}_\gamma + \hat{a}_{w_\kappa\times\gamma})]$	`2r11`	$(v^2-v^{-2})[\hat{a}_\gamma + v^{-2}(\hat{a}_{\gamma^1_\kappa} + a_{\gamma^2_\kappa})]$
`2i12`	$\sum_{\gamma'\mid\gamma'\xrightarrow{\kappa}\gamma} \epsilon(\gamma',\gamma)[\hat{a}_{\gamma'} + v^{-2}\sum_{\mu\mid\gamma'\xrightarrow{\kappa}\mu} \epsilon(\gamma',\mu)\hat{a}_\mu]$	`2r21`	$(v^2-v^{-2})[\hat{a}_\gamma + v^{-2}\sum_{\gamma'\mid\gamma'\xrightarrow{\kappa}\gamma} \epsilon(\gamma',\gamma)\hat{a}_{\gamma'}$
`2i22`	$[\hat{a}_{\gamma^\kappa_1} + v^{-2}\hat{a}_\gamma] + [\hat{a}_{\gamma^\kappa_2} + v^{-2}\hat{a}_\gamma]$	`2r22`	$v^2[\hat{a}_\gamma + v^{-2}\hat{a}_{\gamma_\kappa}]$ $-v^{-2}[\hat{a}_{w_\kappa\times\gamma} + v^{-2}\hat{a}_{\gamma_\kappa}]$
`2rn`	0	`2ic`	$(v^2+v^{-2})\hat{a}_\gamma$
`3C+`	$[\hat{a}_{w_\kappa\times\gamma} + v^{-3}\hat{a}_\gamma]$	`3C-`	$v^3[\hat{a}_\gamma + v^{-3}\hat{a}_{w_\kappa\times\gamma}]$
`3Ci`, `3i`	$(v+v^{-1})[\hat{a}_{\gamma^\kappa} + v^{-2}\hat{a}_\gamma]$	`3Cr`, `3r`	$v(v^2-v^{-2})[\hat{a}_\gamma + v^{-2}\hat{a}_{\gamma_\kappa}]$
`3rn`	0	`3ic`	$(v^3+v^{-3})[\hat{a}_\gamma]$

{table:Tagamma}

We're going to simplify this table - see Table 9.1.3.

Remark 8.2.2 The identity in the `2i12` case is tricky, let's write it out. Suppose $t_\gamma(\kappa)$ =`2i12`, and γ is one member of the ordered pair (γ_1, γ_2). Simlarly γ^κ is an ordered pair (γ'_1, γ'_2).

The formula from Section 8.1 is:

(8.2.3)(a) $$2v^{-2}a_\gamma + \epsilon(\gamma'_1,\gamma)a_{\gamma'_1} + \epsilon(\gamma'_2,\gamma)a_{\gamma'_2}$$

whereas Table 8.2.1 gives:

$$\epsilon(\gamma'_1,\gamma)[a_{\gamma'_1} + v^{-2}(\epsilon(\gamma'_1,\gamma_1)a_{\gamma_1} + \epsilon(\gamma'_1,\gamma_2)a_{\gamma_2})]+$$
$$\epsilon(\gamma'_2,\gamma)[a_{\gamma'_2} + v^{-2}(\epsilon(\gamma'_2,\gamma_1)a_{\gamma_1} + \epsilon(\gamma'_2,\gamma_2)a_{\gamma_2})].$$

Using the definition of ϵ this equals:

(8.2.3)(b) $$[a_{\gamma'_1} + v^{-2}(a_{\gamma_1} + a_{\gamma_2})] + \epsilon(\gamma'_2,\gamma)[a_{\gamma'_2} + v^{-2}(a_{\gamma_1} - a_{\gamma_2})].$$

Plugging $\gamma = \gamma_1$ or γ_2 in to (a) and (b) and comparing confirms the identity.

9 Image of $\widehat{T}_\kappa$

9.1 $\widehat{T}_\kappa(\hat{a}_\gamma)$

{l:kappadescents}

Lemma 9.1.1 *Fix $\kappa \in \overline{S}$.*

(1) The image of $\widehat{T}_\kappa$ is equal to the $(v^{\ell(\kappa)} + v^{-\ell(\kappa)})$ eigenspace of $\widehat{T}_\kappa$. This is also equal to the kernel of $T_\kappa - v^{\ell(\kappa)}$.

(2) Suppose $\kappa \in \tau(\lambda)$. For each λ' satisfying $\lambda \xrightarrow{\kappa} \lambda'$, the sign $\epsilon(\lambda, \lambda') = \pm 1$ (Definition 5.2.1) is the unique integer such that

$$\hat{a}^\kappa_\lambda = \hat{a}_\lambda + v^{\ell(\lambda')-\ell(\lambda)} \sum_{\lambda' \mid \lambda \xrightarrow{\kappa} \lambda'} \epsilon(\lambda, \lambda')\hat{a}_{\lambda'} \tag{9.1.2}$$

belongs to the image of $\widehat{T}_\kappa$.

(3) The elements

$$\{\hat{a}^\kappa_\lambda \mid \gamma \in \mathcal{D}^\sigma, \kappa \in \tau(\gamma)\}$$

form a basis of the image of $\widehat{T}_\kappa$.

Part (1) follows from the quadratic relation (2.3). Statements (2) and (3) follow from an examination of Table 8.2.1. In each entry of the table the terms in square brackets are the $\hat{a}_\delta$ (not including the $v^{\mathrm{def}_\delta(\kappa)}$ term). This amounts to the fact that we can rewrite Table 8.2.1 as in Table 9.1.3.

Recall $\epsilon(\delta, \delta') = 1$ except in cases `2i12/2r21`.

Table 8.2.1 now simplifies.

Table 9.1.3

$t_\gamma(\kappa)$	$\widehat{T}_\kappa(\hat{a}_\gamma)$	$t_\gamma(\kappa)$	$\widehat{T}_\kappa(\hat{a}_\gamma)$
1C+	$\hat{a}^\kappa_{w_\kappa \times \gamma}$	1C-	$v\hat{a}^\kappa_\gamma$
1i1	$\hat{a}^\kappa_{\gamma^\kappa}$	1r1f	$(v - v^{-1})\hat{a}^\kappa_\gamma$
1i2f	$\hat{a}^\kappa_{\gamma_1^\kappa} + \hat{a}^\kappa_{\gamma_2^\kappa}$	1r2	$v\hat{a}^\kappa_\gamma - v^{-1}\hat{a}^\kappa_{w_\kappa \times \gamma}$
1i2s	0	1r1s	$(v - v^{-1})\hat{a}^\kappa_\gamma$
1rn	0	1ic	$(v + v^{-1})\hat{a}^\kappa_\gamma$
2C+	$\hat{a}^\kappa_{w_\kappa \times \gamma}$	2C-	$v^2\hat{a}^\kappa_\gamma$
2Ci	$(v + v^{-1})\hat{a}^\kappa_{\gamma^\kappa}$	2Cr	$v(v - v^{-1})\hat{a}^\kappa_\gamma$
2i11	$\hat{a}_{\gamma^\kappa}$	2r22	$v^2\hat{a}^\kappa_\gamma - v^{-2}\hat{a}^\kappa_{w_\kappa \times \gamma}$
2i12	$\sum_{\gamma' \xrightarrow{\kappa} \gamma} \epsilon(\gamma', \gamma)\hat{a}^\kappa_{\gamma'}$	2r21	$(v^2 - v^{-2})\hat{a}^\kappa_\gamma$
2i22	$\hat{a}^\kappa_{\gamma_1^\kappa} + \hat{a}^\kappa_{\gamma_2^\kappa}$	2r11	$(v^2 - v^{-2})\hat{a}^\kappa_\gamma$
2rn	0	2ic	$(v^2 + v^{-2})\hat{a}^\kappa_\gamma$
3C+	$\hat{a}^\kappa_{w_\kappa \times \gamma}$	3C-	$v^3\hat{a}^\kappa_\gamma$
3Ci, 3i	$(v + v^{-1})\hat{a}^\kappa_{\gamma^\kappa}$	3Cr, 3r	$v(v^2 - v^{-2})\hat{a}^\kappa_\gamma$
3rn	0	3ic	$(v^3 + v^{-3})\hat{a}^\kappa_\gamma$

{table:Tagammakapp

Those extra powers of v in cases 2CR,3Cr,3r are important. Suppose $\kappa \in \tau(\lambda)$. Then $\ell(\lambda')$ is the same for all $\lambda \xrightarrow{\kappa} \lambda'$; typically (always in the classical case) $\ell(\lambda) - \ell(\lambda') = \ell(\kappa)$. In general $\ell(\lambda) - \ell(\lambda') \le \ell(\kappa)$.

{d:defect}

Definition 9.1.4 *Suppose $\lambda \xrightarrow{\kappa} \lambda'$. Define the κ-defect of λ and λ' to be*

(9.1.5) $$\operatorname{def}_\lambda(\kappa) = \operatorname{def}_\kappa(\lambda') = \ell(\kappa) - \ell(\lambda) + \ell(\lambda').$$

(If $\{\lambda' \mid \lambda \xrightarrow{\kappa} \lambda'\} = \emptyset$ define $\operatorname{def}(\kappa, \lambda) = 0$).

Checking the cases gives:

{l:casesofd}

Lemma 9.1.6

$$\operatorname{def}_\lambda(\kappa) = \begin{cases} 1 & t_\lambda(\kappa) = \texttt{2Ci}, \texttt{3Ci}, \texttt{3i};\ \texttt{2Cr}, \texttt{3Cr}, \texttt{3r} \\ 0 & \textit{else} \end{cases}$$

We can now write Table 9.1.3 more concisely. For $\kappa \in \tau(\gamma)$ define:

$$\zeta_\kappa(\gamma) = \begin{cases} 1 & t_\gamma(\kappa) = \mathtt{1ic,2ic,3ic} \\ 0 & t_\gamma(\kappa) = \mathtt{1C-,2C-,3C-} \\ -1 & \text{otherwise} \end{cases}$$

{l:Tkappaagamma1}

Lemma 9.1.7 *Fix $\kappa \in \overline{S}$, $\gamma \in \mathcal{D}^\sigma$, and set $d = \mathrm{def}_\gamma(\kappa)$. Then*

$$\widehat{T}_\kappa(\hat{a}_\gamma) = \begin{cases} (v+v^{-1})^d \sum_{\gamma' \xrightarrow{\kappa} \gamma} \epsilon(\gamma',\gamma)\hat{a}^\kappa_{\gamma'} & \kappa \notin \tau(\gamma) \\ v^d[v^{\ell(\kappa)-d}\hat{a}^\kappa_\gamma + \zeta_\kappa(\gamma)v^{-\ell(\kappa)+d}\hat{a}^\kappa_{w_\kappa \times \gamma}] & \kappa \in \tau(\gamma) \end{cases} \tag{9.1.8}$$

In the second case:

(1) if $t_\gamma(\kappa) =$ `1r2,2r22` then $w_\kappa \times \gamma \neq \gamma$, and $\kappa \in \tau(w_\kappa \times \gamma)$ - there are two terms;

(2) if $t_\gamma(\kappa) =$ `1C-,2C-,3C-` then $w_\kappa \times \gamma \neq \gamma$, but $\kappa \notin \tau(w_\kappa \times \gamma)$ - since $\zeta = 0$ there is only one term;

(3) in all other cases $w_\kappa \times \gamma = \gamma$ (there is one term with a coefficient of $v^d(v^{\ell(\kappa)-d} \pm v^{-\ell(\kappa)+d})$.

9.2 $\widehat{T}_\kappa(\widehat{C}_\lambda)$ in terms of $\hat{a}^\kappa_\gamma$

We can now compute $\widehat{T}_\kappa(\widehat{C}_\lambda)$ in the basis of $\hat{a}^\kappa_\gamma$ (and the unkown $\widehat{P}^\sigma(\gamma,\lambda)$).

Write $\widehat{C}_\lambda = \sum_{\gamma\le\lambda} \widehat{P}^\sigma(\gamma,\lambda)\hat{a}_\gamma$, $\widehat{T}_\kappa(\widehat{C}_\lambda) = \sum_{\gamma\le\lambda} \widehat{P}^\sigma(\gamma,\lambda)\widehat{T}_\kappa(\hat{a}_\gamma)$. The condition $\gamma \le \lambda$ is superfluous because of the $\widehat{P}^\sigma(\gamma,\lambda)$ term. Apply Lemma 9.1.7.

$$
\begin{aligned}
\widehat{T}_\kappa(\widehat{C}_\lambda) &= \sum_\gamma P(\gamma,\lambda)\widehat{T}_\kappa(\hat{a}_\gamma)\\
&= \sum_{\gamma|\kappa\notin\tau(\gamma)} P(\gamma,\lambda)\widehat{T}_\kappa(\hat{a}_\gamma) + \sum_{\gamma|\kappa\in\tau(\gamma)} P(\gamma,\lambda)\widehat{T}_\kappa(\hat{a}_\gamma)\\
&= \sum_{\gamma|\kappa\notin\tau(\gamma)} \big[P(\gamma,\lambda)(v+v^{-1})^{\mathrm{def}_\gamma(\kappa)} \sum_{\gamma'|\gamma'\xrightarrow{\kappa}\gamma} \epsilon(\gamma',\gamma)\hat{a}^\kappa_{\gamma'}\big]+\\
&\quad \sum_{\gamma|\kappa\in\tau(\gamma)} v^{\mathrm{def}_\gamma(\kappa)}\big[P(\gamma,\lambda)(v^{\ell(\kappa)-\mathrm{def}_\gamma(\kappa)}\hat{a}^\kappa_\gamma + \zeta_\kappa(\gamma)v^{-\ell(\kappa)+\mathrm{def}_\gamma(\kappa)}\hat{a}^\kappa_{w_\kappa\times\gamma})\big]\\
&= \sum_{\gamma'|\kappa\in\tau(\gamma')} \big[(v+v^{-1})^{\mathrm{def}_\kappa(\gamma')} \sum_{\gamma|\gamma'\xrightarrow{\kappa}\gamma} P(\gamma,\lambda)\epsilon(\gamma',\gamma)\big]\hat{a}^\kappa_{\gamma'}+\\
&\quad \sum_{\gamma|\kappa\in\tau(\gamma)} v^{\mathrm{def}_\gamma(\kappa)}\big[P(\gamma,\lambda)v^{\ell(\kappa)-\mathrm{def}_\gamma(\kappa)}\hat{a}^\kappa_\gamma + P(\gamma,\lambda)\zeta_\kappa(\gamma)v^{-\ell(\kappa)+\mathrm{def}_\gamma(\kappa)}\hat{a}^\kappa_{w_\kappa\times\gamma}\big]
\end{aligned}
$$

Interchange γ,γ' in the first sum to conclude:

$$
\begin{aligned}
\widehat{T}_\kappa(\widehat{C}_\lambda) &= \sum_{\gamma|\kappa\in\tau(\gamma)} \big[(v+v^{-1})^{\mathrm{def}_\gamma(\kappa)} \sum_{\gamma'|\gamma\xrightarrow{\kappa}\gamma'} P(\gamma',\lambda)\epsilon(\gamma,\gamma')\big]\hat{a}^\kappa_\gamma+\\
&\quad \sum_{\gamma|\kappa\in\tau(\gamma)} v^{\mathrm{def}_\gamma(\kappa)}\big[v^{\ell(\kappa)-\mathrm{def}_\gamma(\kappa)}P(\gamma,\lambda) + \zeta_\kappa(\gamma)v^{-\ell(\kappa)+\mathrm{def}_\gamma(\kappa)}P(w_\kappa\times\gamma,\lambda)\big]\hat{a}^\kappa_\gamma
\end{aligned}
$$

{l:coefficient}

Lemma 9.2.1 *Fix γ with $\kappa\in\tau(\gamma)$. The coefficient of $\hat{a}^\kappa_\gamma$ in $\widehat{T}_\kappa(\widehat{C}_\lambda)$ is*

$$
\begin{gathered}
v^{\mathrm{def}_\gamma(\kappa)}\big[v^{\ell(\kappa)-\mathrm{def}_\gamma(\kappa)}\widehat{P}^\sigma(\gamma,\lambda)+\\
\zeta_\kappa(\gamma)v^{-\ell(\kappa)+\mathrm{def}_\gamma(\kappa)}\widehat{P}^\sigma(w_\kappa\times\gamma,\lambda)]+\\
(v+v^{-1})^{\mathrm{def}_\gamma(\kappa)}\sum_{\gamma'|\gamma\xrightarrow{\kappa}\gamma'}\widehat{P}^\sigma(\gamma',\lambda)\epsilon(\gamma',\gamma)
\end{gathered}
\tag{9.2.2}
$$

Here is the information needed to make this explicit. Assume $\kappa\in\tau(\gamma)$.

If $t_\gamma(\kappa)$ =`1r2,2r22,1C-,2C-,3C-` then $w_\kappa\times\gamma\neq\gamma$. In all other cases $w_\kappa\times\gamma=\gamma$.

We need $\{\gamma' \mid \gamma\xrightarrow{\kappa}\gamma'\}$:

(1) $t_\gamma(\kappa)$ =`1C-,2C-,3C-`: $w_\kappa\times\gamma$;

(2) $t_\gamma(\kappa) =$`1r2,2Cr,2r22,3Cr,3r`: γ_κ (single valued);

(3) $t_\gamma(\kappa) =$`1r1f,2r21,2r11`: $\{\gamma_\kappa^1, \gamma_\kappa^2\}$ (double valued);

(4) $t_\gamma(\kappa) =$`1r1s,1ic,2ic,3ic`: none

The defect $\mathrm{def}_\gamma(\kappa)$ is 1 if $t_\gamma(\kappa) =$`2Cr,3Cr,3r`, and 0 otherwise.

$$\zeta_\kappa(\gamma) = \begin{cases} 1 & t_\gamma(\kappa) = \texttt{1ic,2ic,3ic} \\ 0 & t_\gamma(\kappa) = \texttt{1C-,2C-,3C-} \\ -1 & \text{otherwise} \end{cases}$$

Table 9.2.3

coefficient of $\hat{a}_\gamma^\kappa$ in $\widehat{T}_\kappa(\widehat{C}_\lambda)$		
$t_\gamma(\kappa)$	first term on the RHS of (9.2.2)	second term on RHS of (9.2.2)
`1C-`	$v\widehat{P}^\sigma(\gamma,\lambda)$	$\widehat{P}^\sigma(w_\kappa \times \gamma,\lambda)$
`1r1f`	$(v-v^{-1})\widehat{P}^\sigma(\gamma,\lambda)$	$\widehat{P}^\sigma(\gamma_\kappa^1,\lambda)+\widehat{P}^\sigma(\gamma_\kappa^2,\lambda)$
`1r1s`	$(v-v^{-1})\widehat{P}^\sigma(\gamma,\lambda)$	
`1r2`	$v\widehat{P}^\sigma(\gamma,\lambda)-v^{-1}\widehat{P}^\sigma(w_\kappa \times \gamma,\lambda)$	$\widehat{P}^\sigma(\gamma_\kappa,\lambda)$
`1ic`	$(v+v^{-1})\widehat{P}^\sigma(\gamma,\lambda)$	
`2C-`	$v^2\widehat{P}^\sigma(\gamma,\lambda)$	$\widehat{P}^\sigma(w_\kappa \times \gamma,\lambda)$
`2Cr`	$v(v-v^{-1})\widehat{P}^\sigma(\gamma,\lambda)$	$(v+v^{-1})\widehat{P}^\sigma(\gamma_\kappa,\lambda)$
`2r22`	$v^2\widehat{P}^\sigma(\gamma,\lambda)-v^{-2}\widehat{P}^\sigma(w_\kappa \times \gamma,\lambda)$	$\widehat{P}^\sigma(\gamma_\kappa,\lambda)$
`2r21`	$(v^2-v^{-2})\widehat{P}^\sigma(\gamma,\lambda)$	$\sum_{\gamma' \mid \gamma \xrightarrow{\kappa} \gamma'} \epsilon(\gamma,\gamma')\widehat{P}^\sigma(\gamma',\lambda)$
`2r11`	$(v^2-v^{-2})\widehat{P}^\sigma(\gamma,\lambda)$	$\widehat{P}^\sigma(\gamma_\kappa^1,\lambda)+\widehat{P}^\sigma(\gamma_\kappa^2,\lambda)$
`2ic`	$(v^2+v^{-2})\widehat{P}^\sigma(\gamma,\lambda)$	
`3C-`	$v^3\widehat{P}^\sigma(\gamma,\lambda)$	$\widehat{P}^\sigma(w_\kappa \times \gamma,\lambda)$
`3Cr`	$v(v^2-v^{-2})\widehat{P}^\sigma(\gamma,\lambda)$	$(v+v^{-1})\widehat{P}^\sigma(\gamma_\kappa,\lambda))$
`3r`	$v(v^2-v^{-2})\widehat{P}^\sigma(\gamma,\lambda)$	$(v+v^{-1})\widehat{P}^\sigma(\gamma_\kappa,\lambda)$
`3ic`	$(v^3+v^{-3})\widehat{P}^\sigma(\gamma,\lambda)$	

{table:akappagamma

Here is a condensed version of this table. Let $k = \ell(\kappa)$.

Table 9.2.4

coefficient of $\hat{a}^\kappa_\gamma$ in $\widehat{T}_\kappa(\widehat{C}_\lambda)$		
$t_\gamma(\kappa)$	first term on the RHS of (9.2.2)	second term on RHS of (9.2.2)
`1C-,2C-,3C-`	$v^k \widehat{P}^\sigma(\gamma, \lambda)$	$\widehat{P}^\sigma(w_\kappa \times \gamma, \lambda)$
`1ic,2ic,3ic`	$(v^k + v^{-k})\widehat{P}^\sigma(\gamma, \lambda)$	
`2Cr,3Cr,3r`	$v(v^{k-1} - v^{-k+1})\widehat{P}^\sigma(\gamma, \lambda)$	$(v + v^{-1})\widehat{P}^\sigma(\gamma_\kappa, \lambda)$
`1r1f,2r11`	$(v^k - v^{-k})\widehat{P}^\sigma(\gamma, \lambda)$	$\widehat{P}^\sigma(\gamma^1_\kappa, \lambda) + \widehat{P}^\sigma(\gamma^2_\kappa, \lambda)$
`1r1s`	$(v - v^{-1})\widehat{P}^\sigma(\gamma, \lambda)$	
`1r2,2r22`	$v^k \widehat{P}^\sigma(\gamma, \lambda) - v^{-k}\widehat{P}^\sigma(w_\kappa \times \gamma, \lambda)$	$\widehat{P}^\sigma(\gamma_\kappa, \lambda)$
`2r21`	$(v^2 - v^{-2})\widehat{P}^\sigma(\gamma, \lambda)$	$\sum_{\gamma' \mid \gamma \xrightarrow{\kappa} \gamma'} \epsilon(\gamma, \gamma')\widehat{P}^\sigma(\gamma', \lambda)$

{table:akappagamma

9.3 $\widehat{T}_\kappa(\widehat{C}_\mu)$ in terms of $\widehat{C}_\gamma$

{l:tauinv}

Lemma 9.3.1 *Suppose $\gamma \in \mathcal{D}^\sigma$, $\kappa \in \overline{S}$. Then $\kappa \in \tau(\lambda)$ iff*

$$\widehat{T}_\kappa \widehat{C}_\lambda = (v^{\ell(\kappa)} + v^{-\ell(\kappa)})\widehat{C}_\lambda.$$

This is the way "descent" is defined. In the geometric language of [5], the condition means that the corresponding perverse sheaf is pulled back from the partial flag variety of type κ. Compare [4, Theorem 4.4(c)] and [6, Lemma 6.7].

Recall the image of $\widehat{T}_\kappa$ has $\{\hat{a}^\kappa_\gamma \mid \gamma \in \mathcal{D}^\sigma, \kappa \in \tau(\gamma)\}$ as a basis. We can use $\{\widehat{C}_\gamma \mid \gamma \in \mathcal{D}^\sigma, \kappa \in \tau(\gamma)\}$ instead.

{l:basis}

Lemma 9.3.2 *Fix $\kappa \in \overline{S}$.*

(1) Suppose $\mu \in \mathcal{D}^\sigma$, and $\kappa \in \tau(\mu)$. Then

$$\widehat{C}_\mu = \sum_{\gamma \mid \kappa \in \tau(\gamma)} \widehat{P}^\sigma(\gamma, \mu)\hat{a}^\kappa_\gamma; \tag{9.3.3}$$

The coefficient polynomials are exactly the ones from Theorem 7.5. In particular $\widehat{C}_\mu$ is in the image of $\widehat{T}_\kappa$.

(2) The elements

$$\{\widehat{C}_\mu \mid \mu \in \mathcal{D}^\sigma,\ \kappa \in \tau(\mu)\}$$

form a basis of the image of $\widehat{T}_\kappa$.

Proof. For (1), write {e:easy}

$$\widehat{C}_\mu = \sum_{\gamma|\kappa\in\tau(\gamma)} \widehat{P}^\sigma(\gamma,\mu)\hat{a}_\gamma + \sum_{\gamma|\kappa\notin\tau(\gamma)} \widehat{P}^\sigma(\gamma,\mu)\hat{a}_\gamma. \tag{9.3.4)(a}$$

By Lemmas 9.3.1 and 9.1.1(3) we can also write $\widehat{C}_\mu = \sum_{\gamma|\kappa\in\tau(\gamma)} \widehat{\mathcal{R}}^\sigma(\gamma,\mu)\hat{a}^\kappa_\gamma$ for some $\widehat{\mathcal{R}}^\sigma(\gamma,\mu) \in \mathbb{Z}[v,v^{-1}]$. Plugging in the definition of $\hat{a}_\gamma$ (Lemma 9.1.1(2)) gives

$$\widehat{C}_\mu = \sum_{\gamma|\kappa\in\tau(\gamma)} \widehat{\mathcal{R}}^\sigma(\gamma,\mu)[\hat{a}_\gamma + v^{\ell(\gamma')-\ell(\gamma)} \sum_{\gamma'|\gamma\xrightarrow{\kappa}\gamma'} \epsilon(\gamma,\gamma')\hat{a}_{\gamma'}]. \tag{9.3.4)(b}$$

Since $\kappa \notin \tau(\gamma')$ for each term in the last sum, comparing coefficients of $\hat{a}_\gamma$ ($\kappa \in \tau(\gamma)$) in (a) and (b) gives $\widehat{\mathcal{R}}(\gamma,\mu) = \widehat{P}^\sigma(\gamma,\mu)$. This gives (1), and (2) follows. ☐

Comparing the coefficients of $\hat{a}_\gamma$ with $\kappa \notin \tau(\gamma)$ gives the "easy" recurrence relations for the $\widehat{P}^\sigma(\gamma,\lambda)$. See Section 10.

Next we want to compute $\widehat{T}_\kappa\widehat{C}_\lambda$ in the basis $\widehat{C}_\gamma$. When $\kappa \in \tau(\lambda)$ this is given in Lemma 9.3.1. We turn now to the case $\kappa \notin \tau(\lambda)$.

{l:sumtau}

Lemma 9.3.5 (compare [4, Theorem 4.4(a,b)]**)** *Suppose $\kappa \notin \tau(\lambda)$. Then*

$$\widehat{T}_\kappa\widehat{C}_\lambda = \sum_{\gamma|\kappa\in\tau(\gamma)} m_\kappa(\gamma,\lambda)\widehat{C}_\gamma \tag{9.3.6}$$

for some $m_\kappa(\gamma,\lambda) \in \mathbb{Z}[v,v^{-1}]$. Each $m_\kappa(\gamma,\lambda)$ is self-dual, and is of the form

$$m_\kappa(\gamma,\lambda) = \begin{cases} m_{\kappa,0}(\gamma,\lambda) & \ell(\kappa)=1 \\ m_{\kappa,0}(\gamma,\lambda) + m_{\kappa,1}(\gamma,\lambda)(v+v^{-1}) & \ell(\kappa)=2 \\ m_{\kappa,0}(\gamma,\lambda) + m_{\kappa,1}(\gamma,\lambda)(v+v^{-1}) + m_{\kappa,2}(\gamma,\lambda)(v^2+v^{-2}) & \ell(\kappa)=3 \end{cases}$$

for some integers $m_{\kappa,i}(\gamma,\lambda)$.

Proof. The existence of $m_\kappa(\gamma,\delta)$ is (2) of Lemma 9.3.2. That the left side of (9.3.6) is self-dual is [4, 4.8(e)], and since $\widehat{C}_\delta$ is self-dual this implies $m_\kappa(\gamma,\delta)$ is self-dual.

The highest order term of $m_\kappa(\gamma,\lambda)$ is $v^{\ell(\kappa)-1}$. This follows by downward induction on $\ell(\gamma)$. See [4, pg. 17 (Proof of Theorem 4.4)]. This gives the remaining assertion. □

Remark 9.3.7 It is easy to see $m_\kappa(\gamma,\lambda)\neq 0$ implies $\gamma\overset{\kappa}{\to}\lambda$, or $\gamma<\lambda$, or $\gamma\overset{\kappa}{\to}\gamma'$ for some $\gamma'<\lambda$. We make this more precise in Lemmas 9.4.1 and 9.4.5.

In the classical setting $\mu(\gamma,\lambda)$ is defined to be the coefficient of the top degree term in $P(\gamma,\lambda)$, i.e. $q^{\frac{1}{2}(\ell(\lambda)-\ell(\gamma)-1)}$. Furthermore if $\gamma<\lambda$ then $m(\gamma,\lambda)=\mu(\gamma,\lambda)$.

With our normalization the top degree term in $\widehat{P}^\sigma(\gamma,\lambda)$ is v^{-1}, which is zero unless $\ell(\lambda)-\ell(\gamma)$ is odd. We need a generalization to take κ of length $2,3$ into account.

Definition 9.3.8 *For $i=-1,-2,-3$ let $\widehat{\mu}^\sigma_i(\gamma,\lambda)$ be the coefficient of v^i in $\widehat{P}^\sigma(\gamma,\lambda)$.*

So

$$\widehat{P}^\sigma(\gamma,\lambda)=\widehat{\mu}^\sigma_{-3}(\gamma,\lambda)v^{-3}+\widehat{\mu}^\sigma_{-2}(\gamma,\lambda)v^{-2}+\widehat{\mu}^\sigma_{-1}(\gamma,\lambda)v^{-1}\pmod{v^{-4}}.$$

It is clear that

$$\widehat{\mu}^\sigma_{-k}(\gamma,\lambda)=0\quad\text{unless }\ell(\gamma)-\ell(\gamma)=k\pmod 2. \tag{9.3.9}$$

We can now state the main result of this section.

{t:T_kappa}

Theorem 9.3.10 *Suppose $\kappa\notin\tau(\lambda)$. Then*

$$\widehat{T}_\kappa\widehat{C}_\lambda=\sum_{\gamma|\kappa\in\tau(\gamma)}m_\kappa(\gamma,\lambda)\widehat{C}_\gamma.$$

for coefficients $m_\kappa(\gamma,\lambda)$ given as follows.

(1) If $\gamma \xrightarrow{\kappa} \lambda$ *then*

$$m_\kappa(\gamma,\lambda) = (v+v^{-1})^{\mathrm{def}_\kappa(\gamma)}\epsilon(\gamma,\lambda) = \begin{cases} \epsilon(\gamma,\lambda) & \mathrm{def}_\lambda(\kappa)=0 \\ (v+v^{-1}) & \mathrm{def}_\lambda(\kappa)=1 \end{cases} \tag{9.3.11}$$

(recall $\epsilon(\lambda,\gamma)=\pm1$*, and is* 1 *unless* $t_\gamma(\kappa) =$`2r21`*, cf. Definition 5.2.1).*

(2) Assume $\gamma \overset{\kappa}{\not\to} \lambda$*, and* $\ell(\kappa)=1$*. Then*

$$m_\kappa(\gamma,\lambda) = \widehat{\mu}^\sigma_{-1}(\gamma,\lambda).$$

(3) Assume $\gamma \overset{\kappa}{\not\to} \lambda$*, and* $\ell(\kappa)=2$*.*

(a) If $\ell(\gamma) \not\equiv \ell(\lambda) \pmod 2$ *then*

$$m_\kappa(\gamma,\lambda) = \widehat{\mu}^\sigma_{-1}(\gamma,\lambda)(v+v^{-1}).$$

(b) If $\ell(\gamma) \equiv \ell(\lambda) \pmod 2$ *then*

$$m_\kappa(\gamma,\lambda) = \widehat{\mu}^\sigma_{-2}(\gamma,\lambda) - \sum_{\substack{\delta \\ \kappa\in\tau(\delta) \\ \gamma<\delta<\lambda}} \widehat{\mu}^\sigma_{-1}(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta,\lambda)$$
$$- \begin{Bmatrix} \widehat{\mu}^\sigma_{-1}(\gamma,\lambda^\kappa) & t_\lambda(\kappa)=2Ci \\ 0 & \mathit{else} \end{Bmatrix} + \begin{Bmatrix} \widehat{\mu}^\sigma_{-1}(\gamma_\kappa,\lambda) & t_\gamma(\kappa)=2Cr \\ 0 & \mathit{else} \end{Bmatrix}$$

(4) Assume $\gamma \overset{\kappa}{\not\to} \lambda$*, and* $\ell(\kappa)=3$*.*

(a) If $\ell(\gamma) \equiv \ell(\lambda) \pmod 2$ *then*

$$m_\kappa(\gamma,\lambda) = \big[\widehat{\mu}^\sigma_{-2}(\gamma,\lambda) - \sum_{\substack{\delta \\ \kappa\in\tau(\delta) \\ \gamma<\delta<\lambda}} \widehat{\mu}^\sigma_{-1}(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta,\lambda)\big](v+v^{-1})$$

(b) If $\ell(\gamma) \not\equiv \ell(\lambda) \pmod 2$ *then*

$$m_\kappa(\gamma,\lambda) = \widehat{\mu}^\sigma_{-3}(\gamma,\lambda)(v^2+v^{-2}) + \sum_{\substack{\delta,\phi \\ \kappa\in\tau(\delta),\kappa\in\tau(\phi) \\ \gamma<\delta<\phi<\lambda}} \widehat{\mu}^\sigma_{-1}(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta,\phi)\widehat{\mu}^\sigma_{-1}(\phi,\lambda) + - \sum_{\substack{\delta \\ \kappa\in\tau(\delta) \\ \gamma<\delta<\lambda}} \left[\widehat{\mu}^\sigma_{-1}(\gamma,\delta)\widehat{\mu}^\sigma_{-2}(\delta,\lambda) + \widehat{\mu}^\sigma_{-2}(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta,\lambda)\right]$$

$$- \begin{cases} \widehat{\mu}^\sigma_{-1}(\gamma,\lambda^\kappa) & t_\lambda(\kappa) = 3Ci \textit{ or } 3i \\ 0 & \textit{else} \end{cases} + \begin{cases} \widehat{\mu}^\sigma_{-1}(\gamma_\kappa,\lambda) & t_\gamma(\kappa) = 3Cr \textit{ or } 3r \\ 0 & \textit{else} \end{cases}$$

I've written the proof in great length for the sake of finding errors. See the Appendix.

Theorem 9.3.10 gives us a basic identity which we use repeatedly.

Suppose $\kappa \notin \tau(\lambda)$. By Theorem 9.3.10 and (7.6):

$$\begin{aligned} \widehat{T}_\kappa \widehat{C}_\lambda &= \sum_{\delta|\kappa\in\tau(\delta)} m_\kappa(\delta,\lambda)\widehat{C}_\delta \\ &= \sum_{\delta|\kappa\in\tau(\delta)} m_\kappa(\delta,\lambda) \sum_\gamma \widehat{P}^\sigma(\gamma,\lambda)\hat{a}_\gamma \\ &= \sum_\gamma \Big[\sum_{\delta|\kappa\in\tau(\delta)} \widehat{P}^\sigma(\gamma,\delta) m_\kappa(\delta,\lambda)\Big]\hat{a}_\gamma \end{aligned} \tag{9.3.12}$$

{p:basicidentity}

Proposition 9.3.13 *Fix* $\kappa \in \overline{S}$, $\gamma,\lambda \in \mathcal{D}^\sigma$, *with* $\kappa \notin \tau(\lambda)$. *Then*

$$\sum_{\delta|\kappa\in\tau(\delta)} \widehat{P}^\sigma(\gamma,\delta) m_\kappa(\delta,\lambda) = \textit{multiplicity of } \hat{a}_\gamma \textit{ in } \widehat{T}_\kappa(\widehat{C}_\lambda) \tag{9.3.14}$$

If $\kappa \in \tau(\gamma)$ *the same equality holds with* $\hat{a}^\kappa_\gamma$ *on the right hand side.*

9.4 Nonvanishing of $m_\kappa(\gamma,\lambda)$

It is important to know when $m_\kappa(\gamma,\lambda)$ can be nonzero.

{l:mnotzero}

Lemma 9.4.1 *Assume $\kappa\in\tau(\gamma),\kappa\notin\tau(\lambda)$, and $m_\kappa(\gamma,\lambda)\neq 0$. Then one of the following conditions holds:*

(a) $\gamma\overset{\kappa}{\to}\lambda$

(b) $\gamma<\lambda$

(c) $\mathrm{def}_\gamma(\kappa)=1$, $\gamma\not<\lambda$, *and* $\widehat{\mu}^\sigma_{-1}(\gamma_\kappa,\lambda)\neq 0$.

(c′) $\mathrm{def}_\lambda(\kappa)=1$, $\gamma\not<\lambda$, *and* $\widehat{\mu}^\sigma_{-1}(\gamma,\lambda^\kappa)\neq 0$.

See Definition 9.1.4 for $\mathrm{def}_\gamma(\kappa)$. Compare [6, Lemma 6.7], and [3, Section 3,II].

Remark 9.4.2 In the classical case either $\gamma\overset{\kappa}{\to}\lambda$, or $m(\gamma,\lambda)=\mu(\gamma,\lambda)$ (the top degree term of $P(\gamma,\lambda)$), which is nonzero only if $\gamma<\lambda$. So cases (c), (c′) don't occur. Since they allow $m_\kappa(\gamma,\lambda)\neq 0$ for some $\gamma\not<\lambda$, these cause some trouble.

Proof. Consulting the cases in the Theorem, if $m_\kappa(\gamma,\lambda)\neq 0$ then either:

(1) $\gamma\overset{\kappa}{\to}\lambda$ (Case (1) of the Theorem)

(2) Some $\widehat{\mu}^\sigma_{-k}(\gamma,\delta)\neq 0$ with $\delta\le\lambda$. This implies $\gamma<\lambda$ ($\gamma\neq\lambda$ since they have opposite τ-invariants).

(3) One of the terms in braces in Cases (3b) or (4b) is nonzero.

The cases `2Cr, 3Cr, 3r, 2Ci, 3Ci, 3i` are exactly the ones in which the defect is 1, and since $\widehat{\mu}^\sigma_{-1}(\gamma,\lambda^\kappa)\neq 0$ or $\widehat{\mu}^\sigma_{-1}(\gamma_\kappa,\lambda)\neq 0$ this gives the result. □

{d:kappaless}

Definition 9.4.3 *Suppose $\kappa\in\tau(\gamma),\kappa\notin\tau(\lambda)$. We say $\gamma\overset{\kappa}{\prec}\lambda$ if one of conditions (b,c,c′) of the Lemma hold:*

(b) $\gamma<\lambda$

(c) $\mathrm{def}_\gamma(\kappa)=1$, $\gamma\not<\lambda$, *and* $\widehat{\mu}^\sigma_{-1}(\gamma_\kappa,\lambda)\neq 0$.

(c′) $\mathrm{def}_\lambda(\kappa) = 1$, $\gamma \not< \lambda$, *and* $\widehat{\mu}^\sigma_{-1}(\gamma, \lambda^\kappa) \neq 0$.

Thus (9.3.6) becomes

$$\widehat{T}_\kappa \widehat{C}_\lambda = \sum_{\substack{\gamma \mid \kappa \in \tau(\gamma) \\ \gamma \stackrel{\kappa}{\to} \lambda}} m_\kappa(\gamma, \lambda) \widehat{C}_\gamma + \sum_{\substack{\gamma \mid \kappa \in \tau(\gamma) \\ \gamma \stackrel{\kappa}{\prec} \lambda}} m_\kappa(\gamma, \lambda) \widehat{C}_\gamma \tag{9.4.4}$$

We want to replace (b),(c),(c′) with (weaker) conditions in terms of length. Obviously (b) implies $\ell(\gamma) < \ell(\lambda)$. Suppose $\ell(\gamma) \geq \ell(\lambda)$, and (c) or (c′) holds. This is quite rare.

Consider Case (c). We're assuming $\ell(\gamma_\kappa) < \ell(\lambda) \leq \ell(\gamma)$. It is hard to satisfy this. Subtract $\ell(\gamma_\kappa)$ from each term, and use $\ell(\gamma_\kappa) = \ell(\gamma) - \ell(\kappa) + 1$, to see

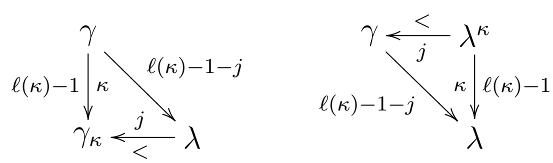
$$0 < \ell(\lambda) - \ell(\gamma_\kappa) \leq \ell(\kappa) - 1 \in \{1, 2\}$$

But $\widehat{\mu}^\sigma_{-1}(\gamma_\kappa, \lambda) \neq 0$ implies $\ell(\lambda) - \ell(\gamma_\kappa)$ is odd, so it equals 1, and

$$\ell(\gamma) = \ell(\lambda) + \ell(\kappa) - 2,\ \ell(\gamma_\kappa) = \ell(\lambda) - 1.$$

Case (c′) is similar: $t_\lambda(\kappa) =$`2Cr,3Cr,3r`, $\ell(\gamma) = \ell(\lambda) + \ell(\kappa) - 2$, and $\ell(\lambda^\kappa) = \ell(\gamma) + 1$.

These are illustrated by the following pictures. An arrow with a label: $\alpha \underset{k}{\to} \beta$ indicates $k = \ell(\alpha) - \ell(\beta)$.



The preceding argument shows that $j = 1$ in both cases.

This gives a nonvanishing criterion in terms of length.

`{l:kappalesslength`

Lemma 9.4.5 *Assume* $\kappa \in \tau(\gamma), \kappa \notin \tau(\lambda)$, *and* $\gamma \stackrel{\kappa}{\prec} \lambda$. *Then one of the following conditions holds:*

(b) $\ell(\gamma) < \ell(\lambda)$

(c) $\ell(\gamma) = \ell(\lambda) + \ell(\kappa) - 2$, $\mathrm{def}_\gamma(\kappa) = 1$

(c′) $\ell(\gamma) = \ell(\lambda) + \ell(\kappa) - 2$, $\mathrm{def}_\lambda(\kappa) = 1$

Remark 9.4.6 Explicitly, Cases (c) and (c′) of the Lemma are: {r:rare}

(1) $\ell(\kappa) = 2$, $t_\gamma(\kappa) =$`2Cr` or $t_\lambda(\kappa) =$`2Ci`, and $\ell(\gamma) = \ell(\lambda)$;

(2) $\ell(\kappa) = 3$, $t_\gamma(\kappa) =$`3Cr,3r` or $t_\lambda(\kappa) =$`3Ci,3i`, and $\ell(\gamma) = \ell(\lambda) + 1$.

Cases (a,b,c,c′) in Lemmas 9.4.1 and 9.4.5 don't precisely line up. There can be γ in case (c) or (c′) of Lemma 9.4.1, so $\gamma \not< \lambda$, but $\ell(\gamma) < \ell(\lambda)$, putting it in case (b) of Lemma 9.4.5.

10 Computing $\widehat{P}^\sigma(\gamma, \mu)$

{s:recursion}

10.1 Easy recurrence relations

{s:easy}

Recall (9.3.4)(a) and (b)

$$\widehat{C}_\mu = \sum_{\gamma|\kappa\in\tau(\gamma)} \widehat{P}^\sigma(\gamma,\mu)\hat{a}_\gamma + \sum_{\gamma|\kappa\notin\tau(\gamma)} \widehat{P}^\sigma(\gamma,\mu)\hat{a}_\gamma.$$

and

$$\widehat{C}_\mu = \sum_{\gamma|\kappa\in\tau(\gamma)} \widehat{P}^\sigma(\gamma,\mu)\hat{a}_\gamma + \sum_{\gamma'|\kappa\notin\tau(\gamma')} \big[\sum_{\gamma|\gamma\xrightarrow{\kappa}\gamma'} v^{\ell(\gamma')-\ell(\gamma)}\widehat{P}^\sigma(\gamma,\mu)\epsilon(\gamma,\gamma')\big]\hat{a}_{\gamma'}.$$

Equate the coefficients of $\hat{a}_\gamma$ ($\kappa \notin \tau(\gamma)$), and use (9.1.5) to conclude the "easy" relations:

{l:easyrecursion}

Lemma 10.1.1 *Suppose $\kappa \notin \tau(\gamma), \kappa \in \tau(\mu)$. Then*

(10.1.2) $$\boxed{\widehat{P}^\sigma(\gamma,\mu) = v^{-\ell(\kappa)+\mathrm{def}_\gamma(\kappa)} \sum_{\gamma'|\gamma'\xrightarrow{\kappa}\gamma} \epsilon(\gamma',\gamma)\widehat{P}^\sigma(\gamma',\mu)}$$

We will compute $\widehat{P}^\sigma(\gamma,\mu)$ (and $m_\kappa(\gamma,\mu)$) by induction on length as follows. To compute $\widehat{P}^\sigma(\gamma,\mu)$ we may assume we know:

(10.1.3) $$\begin{aligned}&\widehat{P}^\sigma(*,\mu') \text{ if } \ell(\mu') < \ell(\mu)\\ &\widehat{P}^\sigma(\gamma',\mu) \text{ if } \ell(\gamma') > \ell(\gamma).\end{aligned}$$

We know the right hand side of (10.1.2) by the inductive assumption. If there is only one term on the right hand side $\widehat{P}^\sigma(\gamma,\mu)$ is equal (up to a power of v) to a polynomial we have already computed. Otherwise $\widehat{P}^\sigma(\gamma,\mu)$ is the sum of two terms.

Definition 10.1.4 *Suppose $\gamma < \mu$. Then (γ,μ) is:*

- *extremal if $\kappa \in \tau(\mu) \Rightarrow \kappa \in \tau(\gamma)$*
- *primitive if $\kappa \in \tau(\mu) \Rightarrow \kappa \in \tau(\gamma)$ or $\kappa \notin \tau(\gamma)$, $|\{\gamma' \mid \gamma' \overset{\kappa}{\to} \gamma\}| = 2$.*

I find the converses more natural:

Definition 10.1.5 *Suppose $\gamma < \mu$. Then (γ,μ) is:*

- *non-extremal if there exists $\kappa \in \tau(\mu), \kappa \notin \tau(\gamma)$.*
- *non-primitive if there exists $\kappa \in \tau(\mu), \kappa \notin \tau(\gamma)$ and $|\{\gamma' \mid \gamma' \overset{\kappa}{\to} \gamma\}| < 2$.*

Explicitly (γ,μ) is:

- *non-primitive if there exists $\kappa \in \tau(\mu), \kappa \notin \tau(\gamma)$ and $t_\gamma(\kappa) \neq$*`1i2f,2i12`*.*

Thus *extremal* $\subset$ *primitive* and *non-primitive* $\subset$ *non-extremal.*

If (γ,μ) is non-primitive, (10.1.2) writes $P(\gamma,\mu) = v^k P(\gamma',\mu)$.

10.2 Direct Recursion Relations

Recall Proposition 9.3.13: {e:direct}

(10.2.1)(a)
$$\sum_{\delta|\kappa\in\tau(\delta)} \widehat{P}^\sigma(\gamma,\delta) m_\kappa(\delta,\lambda) = \text{multiplicity of } \hat{a}_\gamma \text{ in } \widehat{T}_\kappa(\widehat{C}_\lambda),$$

and if $\kappa \in \tau(\gamma)$ the same equality holds with $\hat{a}^\kappa_\gamma$ on the right hand side.

By (9.4.4) the left hand side is:

(10.2.1)(b)
$$\sum_{\substack{\delta|\kappa\in\tau(\delta)\\ \delta\overset{\kappa}{\to}\lambda}} \widehat{P}^\sigma(\gamma,\delta) m_\kappa(\delta,\lambda) + \sum_{\substack{\delta|\kappa\in\tau(\delta)\\ \delta\overset{\kappa}{\prec}\lambda}} \widehat{P}^\sigma(\gamma,\delta) m_\kappa(\delta,\lambda)$$

We introduce some notation for the final sum. See [6, after Lemma 6.7].

Definition 10.2.2 *For $\kappa \notin \tau(\lambda)$ define:* {d:U}

$$\widehat{U}_\kappa(\gamma, \lambda) = \sum_{\substack{\delta \mid \kappa \in \tau(\delta) \\ \delta \overset{\kappa}{\prec} \lambda}} \widehat{P}^\sigma(\gamma, \delta) m_\kappa(\delta, \lambda).$$

By (10.2.1)(a) and (b) we see:

Lemma 10.2.3 *Fix γ, λ, with $\kappa \notin \tau(\lambda)$. Then* {l:murecursion1}

$$\sum_{\delta \mid \delta \overset{\kappa}{\to} \lambda} \widehat{P}^\sigma(\gamma, \delta) m_\kappa(\delta, \lambda) = [\textit{coefficient of } \hat{a}_\gamma \textit{ in } \widehat{T}_\kappa \widehat{C}_\lambda] - \widehat{U}_\kappa(\gamma, \lambda). \tag{10.2.4}$$

If $\kappa \in \tau(\gamma)$ we can replace the term in brackets with

$$\textit{coefficient of } \hat{a}_\gamma^\kappa \textit{ in } \widehat{T}_\kappa \widehat{C}_\lambda$$

We first dispense with a case which won't be used until Section 11. In the setting of the Lemma, if $t_\lambda(\kappa) =$`1i2s,1ic,2ic,3ic` then, even though $\kappa \in \tau(\lambda)$, there are no δ occuring in the sum on the left hand side.

Lemma 10.2.5 *Assume $t_\lambda(\kappa) =$ `1i2s,1rn,2rn,3rn`. Equivalently $\kappa \notin \tau(\lambda)$ but there does not exist δ satisfying $\delta \overset{\kappa}{\to} \lambda$. Then* {l:lhsempty}

$$\textit{coefficient of } \hat{a}_\gamma \textit{ in } \widehat{T}_\kappa \widehat{C}_\lambda = \widehat{U}_\kappa(\gamma, \lambda)$$

We turn now to the main case, in which the left hand side of (10.2.4) is nonempty. In this case this sum has 1 or 2 terms. We're mainly interested when it has 1 term, in which case it gives a formula for $\widehat{P}^\sigma(\gamma, \mu)$. For this reason it is convenient to change variables. This gives the main result.

Proposition 10.2.6 *Suppose $\kappa \in \tau(\gamma)$, $\kappa \in \tau(\mu)$, and $t_\mu(\kappa) \neq$`1r1s, 1ic, 2ic, 3ic`. Choose λ satisfying $\mu \overset{\kappa}{\to} \lambda$. Then* {p:recursion}

$$\boxed{\sum_{\mu' \mid \mu' \overset{\kappa}{\to} \lambda} \widehat{P}^\sigma(\gamma, \mu') m_\kappa(\mu', \lambda) = [\textit{coefficient of } \hat{a}_\gamma^\kappa \textit{ in } \widehat{T}_\kappa \widehat{C}_\lambda] - \widehat{U}_\kappa(\gamma, \lambda)} \tag{10.2.7}$$

The sum on the left hand side is over $\{\mu, w_\kappa \times \mu\}$ if $t_\mu(\kappa) =$`1r2, 2r22, 2r21`, and just μ otherwise.

This is our main recursion relation. We analyse its effectiveness in the next section.

Since we are assuming $\kappa \in \tau(\gamma)$ we have replaced $\hat{a}_\gamma$ with $\hat{a}^\kappa_\gamma$ as in Lemma 10.2.3.

Note that the term $m_\kappa(\mu, \lambda)$ is computed by the first case of Theorem 9.3.10, i.e. $\mu \stackrel{\kappa}{\to} \lambda$, and either equals $\epsilon(\mu, \lambda) = \pm 1$ (see Definition 5.2.1) or $(v + v^{-1})$.

Consulting Theorem 9.3.10 we make the left side of the Proposition explicit.

Table 10.2.8

$t_\mu(\kappa)$	LHS of (10.2.7)
`1C-,1r1f,2C-,2r11,3C-`	$\widehat{P}^\sigma(\gamma, \mu)$
`1r2,2r22`	$\widehat{P}^\sigma(\gamma, \mu) + \widehat{P}^\sigma(\gamma, w_\kappa \times \mu)$
`2r21`	$\sum_{\mu' \mid \mu \stackrel{\kappa}{\to} \mu'} \epsilon(\mu, \mu')\widehat{P}^\sigma(\gamma, \mu')$
`2Cr,3Cr,3r`	$(v + v^{-1})\widehat{P}^\sigma(\gamma, \mu)$

In the `2r21` case, recall μ comes in an ordered pair (μ_1, μ_2). Then $\mu \stackrel{\kappa}{\to} \lambda$ says that λ is one member of the ordered pair $\mu_\kappa = (\lambda_1, \lambda_2)$.

We turn to the right hand side of (10.2.7). The first term is given by Lemma 9.2.1. Let $k = \ell(\kappa)$. This is copied from Table 9.2.4, which is a condensed version of Table 9.2.3.

{`table:condensed2`}

Table 10.2.9

$t_\gamma(\kappa)$	first term on RHS of (10.2.7)
1C-,2C-,3C-	$v^k\widehat{P}^\sigma(\gamma,\lambda)+\widehat{P}^\sigma(w_\kappa\times\gamma,\lambda)$
1ic,2ic,3ic	$(v^k+v^{-k})\widehat{P}^\sigma(\gamma,\lambda)$
2Cr,3Cr,3r	$v(v^{k-1}-v^{-k+1})\widehat{P}^\sigma(\gamma,\lambda)+(v+v^{-1})\widehat{P}^\sigma(\gamma_\kappa,\lambda)$
1r1f,2r11	$(v^k-v^{-k})\widehat{P}^\sigma(\gamma,\lambda)+\widehat{P}^\sigma(\gamma^1_\kappa,\lambda)+\widehat{P}^\sigma(\gamma^2_\kappa,\lambda)$
1r1s	$(v-v^{-1})\widehat{P}^\sigma(\gamma,\lambda)$
1r2,2r22	$v^k\widehat{P}^\sigma(\gamma,\lambda)-v^{-k}\widehat{P}^\sigma(w_\kappa\times\gamma,\lambda)+\widehat{P}^\sigma(\gamma_\kappa,\lambda)$
2r21	$(v^2-v^{-2})\widehat{P}^\sigma(\gamma,\delta)+\sum_{\gamma'\mid\gamma\overset{\kappa}{\to}\gamma'}\epsilon(\gamma,\gamma')\widehat{P}^\sigma(\gamma',\delta)$

10.3 Analysis of the Recursion

{s:analysis}

Fix γ,μ with $\kappa\in\tau(\gamma)$ and $t_\mu(\kappa)$ =1C-, 1r1f, 2C-, 2r11, 3C-, 2Cr, 3Cr, 3r. Choose λ with $\mu\overset{\kappa}{\to}\lambda$. Then Proposition 10.2.6 says

(10.3.1) $$\widehat{P}^\sigma(\gamma,\mu)=\text{coefficient of } \hat{a}^\kappa_\gamma \text{ in } \widehat{T}_\kappa\widehat{C}_\lambda-\widehat{U}_\kappa(\gamma,\lambda).$$

Recall $\widehat{U}_\kappa(\gamma,\lambda)$ is given in Definition 10.2.2. We analyse how this fits in our recursive scheme.

Recall to compute $P(\gamma,\mu)$ we may assume we know:

(10.3.2) $$\begin{aligned}&\widehat{P}^\sigma(*,\mu') \text{ if } \ell(\mu')<\ell(\mu)\\&\widehat{P}^\sigma(\gamma',\mu) \text{ if } \ell(\gamma')>\ell(\gamma).\end{aligned}$$

To compute the first term right hand side of (10.3.1) we only need various $\widehat{P}^\sigma(*,\lambda)$, which we know by induction since $\ell(\lambda)<\ell(\mu)$. See Table 10.2.9. This leaves $\widehat{U}_\kappa(\gamma,\lambda)$.

{l:U}
{e:U}

Lemma 10.3.3 *Fix κ. For all γ,λ with $\kappa\notin\tau(\lambda)$:*

(10.3.4)(a) $$\begin{aligned}\widehat{U}_\kappa(\gamma,\lambda)=&\sum_{\substack{\delta\mid\kappa\in\tau(\delta)\\ \ell(\gamma)\le\ell(\delta)<\ell(\lambda)}}\widehat{P}^\sigma(\gamma,\delta)m_\kappa(\delta,\lambda)+\\&\sum_{\substack{\delta\mid t_\delta(\kappa)=\texttt{2Cr},\texttt{3Cr},\texttt{3r}\\ \ell(\delta)=\ell(\lambda)+\ell(\kappa)-2}}\widehat{P}^\sigma(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta_\kappa,\lambda)+\\&\sum_{\substack{\delta\mid\kappa\in\tau(\delta)\\ \ell(\delta)=\ell(\lambda)+\ell(\kappa)-2}}\widehat{P}^\sigma(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta,\lambda^\kappa)\end{aligned}$$

The final sum occurs only if $t_\gamma(\lambda)$ =`2Ci,3Ci,3i`, *i.e.* $\mathrm{def}_\lambda(\kappa) = 1$. *Furthermore the first sum can be further decomposed*

$$\text{(10.3.4)(b)} \qquad \widehat{U}_\kappa(\gamma,\lambda) = m_\kappa(\gamma,\lambda) + \sum_{\substack{\delta|\kappa\in\tau(\delta)\\ \ell(\gamma)<\ell(\delta)<\ell(\lambda)}} \widehat{P}^\sigma(\gamma,\delta)m_\kappa(\delta,\lambda)$$

with the first term included only if $\kappa \in \tau(\gamma)$.

Proof. Recall $\widehat{U}_\kappa(\gamma,\lambda)$ is defined by the sum over $\delta \overset{\kappa}{\prec} \lambda$. Of course $\widehat{P}^\sigma(\gamma,\delta) \neq 0$ implies $\gamma \leq \delta$. Therefore, by the definition of $\overset{\kappa}{\prec}$, $\widehat{U}_\kappa(\gamma,\lambda)$ is the sum over δ satisfying $\kappa \in \tau(\delta)$, and one of:

(1) $\gamma \leq \delta < \lambda$

(2) $\mathrm{def}_\delta(\kappa) = 1, \delta \not< \lambda, \delta_\kappa < \lambda$

(3) $\mathrm{def}_\lambda(\kappa) = 1, \delta \not< \lambda, \delta < \lambda^\kappa$

Every such term appears in (10.3.4). Conversely any nonzero term in (10.3.4) appears in one of these cases. □

{p:directrecursion

Proposition 10.3.5 (Direction Recursion) *Suppose* $\kappa \in \tau(\gamma)$, $\kappa \in \tau(\mu)$, *and* $t_\mu(\kappa)$ =`1C-, 1r1f, 2C-, 2r11, 3C-, 2Cr, 3Cr, 3r`. *By induction we know all terms necessary to compute* $\widehat{P}^\sigma(\gamma,\mu)$ *unless* $t_\mu(\kappa)$ =`2Cr,3Cr,3r` *and* $\ell(\mu) - \ell(\gamma)$ *is odd. In this case:*

$$(v + v^{-1})\widehat{P}^\sigma(\gamma,\mu) = \widehat{\mu}^\sigma_{-1}(\gamma,\mu) + (*)$$

where all terms in (*) are known. This can be solved for $\widehat{P}^\sigma(\gamma,\mu)$.

Remark 10.3.6 This computes $\widehat{P}^\sigma(\gamma,\mu)$ ($\kappa \in \tau(\gamma),\tau(\mu)$) unless:

(1) there is no λ with $\mu \overset{\kappa}{\to} \lambda$: `1r1s,1ic,2ic,3ic`

(2) $\mu \overset{\kappa}{\to} \lambda$ for some λ, but $|\{\mu'|\mu' \overset{\kappa}{\to} \lambda\}| = 2$: `1r2,2r22,2r21`.

See Section 12.

Proof.

Since $\kappa \in \tau(\gamma)$, and λ is not of type `2Ci,3Ci,3i`, Lemma 10.3.3 says:

$$\widehat{U}_\kappa(\gamma,\lambda) = m(\gamma,\lambda) + \sum_{\substack{\delta|\kappa\in\tau(\delta)\\ \ell(\gamma)<\ell(\delta)<\ell(\lambda)}} \widehat{P}^\sigma(\gamma,\delta)m_\kappa(\delta,\lambda) + \sum_{\substack{\delta|t_\delta(\kappa)=\mathtt{2Cr,3Cr,3r}\\ \ell(\delta)=\ell(\lambda)+\ell(\kappa)-2}} \widehat{P}^\sigma(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta_\kappa,\lambda) \tag{10.3.7}$$

In the first sum $\ell(\delta) < \ell(\lambda) < \ell(\mu)$. In the second $\ell(\delta) = \ell(\lambda)+\ell(\kappa)-2 < \ell(\lambda)+\ell(\kappa)-\mathrm{def}_\mu(\kappa) = \ell(\mu)$. So we know all terms $\widehat{P}^\sigma(\gamma,\delta)$ occuring by the inductive hypothesis.

We also need various $m_\kappa(\delta,\lambda)$ with $\ell(\gamma) < \ell(\delta) < \ell(\lambda)$. By Theorem 9.3.10 this requires some terms $\widehat{P}^\sigma(*,\lambda')$ with $\ell(\lambda') \le \ell(\lambda)$, which we know by induction. We also need some terms $\widehat{\mu}^\sigma_{-1}(\delta_\kappa,\lambda)$ (when $t_\delta(\kappa)$ =`2Cr,3Cr,3r`). Since $\ell(\lambda) < \ell(\mu)$ we know all of these terms by the inductive hypothesis. We also need to know terms of the form $\widehat{\mu}^\sigma_{-1}(\delta,\lambda^\kappa)$ (when $t_\lambda(\kappa)$ =`2Ci,3Ci,3i`). Since $\lambda^\kappa = \mu$, this requires $\widehat{P}^\sigma(\delta,\mu)$ with $\ell(\delta) > \ell(\gamma)$, which we know.

This takes care of all terms except the lead term $m_\kappa(\gamma,\lambda)$.

To compute $m_\kappa(\gamma,\lambda)$ we again need various $\widehat{P}^\sigma(*,\lambda')$ ($\ell(\lambda') < \ell(\lambda)$) and $\widehat{\mu}^\sigma_{-1}(\gamma_\kappa,\lambda)$, all of which we know. This leaves only $\widehat{\mu}^\sigma_{-1}(\gamma,\lambda^\kappa) = \widehat{\mu}^\sigma_{-1}(\gamma,\mu)$, when $t_\lambda(\kappa)$ =`2Ci, 3Ci, 3i`. Consulting the cases of Theorem 9.3.10, we see this term appears if $\ell(\kappa) = \ell(\gamma) - \ell(\lambda) \pmod 2$, in which case $\widehat{\mu}^\sigma_{-1}(\gamma,\mu)$ is the constant term of $m_\kappa(\gamma,\mu)$. We conclude that the right hand side of the formula in Proposition 10.2.6 is of the form $\widehat{\mu}^\sigma_{-1}(\gamma,\mu) + (*)$ where $(*)$ is known by induction. Also, the left hand side is $(v+v^{-1})\widehat{P}^\sigma(\gamma,\mu)$. This implies $\widehat{P}^\sigma(\gamma,\nu) \in v^{-1}\mathbb{Z}[v^{-2}]$ (which also follows by checking lengths).

The remaining case is provided by the next Lemma. □

{l:topterm}

Lemma 10.3.8 *Suppose $f(v) \in v^{-1}\mathbb{Z}[v^{-2}]$, and we know all terms of $f(v)(v+v^{-1})$ except the constant term. Then we can compute $f(v)$.*

Proof. Write $f(v) = c_n v^{-1} + c_{n-1}v^{-3} + \cdots + c_0 v^{-(2n+1)}$, and

$$(v+v^{-1})f(v) = b_{n+1} + b_n v^{-2} + v_{n-1}v^{-4} + \cdots + b_0 v^{-2n-2}$$

and we know $b_0, \ldots, b_n$. Starting at v^{-2n-2} we see $c_0 = b_0, (c_1 + c_0) = b_1, (c_2 + c_1) = b_2, \ldots$. This is easy to solve for c_i: $c_0 = b_0, c_1 = b_0 - b_1, c_2 = b_2 - b_1 + b_0, \ldots$. That is:

$$c_k = (-1)^k \sum_0^k (-1)^j b_j \quad (0 \leq k \leq n).$$

$\square$

Remark 10.3.9 It might be easier to think about this if we multiply by v^{2n+2} and replace v^2 with q. This gives

$$(1+q)(c_0 + c_1 q^2 + \cdots + c_n q^n) = b_0 + b_1 q + \cdots + b_{n+1} q^{n+1}$$

If we know all terms on the right hand side except for b_{n+1}, we can find all c_i.

11 New Recursion Relations

{s:new}

We return now to the setting of Lemma 10.2.3, and the case we skipped earlier.

{l:lhsempty2}

Lemma 11.1 *Assume* $t_\lambda(\kappa) =$ `1i2s,1rn,2rn,3rn`. *Equivalently* $\kappa \notin \tau(\lambda)$ *but there does not exist* λ' *satisfying* $\lambda' \xrightarrow{\kappa} \lambda$. *Then for any* γ*:*

$$\sum_\mu \widehat{P}^\sigma(\mu,\lambda)(\textit{multiplicity of } \hat{a}_\gamma \textit{ in } \widehat{T}_\kappa(\hat{a}_\mu)) = \widehat{U}_\kappa(\gamma,\lambda) \tag{11.2}$$

This is an immediate consequence of Lemma 10.2.5, which says that under this assumption coefficient of $\hat{a}_\gamma$ in $\widehat{T}_\kappa \widehat{C}_\lambda = \widehat{U}_\kappa(\gamma,\lambda)$.

The left hand side has at most 3 terms, which can be read off from the tables in Section 8.1, or Table 8.2.1. One of the terms is a polynomial times $\widehat{P}^\sigma(\gamma,\lambda)$, and we wish to solve for $\widehat{P}^\sigma(\gamma,\lambda)$.

If $\kappa \notin \tau(\gamma)$ then the only possibilities for μ are $\mu = \gamma, \mu = w_\kappa \times \gamma$, or $\mu \xrightarrow{\kappa} \gamma$. These are all well suited to using induction to computing $\widehat{P}^\sigma(\gamma,\lambda)$, unless $t_\gamma(\kappa) =$`1i2s,1rn,2rn,3rn`, in which case the coefficient of $\widehat{P}^\sigma(\gamma,\lambda)$ is 0.

If $\kappa \in \tau(\gamma)$ then $\mu = \gamma, \mu = w_\kappa \times \gamma$ or $\gamma \xrightarrow{\kappa} \mu$. If $\gamma \xrightarrow{\kappa} \mu$ then $\gamma > \mu$ and this is not well suited to our inductive hypothesis. So this case is only effective if there is no such μ, i.e. $t_\gamma(\kappa) =$`1r1s,1ic,2ic,3ic`.

Here are the resulting formulas.

{table:newnotkappa

Table 11.3

Formula (11.2): $\kappa \notin \tau(\gamma)$	
$t_\gamma(\kappa)$	LHS
`1C+`	$v^{-1}\widehat{P}^\sigma(\gamma,\lambda) + \widehat{P}^\sigma(w_\kappa \times \gamma, \lambda)$
`1i1`	$v^{-1}\widehat{P}^\sigma(\gamma,\lambda) + v^{-1}\widehat{P}^\sigma(w_\kappa \times \gamma, \lambda) + (1-v^{-2})\widehat{P}^\sigma(\gamma^\kappa,\lambda)$
`1i2f`	$2v^{-1}\widehat{P}^\sigma(\gamma,\lambda) + v^{-1}(v-v^{-1})(\widehat{P}^\sigma(\gamma_1^\kappa,\lambda) + \widehat{P}^\sigma(\gamma_2^\kappa,\lambda))$
`1i2s`	0
`1rn`	0
`2C+`	$v^{-2}\widehat{P}^\sigma(\gamma,\lambda) + \widehat{P}^\sigma(w_\kappa \times \gamma, \lambda)$
`2Ci`	$v^{-1}(v+v^{-1})\widehat{P}^\sigma(\gamma,\lambda) + (v-v^{-1})\widehat{P}^\sigma(\gamma^\kappa,\lambda)$
`2i11`	$v^{-2}(\widehat{P}^\sigma(\gamma,\lambda) + \widehat{P}^\sigma(w_\kappa \times \gamma, \lambda)) + (1-v^{-4})\widehat{P}^\sigma(\gamma^\kappa,\lambda)$
`2i12`	$2v^{-2}\widehat{P}^\sigma(\gamma,\lambda) + (1-v^{-4}) \sum_{\gamma' \mid \gamma' \xrightarrow{\kappa} \gamma} \epsilon(\gamma',\gamma)\widehat{P}^\sigma(\gamma',\lambda)$
`2i22`	$2v^{-2}\widehat{P}^\sigma(\gamma,\lambda) + (1-v^{-4})(\widehat{P}^\sigma(\gamma_1^\kappa,\lambda) + \widehat{P}^\sigma(\gamma_2^\kappa,\lambda))$
`2rn`	0
`3C+`	$v^{-3}\widehat{P}^\sigma(\gamma,\lambda) + \widehat{P}^\sigma(w_\kappa \times \gamma, \lambda)$
`3Ci,3i`	$v^{-2}(v+v^{-1})\widehat{P}^\sigma(\gamma,\lambda) + (v^2-v^{-2})v^{-1}\widehat{P}^\sigma(\gamma^\kappa,\lambda)$
`3rn`	0

{table:newkappa}

Table 11.4

Formula (11.2): $\kappa\in\tau(\gamma)$	
$t_\gamma(\kappa)$	LHS
`1r1s`	$(v-v^{-1})\widehat{P}^\sigma(\gamma,\lambda)$
`1ic`	$(v+v^{-1})\widehat{P}^\sigma(\gamma,\lambda)$
`2ic`	$(v^2+v^{-2})\widehat{P}^\sigma(\gamma,\lambda)$
`3ic`	$(v^3+v^{-3})\widehat{P}^\sigma(\gamma,\lambda)$

Solve these for $\widehat{P}^\sigma(\gamma,\lambda)$.

Lemma 11.5 *Assume* $t_\lambda(\kappa) =$`1i2s,1rn,2rn,3rn`. *Then:*

{`table:new`}

Table 11.6

Formula for $\widehat{P}^\sigma(\gamma,\lambda)$, $t_\lambda(\kappa)=$1i2s,1rn,2rn,3rn.	
$t_\gamma(\kappa)$	Formula for $\widehat{P}^\sigma(\gamma,\lambda)$
$\kappa\notin\tau(\gamma)$	
1C+	$\widehat{P}^\sigma(\gamma,\lambda)=-v\widehat{P}^\sigma(w_\kappa\times\gamma,\lambda)+v\widehat{U}_\kappa(\gamma,\lambda)$
1i1	$\widehat{P}^\sigma(\gamma,\lambda)+\widehat{P}^\sigma(w_\kappa\times\gamma,\lambda)=-(v-v^{-1})\widehat{P}^\sigma(\gamma^\kappa,\lambda)+v\widehat{U}_\kappa(\gamma,\lambda)$
1i2f	$\widehat{P}^\sigma(\gamma,\lambda)=-2(v-v^{-1})(\widehat{P}^\sigma(\gamma_1^\kappa,\lambda)+\widehat{P}^\sigma(\gamma_2^\kappa,\lambda))+2v\widehat{U}_\kappa(\gamma,\lambda)$
1i2s	none
1rn	none
2C+	$\widehat{P}^\sigma(\gamma,\lambda)=-v^2\widehat{P}^\sigma(w_\kappa\times\gamma,\lambda)+v^2\widehat{U}_\kappa(\gamma,\lambda)$
2Ci	$(v+v^{-1})\widehat{P}^\sigma(\gamma,\lambda)=-v(v-v^{-1})\widehat{P}^\sigma(\gamma^\kappa,\lambda)+v\widehat{U}_\kappa(\gamma,\lambda)$
2i11	$\widehat{P}^\sigma(\gamma,\lambda)+\widehat{P}^\sigma(w_\kappa\times\gamma,\lambda)=-(v^2-v^{-2})\widehat{P}^\sigma(\gamma^\kappa,\lambda)+v^2\widehat{U}_\kappa(\gamma,\lambda)$
2i12	$2v^{-2}\widehat{P}^\sigma(\gamma,\lambda)=-\frac{1}{2}(v^2-v^{-2})\sum_{\gamma'\mid\gamma'\overset{\kappa}{\to}\gamma}\epsilon(\gamma',\gamma)\widehat{P}^\sigma(\gamma',\lambda)+\frac{1}{2}v^2\widehat{U}_\kappa(\gamma,\lambda)$
2i22	$\widehat{P}^\sigma(\gamma,\lambda)=-\frac{1}{2}(v^2-v^{-2})(\widehat{P}^\sigma(\gamma_1^\kappa,\lambda)+\widehat{P}^\sigma(\gamma_2^\kappa,\lambda))+\frac{1}{2}v^2\widehat{U}_\kappa(\gamma,\delta)$
2rn	none
3C+	$\widehat{P}^\sigma(\gamma,\lambda)=-v^3\widehat{P}^\sigma(w_\kappa\times\gamma,\lambda)+v^3\widehat{U}_\kappa(\gamma,\delta)$
3Ci,3i	$(v+v^{-1})\widehat{P}^\sigma(\gamma,\lambda)=-v(v^2-v^{-2})\widehat{P}^\sigma(\gamma^\kappa,\lambda)+v^2\widehat{U}_\kappa(\gamma,\lambda)$
3rn	none
$\kappa\in\tau(\gamma)$	
1r1s	$(v-v^{-1})\widehat{P}^\sigma(\gamma,\lambda)=\widehat{U}_\kappa(\gamma,\lambda)$
1ic	$(v+v^{-1})\widehat{P}^\sigma(\gamma,\lambda)=\widehat{U}_\kappa(\gamma,\lambda)$
2ic	$(v^2+v^{-2})\widehat{P}^\sigma(\gamma,\lambda)=\widehat{U}_\kappa(\gamma,\lambda)$
3ic	$(v^3+v^{-3})\widehat{P}^\sigma(\gamma,\lambda)=\widehat{U}_\kappa(\gamma,\lambda)$

In every case we know all terms on the RHS by induction, with the possible exception of $\widehat{U}_\kappa(\gamma,\lambda)$. By Lemma 10.3.3, since $t_\lambda(\kappa)\neq$2Ci,3Ci,3i,

(11.7)
$$\widehat{U}_\kappa(\gamma,\lambda)=\sum_{\substack{\delta\mid\kappa\in\tau(\delta)\\ \ell(\gamma)\le\ell(\delta)<\ell(\lambda)}}\widehat{P}^\sigma(\gamma,\delta)m_\kappa(\delta,\lambda)+\sum_{\substack{\delta\mid t_\delta(\kappa)=\texttt{2Cr,3Cr,3r}\\ \ell(\delta)=\ell(\lambda)+\ell(\kappa)-2}}\widehat{P}^\sigma(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta_\kappa,\lambda)$$

The second sum is problematic, so we give it a name.

{d:Udagger}

Definition 11.8 *Suppose $\kappa \notin \tau(\lambda)$. Define:*

$$\widehat{U}^{\dagger}_{\kappa}(\gamma,\lambda) \sum_{\substack{\delta | t_{\delta}(\kappa)=\mathtt{2Cr,3Cr,3r} \\ \ell(\delta)=\ell(\lambda)+\ell(\kappa)-2}} \widehat{P}^{\sigma}(\gamma,\delta)\widehat{\mu}^{\sigma}_{-1}(\delta_{\kappa},\lambda) \tag{11.9}$$

This term doesn't fit well in our recursion scheme, since $\ell(\delta) > \ell(\lambda)$. (In the direct recursion section this wasn't an issue, since we were computing $\widehat{P}^{\sigma}(\gamma,\mu)$, and $\ell(\mu) > \ell(\delta)$.) We're hoping this term is usually 0.

Definition 11.10 *Fix $\kappa \in \overline{S}, \gamma, \lambda \in \mathcal{D}^{\sigma}$, with $\kappa \notin \tau(\lambda)$.*

We say condition A holds for (κ,γ,λ) if

(A) $\kappa \in \tau(\delta)$, $\mathrm{def}_{\delta}(\kappa) = 1$, $\ell(\delta) = \ell(\lambda) + \ell(\kappa) + 2 \Rightarrow \widehat{P}^{\sigma}(\gamma,\delta)\widehat{\mu}^{\sigma}_{-1}(\delta_{\kappa},\lambda) = 0$

This is automatic if $\ell(\kappa) = 1$.

We say condition (†) holds for (κ,γ,λ) if

$$\widehat{U}^{\dagger}_{\kappa}(\gamma,\lambda) = 0 \tag{†}$$

Obviously (A)$\Rightarrow$ (†), although (A) is easier to check. It often holds simply because the set in (A) is empty.

So now we need to compute:

$$\widehat{U}_{\kappa}(\gamma,\lambda) = m_{\kappa}(\gamma,\lambda) + \sum_{\substack{\delta | \kappa \in \tau(\delta) \\ \ell(\gamma)<\ell(\delta)<\ell(\lambda)}} \widehat{P}^{\sigma}(\gamma,\delta) m_{\kappa}(\delta,\lambda) + \widehat{U}^{\dagger}_{\kappa}(\gamma,\lambda) \tag{11.11}$$

with the first term present only if $\kappa \in \tau(\gamma)$.

As in the discussion in Section 10.3 we know all of the terms in the first sum. There is one crucial difference with the direct recursion of Proposition 10.3.5. In that lemma we were computing $\widehat{P}^{\sigma}(\gamma,\mu)$, and we needed $\widehat{P}^{\sigma}(\gamma,\lambda)$, where $\mu \xrightarrow{\kappa} \lambda$. This gave us a little extra room when applying the inductive hypothesis that we know $\widehat{P}^{\sigma}(*,\mu')$ with $\mu' < \mu$.

We consider the terms appearing in the sum in (11.11).

If $\kappa \in \tau(\gamma)$ we don't know the term $m_{\kappa}(\gamma,\lambda)$.

By the inductive hypothesis we know all terms $\widehat{P}^{\sigma}(\gamma,\delta)$ occuring in (11.11). Consider $m_{\kappa}(\delta,\lambda)$. This requires various $\widehat{P}^{\sigma}(*,\lambda')$ with $\ell(\lambda)' < \ell(\lambda)$, and

$\widehat{P}^\sigma(\delta',\lambda)$ with $\ell(\delta')\ge\ell(\delta)>\ell(\gamma)$, all of which we know. The only potential problems are the terms

$$\widehat{P}^\sigma(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta_\kappa,\lambda) \text{ if } \mathrm{def}_\gamma(\kappa)=1$$

which occur in the formulas for $m_\kappa(\delta,\lambda)$ (the opposite case $\widehat{P}^\sigma(\delta,\lambda^\kappa)$ does not occur since $t_\lambda(\kappa)\ne$2Ci,3Ci,3i). If $\ell(\delta_\kappa)>\ell(\gamma)$ we know this by induction. This leaves:

$$\widehat{P}^\sigma(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta_\kappa,\lambda)\quad \ell(\delta_\kappa)\le\ell(\gamma)<\ell(\delta),\mathrm{def}_\gamma(\kappa)=1.$$

This might be an issue. (In the Direct Recursion this was taken care of by the fact that $\lambda<\mu$). As in the discussion after Definition 9.4.3 this term is nonzero only if:

$$\ell(\delta)=\ell(\gamma)+\ell(\kappa)-2,\quad \ell(\gamma_\kappa)=\ell(\delta)-1$$

Definition 11.12 *Fix $\kappa\in\overline{S},\gamma,\lambda\in\mathcal{D}^\sigma$, with $\kappa\notin\tau(\lambda)$. We say condition (B) holds for (κ,γ,λ) if*

(B) $\kappa\in\tau(\delta)$, $\mathrm{def}_\delta(\kappa)=1$, $\ell(\delta)=\ell(\gamma)+\ell(\kappa)+2\Rightarrow\widehat{P}^\sigma(\gamma,\delta)\widehat{\mu}^\sigma_{-1}(\delta_\kappa,\lambda)=0$

This is automatic if $\ell(\kappa)=1$.

Here is the conclusion.

{p:newrecursion}

Proposition 11.13 *Fix $\kappa \in \overline{S}, \gamma, \lambda \in \mathcal{D}^\sigma$, satisfying:*

(1) $t_\gamma(\kappa) =$`1C+, 1i2f, 2C+, 2Ci, 2i12,2i22, 3C+, 3Ci, 3i, 1r1s, 1ic, 2ic, 3ic`.

(2) $t_\lambda(\kappa) =$`1i2s,1rn,2rn,3rn`

Assume conditions (A) and (B) hold. Then the formulas of Table 11.6 give an effective recursion relation for $\widehat{P}^\sigma(\gamma, \lambda)$. It is sufficient to assume Conditions (†) and (B).

If $t_\gamma(\kappa) =$`1i1,2i11` the same result holds, except that we get a formuls for $\widehat{P}^\sigma(\gamma, \lambda) + \widehat{P}^\sigma(w_\kappa \times \gamma, \lambda)$.

Remark 11.14

(1) We allow λ if $\kappa \notin \tau(\lambda)$ but there is no λ' with $\lambda' \xrightarrow{\kappa} \lambda$ (see Lemma 11.1): `1i2s,1rn,2rn,3rn`. This gives the λ of the Proposition.

(2) If $\kappa \notin \tau(\gamma)$ we exclude γ if the formula has two terms on the LHS: `1i1, 2i11`

(3) If $\kappa \notin \tau(\gamma)$ we exclude γ if there is no γ' with $\gamma' \xrightarrow{\kappa} \gamma$: type `1i2s, 1rn, 2rn, 3rn`. Together (2) and (3) leave: `1C+, 1i2f, 2C+, 2Ci, 2i12,2i22, 3C+, 3Ci, 3i`

(4) If $\kappa \in \tau(\gamma)$ we include γ if there is no γ' with $\gamma \xrightarrow{\kappa} \gamma'$: `1r1s,1ic,2ic, 3ic`. (2-4) give the γ of the Proposition.

If $\kappa \notin \tau(\gamma)$ the term $m_\kappa(\gamma, \lambda)$ does not appear in (10.3.4), and the Lemma is evident from the preceding discussion. If $\kappa \in \tau(\gamma)$ we need a generalization of Lemma 10.3.8. Here are the cases.

In the column $m_\kappa(\gamma, \lambda)$ α is an unknown constant. We have written $\widehat{P}^\sigma(\gamma, \lambda) = v^{-1} f(v^{-2})$ or $v^{-2} f(v^{-2})$, depending on the parity of $\ell(\lambda) - \ell(\gamma)$, where f is a polynomial. In the last column g is a polynomial which is known, and α, β, γ are unkown constants. We want to solve for f.

$\ell(\kappa)$	$\ell(\lambda)-\ell(\gamma)$	$m_\kappa(\gamma,\lambda)$	equation
1	even	0	$(v \pm v^{-1})v^{-2}f(v^{-2}) = v^{-1}g(v^{-2})$
1	odd	α	$(v \pm v^{-1})v^{-1}f(v^{-2}) = \alpha + v^{-2}g(v^{-2})$
2	even	α	$(v^2 - v^{-2})v^{-2}f(v^{-2}) = \alpha + v^{-2}g(v^{-2})$
2	odd	$\alpha(v+v^{-1})$	$(v^2 - v^{-2})v^{-1}f(v^{-2}) = \alpha v + \beta v^{-1} + v^{-3}g(v^{-2})$
3	even	$\alpha(v+v^{-1})$	$(v^3 - v^{-3})v^{-2}f(v^{-2}) = \alpha v + \beta v^{-1} + v^{-3}g(v^{-2})$
3	odd	α	$(v^3 - v^{-3})v^{-1}f(v^{-2}) = \beta v^2 + \alpha + \gamma v^{-2} + v^{-4}g(v^{-2})$

{l:topterm2}

Lemma 11.15 *In each case in the table we can solve for f.*

After multiplying by the appropriate power of v, these all come down to:

Lemma 11.16 *Suppose $(1 \pm q^k)f(q) = g(q)$ where $g(q)$ is a polyomial. If we know all but the top k coefficients of g, then we can solve for f.*

12 Guide

{s:guide}

In the following tables, we've indicated which formulas to use in various cases.

(1) *: not primitive, easy recursion

(2) NE: not extremal (but primitive): easy recursion, but has a sum on the right hand side

(3) *0: not primitive, necessarily 0

(4) DR: direct recursion (Proposition 10.3.5)

(5) DR+: new type of direct recursion (Proposition 11.13, $\ell(\kappa) = 1$)

(6) DR+?: new type of direct recursion, but the recursion may not work. See Proposition 11.13, $\ell(\kappa) = 2, 3$; we need conditions (A) and (B).

In (2), {*non-primitive*}⊂{*non-extremal*}; the pairs marked NE are in the second set, but not the first (they are *not non-primitive*).

Type 1

	1C-	1r1f	1r1s	1r2	1ic	1C+	1i1	1i2f	1i2s	1rn
1C-	DR	DR								
1r1f	DR	DR								
1r1s	DR	DR							DR+	DR+
1r2	DR	DR								
1ic	DR	DR							DR+	DR+
1C+	*	*	*	*	*				DR+	DR+
1i1	*	*	*	*	*					
1i2f	NE	NE	NE	NE	NE				DR+	DR+
1i2s	*0	*0	*0	*0	*0					
1rn	*0	*0	*0	*0	*0					

Type 2

	2C-	2Cr	2r22	2r21	2r11	2ic	2C+	2Cif	1i11	1i12	2i22	2rn
2C-	DR	DR			DR							
2Cr	DR	DR			DR							
2r22	DR	DR			DR							
2r21	DR	DR			DR							
2r11	DR	DR			DR							
2ic	DR	DR			DR							DR+?
2C+	*	*	*	*	*	*						DR+?
2Ci	*0	*0	*0	*0	*0	*0						DR+?
2i11	*	*	*	*	*	*						
2i12	NE	NE	NE	NE	NE	NE						DR+?
2i22	*	*	*	*	*	*						DR+?
2rn	*0	*0	*0	*0	*0	*0						

Type 3

	3C-	3Cr	3r	3ic	3C+	3Ci	3i	3rn
3C-	DR	DR	DR					
3Cr	DR	DR	DR					
3r	DR	DR	DR					
3ic	DR	DR	DR					DR+?
3C+	*	*	*	*				DR+?
3Ci	*	*	*	*				DR+?
3i	*	*	*	*				DR+?
3rn	*0	*0	*0	*0				

13 Appendix: Proof of Theorem 9.3.10

Throughout this section we assume $\kappa \in \tau(\gamma), \kappa \notin \tau(\lambda)$. Recall we are trying to find $m_\kappa(\gamma, \lambda)$ such that:

$$\widehat{T}_\kappa \widehat{C}_\lambda = \sum_{\gamma | \kappa \in \tau(\gamma)} m_\kappa(\gamma, \lambda) \widehat{C}_\gamma$$

The main tool is the identity (9.3.14) (for $\kappa \notin \tau(\lambda)$):

$$\sum_{\mu | \kappa \in \tau(\mu)} \widehat{P}^\sigma(\gamma, \mu) m_\kappa(\mu, \lambda) = \text{multiplicity of } \hat{a}_\gamma \text{ in } \widehat{T}_\kappa(\widehat{C}_\lambda) \tag{13.1}$$

The right hand side is given by Table 10.2.9, which we reproduce here.

{table:condensed3}

Table 13.2

$t_\gamma(\kappa)$	RHS of (13.1)
`1C-,2C-,3C-`	$v^k \widehat{P}^\sigma(\gamma, \lambda) + \widehat{P}^\sigma(w_\kappa \times \gamma, \lambda)$
`1ic,2ic,3ic`	$(v^k + v^{-k}) \widehat{P}^\sigma(\gamma, \lambda)$
`2Cr,3Cr,3r`	$v(v^{k-1} - v^{-k+1}) \widehat{P}^\sigma(\gamma, \lambda) + (v + v^{-1}) \widehat{P}^\sigma(\gamma_\kappa, \lambda)$
`1r1f,2r11`	$(v^k - v^{-k}) \widehat{P}^\sigma(\gamma, \lambda) + \widehat{P}^\sigma(\gamma^1_\kappa, \lambda) + \widehat{P}^\sigma(\gamma^2_\kappa, \lambda)$
`1r1s`	$(v - v^{-1}) \widehat{P}^\sigma(\gamma, \lambda)$
`1r2,2r22`	$v^k \widehat{P}^\sigma(\gamma, \lambda) - v^{-k} \widehat{P}^\sigma(w_\kappa \times \gamma, \lambda) + \widehat{P}^\sigma(\gamma_\kappa, \lambda)$
`2r21`	$(v^2 - v^{-2}) \widehat{P}^\sigma(\gamma, \lambda) + \sum_{\gamma' \mid \gamma \xrightarrow{\kappa} \gamma'} \epsilon(\gamma, \gamma') \widehat{P}^\sigma(\gamma', \lambda)$

We are going to look at the top degree terms of both sides.

Write any element of $\mathbb{Z}[v, v^{-1}]$ as $f = f^+ + f^-$ where $f^+ \in \mathbb{Z}[v]$ and $f^- \in v^{-1}\mathbb{Z}[v^{-1}]$. We make frequent use of:

Lemma 13.3 *If $\gamma = \mu$ then*

$$[\widehat{P}^\sigma(\gamma, \mu)m_\kappa(\mu, \lambda)]^+ = m_{\kappa,0}(\mu, \lambda) + m_{\kappa,1}(\mu, \lambda)v + m_{\kappa,2}(\mu, \lambda)v^2$$

If $\gamma < \mu$ then

$$\begin{aligned}\widehat{P}^\sigma(\gamma, \mu)m_\kappa(\mu, \lambda)^+ =&[\widehat{\mu}^\sigma_{-2}(\gamma, \mu)m_{\kappa,2}(\mu, \lambda) + \widehat{\mu}^\sigma_{-1}(\gamma, \mu)m_{\kappa,1}(\mu, \lambda)]+\\ &\widehat{\mu}^\sigma_{-1}(\gamma, \mu)m_{\kappa,2}(\mu, \lambda)v\end{aligned}$$

{l:length1}

Lemma 13.4 *If $\ell(\kappa) = 1$ then*

(13.5)
$$m_\kappa(\gamma, \lambda) = m_{\kappa,0}(\gamma, \lambda) = \begin{cases} 1 & \gamma \xrightarrow{\kappa} \lambda \\ \widehat{\mu}^\sigma_{-1}(\gamma, \lambda) & \gamma < \lambda \\ 0 & \textit{else} \end{cases}$$

Proof. Since $\ell(\kappa) = 1$, $m_\kappa(\mu, \lambda) = m_{\kappa,0}(\mu, \lambda)$ for all μ, and on the left hand side of (13.1) the maximal degree is 0. In each term on the right hand side, $[v\widehat{P}^\sigma(\gamma, \lambda)]^+ = \widehat{\mu}^\sigma_{-1}(\gamma, \lambda)$ and $[v^{-1}\widehat{P}^\sigma(\gamma, \lambda)]^+ = 0$. On the other hand a term $\widehat{P}^\sigma(\mu, w_\kappa \times \gamma), \widehat{P}^\sigma(\mu, \gamma_\kappa)$ or $\widehat{P}^\sigma(\mu, \gamma^i_\kappa)$ contributes 1 if and only if the two arguments are equal. So, $[\quad]^+$ of both sides gives (the last column is $t_\gamma(\kappa)$):

(13.6)
$$m_\kappa(\gamma, \lambda) = \widehat{\mu}^\sigma_{-1}(\gamma, \lambda) + \begin{cases} \delta_{w_\kappa \times \gamma, \lambda} & \texttt{1C-} \\ \delta_{\gamma^1_\kappa \lambda} + \delta_{\gamma^2_\kappa, \lambda} & \texttt{1r1f} \\ 0 & \texttt{1r1s} \\ \delta_{\gamma_\kappa, \lambda} & \texttt{1r2} \\ 0 & \texttt{1ic} \end{cases}$$

Each Kronecker δ after the brace is non-zero precisely when $\gamma \xrightarrow{\kappa} \lambda$, in which case $\gamma > \lambda$, so $\widehat{\mu}^\sigma_{-1}(\gamma, \lambda) = 0$. □

This proves Cases (1) ($\ell(\kappa) = 1$) and (2) of Theorem 9.3.10. Note that $m(\gamma, \lambda) = 0$ unless $\gamma \xrightarrow{\kappa} \lambda$ or $\gamma < \lambda$.

13.1 $\ell(\kappa) = 2$

Take the + part of both sides of (13.1). The left hand side is

$$\text{(13.1.1)(a)} \qquad [m_{\kappa,0}(\gamma,\lambda) + \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu) m_{\kappa,1}(\mu,\lambda)] + m_{\kappa,1}(\gamma,\lambda)v$$

The first and last terms are from $\mu = \gamma$, and the second sum is from all other terms $\mu \neq \gamma$. (Note that the summand is 0 if $\mu = \gamma$.)

The right hand side is

$$\text{(13.1.1)(b)} \qquad \widehat{\mu}^{\sigma}_{-2}(\gamma,\lambda) + \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda)v + \begin{cases} \delta_{w_\kappa\times\gamma,\lambda} & \texttt{2C-} \\ \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) + \delta_{\gamma_\kappa,\lambda}v & \texttt{2Cr} \\ \delta_{\gamma_\kappa,\lambda} & \texttt{2r22} \\ \sum_{\gamma'|\gamma\overset{\kappa}{\to}\gamma'} \epsilon(\gamma,\gamma')\delta_{\gamma',\lambda} & \texttt{2r21} \\ \delta_{\gamma^1_\kappa,\lambda} + \delta_{\gamma^2_\kappa,\lambda} & \texttt{2r11} \\ 0 & 2ic \end{cases}$$

Equating the coefficient of v in (a) and (b) gives

$$\text{(13.1.1)(c)} \qquad m_{\kappa,1}(\gamma,\lambda) = \begin{cases} \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda) & t_\gamma(\kappa) \neq \texttt{2Cr} \\ \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda) & t_\gamma(\kappa) = \texttt{2Cr}, \text{ and } \gamma < \lambda \\ 1 & t_\gamma(\kappa) = \texttt{2Cr}, \text{ and } \gamma \overset{\kappa}{\to} \lambda \\ 0 & t_\gamma(\kappa) = \texttt{2Cr}, \text{ otherwise} \end{cases}$$

We can rewrite this

$$\text{(13.1.1)(d)} \qquad \boxed{m_{\kappa,1}(\gamma,\lambda) = \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda) + \begin{cases} \delta_{\gamma_\kappa,\lambda} & t_\gamma(\kappa) = \texttt{2Cr} \\ 0 & \text{otherwise} \end{cases}}$$

In particular $m_{\kappa,1}(\gamma,\lambda) = 0$ unless $\gamma < \lambda$ or $\gamma \overset{\kappa}{\to} \lambda$.

Now (d) holds for all γ with $\kappa \in \tau(\gamma)$, so we can apply it to all γ occuring the sum in (13.1).

So plug this back in to (a), keep only the constant term, and set this equal to the constant term of (b)
(13.1.1)(e)

$$m_{\kappa,0}(\gamma,\lambda) + \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\left[\widehat{\mu}^{\sigma}_{-1}(\mu,\lambda) + \begin{cases} \delta_{\mu_\kappa,\lambda} & t_\mu(\kappa) = \texttt{2Cr} \\ 0 & \text{else} \end{cases}\right] =$$

$$\widehat{\mu}^{\sigma}_{-2}(\gamma,\lambda) + \begin{cases} \delta_{w_\kappa\times\gamma,\lambda} & \texttt{2C-} \\ \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) & \texttt{2Cr} \\ \delta_{\gamma_\kappa,\lambda} & \texttt{2r22-} \\ \displaystyle\sum_{\gamma'|\gamma\xrightarrow{\kappa}\gamma'} \epsilon(\gamma,\gamma')\delta_{\gamma',\lambda} & \texttt{2r21} \\ \delta_{\gamma^1_\kappa,\lambda} + \delta_{\gamma^2_\kappa,\lambda} & \texttt{2r11} \\ 0 & \texttt{2ic} \end{cases}$$

Note that each Kronecker δ after the brace is 1 iff $\gamma \xrightarrow{\kappa} \lambda$. Also we can put $\epsilon(\gamma,\lambda)$ in front of each such term without changing anything (these are 1 unless $t_\gamma(\kappa) =$2r21). Therefore

$$\begin{aligned} m_{\kappa,0}(\gamma,\lambda) &= \widehat{\mu}^{\sigma}_{-2}(\gamma,\lambda) \\ &\quad - \sum_{\mu} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-1}(\mu,\lambda) \\ &\quad - \sum_{\mu} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu) * \begin{cases} \delta_{\mu_\kappa,\delta} & t_\mu(\kappa) = \texttt{2Cr} \\ 0 & \text{else} \end{cases} \\ &\quad + \begin{cases} \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) & t_\gamma(\kappa) = \texttt{2Cr} \\ \epsilon(\gamma,\lambda) & \gamma \xrightarrow{\kappa} \lambda, t_\gamma(\kappa) \neq \texttt{2Cr} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{13.1.2}$$

Then

(13.1.3)
$$\sum_{\mu} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu) * \begin{cases} \delta_{\mu_\kappa,\lambda} & t_\mu(\kappa) = \texttt{2Cr} \\ 0 & \text{else} \end{cases}$$

is equal to

(13.1.4)
$$\begin{cases} \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda^\kappa) & t_\lambda(\kappa) = \texttt{2Ci} \\ 0 & \text{else} \end{cases}$$

{l:length2}

Lemma 13.1.5 *Assume* $\kappa \in \tau(\gamma), \kappa \notin \tau(\lambda)$, $\ell(\kappa) = 2$. *Then*

(13.1.6)(a)
$$\boxed{\begin{aligned} m_{\kappa,0}(\gamma,\lambda) = \widehat{\mu}^{\sigma}_{-2}(\gamma,\lambda) - &\sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-1}(\mu,\lambda) \\ &- \begin{cases} \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda^\kappa) & t_\lambda(\kappa) = \texttt{2Ci} \\ 0 & \textit{else} \end{cases} \\ &+ \begin{cases} \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) & t_\gamma(\kappa) = \texttt{2Cr} \\ \epsilon(\gamma,\lambda) & \gamma \xrightarrow{\kappa} \lambda, t_\gamma(\kappa) \neq \texttt{2Cr} \\ 0 & \textit{else} \end{cases} \end{aligned}}$$

and

(13.1.6)(b)
$$\boxed{m_{\kappa,1}(\gamma,\lambda) = \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda) + \begin{cases} \delta_{\gamma_\kappa,\lambda} & t_\gamma(\kappa) = \texttt{2Cr} \\ 0 & \textit{otherwise} \end{cases}}$$

Let's look at some cases. First assume $\gamma \xrightarrow{\kappa} \lambda$. In particular $\gamma > \delta$, so the first two terms are 0. A little checking gives

(13.1.7)
$$m_{\kappa,0}(\gamma,\lambda) = \begin{cases} \epsilon(\lambda,\gamma) & t_\lambda(\kappa) \neq \texttt{2Ci} \\ 0 & t_\lambda(\kappa) = \texttt{2Ci} \end{cases}$$

Putting this together with formula (13.1.1)(d) for $m_{\kappa,1}$ we get:

(13.1.8)
$$\boxed{\gamma \xrightarrow{\kappa} \lambda \Rightarrow m_\kappa(\gamma,\lambda) = \begin{cases} v+v^{-1} & t_\lambda(\kappa) = \texttt{2Cr} \\ \epsilon(\gamma,\lambda) & \textit{else} \end{cases}}$$

Assume $\gamma \overset{\kappa}{\not\to} \lambda$. We see:

$$\begin{aligned} m_\kappa(\gamma,\lambda) &= \widehat{\mu}^\sigma_{-2}(\gamma,\lambda) - \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^\sigma_{-1}(\gamma,\mu)\widehat{\mu}^\sigma_{-1}(\mu,\lambda) \\ &\quad - \begin{cases} \widehat{\mu}^\sigma_{-1}(\gamma,\lambda^\kappa) & t_\lambda(\kappa) = \texttt{2Ci} \\ 0 & \text{else} \end{cases} + \begin{cases} \widehat{\mu}^\sigma_{-1}(\gamma_\kappa,\lambda) & t_\gamma(\kappa) = \texttt{2Cr} \\ 0 & \text{else} \end{cases} \\ &\quad + \widehat{\mu}^\sigma_{-1}(\gamma,\lambda)(v+v^{-1}) \end{aligned}$$

If $\ell(\lambda) \not\equiv \ell(\gamma) \pmod 2$ all terms but the last are 0, so

(13.1.9)
$$\boxed{\gamma \overset{\kappa}{\not\to} \lambda, \ell(\gamma) \not\equiv \ell(\lambda) \pmod 2 \Rightarrow m_\kappa(\gamma,\lambda) = \widehat{\mu}^\sigma_{-1}(\gamma,\lambda)(v+v^{-1})}$$

On the other hand $\ell(\gamma) = \ell(\lambda) \pmod 2$ implies the last term is 0, and
(13.1.10)
$$\boxed{\begin{aligned} m_\kappa(\gamma,\lambda) &= \widehat{\mu}^\sigma_{-2}(\gamma,\lambda) - \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^\sigma_{-1}(\gamma,\mu)\widehat{\mu}^\sigma_{-1}(\mu,\lambda) \\ &\quad - \begin{cases} \widehat{\mu}^\sigma_{-1}(\gamma,\lambda^\kappa) & t_\lambda(\kappa) = \texttt{2Ci} \\ 0 & \text{else} \end{cases} + \begin{cases} \widehat{\mu}^\sigma_{-1}(\gamma_\kappa,\lambda) & t_\gamma(\kappa) = \texttt{2Cr} \\ 0 & \text{else} \end{cases} \end{aligned}}$$

I believe this agrees with [4, Theorem 4.4].

Note that all terms of $m(\gamma,\delta)$ are 0 unless $\gamma \xrightarrow{\kappa} \delta$ or $\gamma < \delta$, except possibly the last two.

13.2 $\ell(\kappa) = 3$

We continue to assume $\kappa \in \tau(\gamma), \kappa \notin \tau(\lambda)$. In particular $\gamma \neq \lambda$.

Take the + part of both sides of (13.1). The left hand side is: {e:3main}

(13.2.1)(a)

$$[m_{\kappa,0}(\gamma,\lambda) + \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)m_{\kappa,1}(\mu,\lambda) + \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-2}(\gamma,\mu)m_{\kappa,2}(\mu,\lambda)]$$
$$+ [m_{\kappa,1}(\gamma,\lambda) + \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)m_{\kappa,2}(\mu,\lambda)]v + m_{\kappa,2}(\gamma,\lambda)v^2$$

The right hand side is

(13.2.1)(b)

$$\widehat{\mu}^{\sigma}_{-3}(\gamma,\lambda)+\widehat{\mu}^{\sigma}_{-2}(\gamma,\lambda)v + \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda)v^2 + \begin{cases} \delta_{w_\kappa\times\gamma,\lambda} & \texttt{3C}- \\ \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) + \delta_{\gamma_\kappa,\lambda}v & \texttt{3Cr} \\ \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) + \delta_{\gamma_\kappa,\lambda}v & \texttt{3r} \\ 0 & 3ic \end{cases}$$

Comparing the coefficient of v^2 gives

(13.2.1)(c) $$\boxed{m_{\kappa,2}(\gamma,\lambda) = \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda)}$$

Plugging this in to (13.2.1), the coefficient of v gives

(13.2.1)(d)

$$m_{\kappa,1}(\gamma,\lambda) + \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-1}(\mu,\lambda) = \widehat{\mu}^{\sigma}_{-2}(\gamma,\lambda) + \begin{cases} 0 & \texttt{3C}- \\ \delta_{\gamma_\kappa,\lambda} & \texttt{3Cr} \\ \delta_{\gamma_\kappa,\lambda} & \texttt{3r} \\ 0 & 3ic \end{cases}$$

i.e.

(13.2.1)(e)

$$\boxed{m_{\kappa,1}(\gamma,\lambda) = \widehat{\mu}^{\sigma}_{-2}(\gamma,\lambda) - \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-1}(\mu,\lambda) + \begin{cases} 0 & t_\gamma(\kappa) = \texttt{3C}-, \texttt{3ic} \\ \delta_{\gamma_\kappa,\lambda} & t_\gamma(\kappa) = \texttt{3Cr}, \texttt{3r} \end{cases}}$$

Turn the crank one more time, plugging this in, to compute the constant

term:
(13.2.1)(f)

$$m_{\kappa,0}(\gamma,\lambda) = -\sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)m_{\kappa,1}(\mu,\lambda) - \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-2}(\gamma,\mu)m_{\kappa,2}(\mu,\lambda)+$$

$$\widehat{\mu}^{\sigma}_{-3}(\gamma,\lambda) + \begin{cases} \delta_{w_\kappa\times\gamma,\lambda} & \texttt{3C}- \\ \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) & \texttt{3Cr} \\ \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) & \texttt{3r} \\ 0 & 3ic \end{cases}$$

Plug in (c) and (e):
(13.2.1)(g)

$$m_{\kappa,0}(\gamma,\lambda) =$$

$$-\sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\Bigg[\widehat{\mu}^{\sigma}_{-2}(\mu,\lambda) - \sum_{\phi|\kappa\in\tau(\phi)} \widehat{\mu}^{\sigma}_{-1}(\mu,\phi)\widehat{\mu}^{\sigma}_{-1}(\phi,\lambda)$$

$$+\begin{cases} 0 & t_\lambda(\kappa) = 3C-, \texttt{3ic} \\ \delta_{\lambda_\kappa,\lambda} & t_\lambda(\kappa) = \texttt{3Cr}, \texttt{3r} \end{cases}\Bigg]$$

$$-\Bigg[\sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-2}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-1}(\mu,\lambda)\Bigg] + \widehat{\mu}^{\sigma}_{-3}(\gamma,\lambda) + \begin{cases} \delta_{w_\kappa\times\gamma,\lambda} & t_\gamma(\kappa) = \texttt{3C}- \\ \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) & t_\gamma(\kappa) = \texttt{3Cr} \\ \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) & t_\gamma(\kappa) = \texttt{3r} \\ 0 & t_\gamma(\kappa) = 3ic \end{cases}$$

Note that if $t_\gamma(\kappa) = 3C-$, $\delta_{w_\kappa\times\gamma,\lambda} = 1$ if $\gamma \xrightarrow{\kappa} \lambda$, and 0 otherwise. Also, evaluating

(13.2.1)(h) $$\sum_{\mu} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu) * \begin{cases} 0 & t_\lambda(\kappa) = 3C-, \texttt{3ic} \\ \delta_{\lambda_\kappa,\lambda} & t_\lambda(\kappa) = \texttt{3Cr}, \texttt{3r} \end{cases}$$

as in the length 2 case gives

(13.2.1)(i) $$\begin{cases} \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda^\kappa) & t_\lambda(\kappa) = 3Ci \text{ or } \texttt{3i} \\ 0 & else \end{cases}$$

Inserting this information, moving a few terms around, and taking $\lambda < \lambda$ in all sums as in the previous cases, gives

$$\text{(13.2.1)(j)} \quad \boxed{\begin{aligned} m_{\kappa,0}(\gamma,\lambda) = {} & \widehat{\mu}^{\sigma}_{-3}(\gamma,\lambda) \\ & + \sum_{\substack{\mu|\kappa\in\tau(\mu) \\ \phi|\kappa\in\tau(\phi)}} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-1}(\mu,\phi)\widehat{\mu}^{\sigma}_{-1}(\phi,\lambda)+ \\ & - \sum_{\mu|\kappa\in\tau(\mu)} \left[\widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-2}(\mu,\lambda) + \widehat{\mu}^{\sigma}_{-2}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-1}(\mu,\lambda)\right] \\ & - \begin{cases} \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda^{\kappa}) & t_{\lambda}(\kappa) = \texttt{3Ci} \text{ or } \texttt{3i} \\ 0 & \textit{else} \end{cases} \\ & + \begin{cases} 1 & t_{\gamma}(\kappa) = \texttt{3C}-, \gamma \xrightarrow{\kappa} \lambda \\ 0 & t_{\gamma}(\kappa) = \texttt{3C}-, \gamma \overset{\kappa}{\not\to} \lambda \\ \widehat{\mu}^{\sigma}_{-1}(\gamma_{\kappa},\lambda) & t_{\gamma}(\kappa) = \texttt{3Cr} \\ \widehat{\mu}^{\sigma}_{-1}(\gamma_{\kappa},\lambda) & t_{\gamma}(\kappa) = \texttt{3r} \\ 0 & t_{\gamma}(\kappa) = \texttt{3ic} \end{cases} \end{aligned}}$$

Summarizing the length 3 case:

{l:length3}

Lemma 13.2.2 *Assume $\kappa \in \tau(\gamma), \kappa \notin \tau(\lambda)$, $\ell(\kappa) = 3$. Then $m_{\kappa,2}, m_{\kappa,1}, m_{\kappa,0}$ are given by 13.2.1(c),(e), and (j), respectively.*

Let's look at some cases.

Suppose $\gamma \xrightarrow{\kappa} \lambda$. All $\widehat{\mu}^{\sigma}_{-i}$ terms are 0, and

$$\gamma \xrightarrow{\kappa} \lambda \Rightarrow m_\kappa(\gamma,\lambda) = \begin{cases} 1 & t_\gamma(\kappa) = \texttt{3C-} \\ (v+v^{-1}) & t_\gamma(\kappa) = \texttt{3Cr}, \texttt{3r} \\ 0 & t_\gamma(\kappa) = 3ic \end{cases} \tag{13.2.3}$$

Now assume $\gamma \overset{\kappa}{\not\to} \lambda$, and $\ell(\lambda) \equiv \ell(\gamma) \pmod 2$. Then $m_{\kappa,2}(\gamma,\lambda) = m_{\kappa,0}(\gamma,\lambda) = 0$. The formula for $m_{\kappa,1}$ doesn't simplify, except that last term is 0 since $\gamma \overset{\kappa}{\not\to} \lambda$, so

$$m_\kappa(\gamma,\lambda) = \big[\widehat{\mu}^{\sigma}_{-2}(\gamma,\lambda) - \sum_{\mu|\kappa\in\tau(\mu)} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-1}(\mu,\lambda)\big](v+v^{-1}) \tag{13.2.4}$$

Finally assume $\gamma \overset{\kappa}{\not\to} \lambda$, and $\ell(\lambda) \not\equiv \ell(\gamma) \pmod 2$. Then $m_{\kappa,1}(\gamma,\lambda) = 0$, and $m_{\kappa,0}$ simplifies a little, to give:

$$\begin{aligned} m_\kappa(\gamma,\lambda) = {} & \widehat{\mu}^{\sigma}_{-3}(\gamma,\lambda)(v^2+v^{-2}) \\ & + \sum_{\substack{\mu|\kappa\in\tau(\mu) \\ \phi|\kappa\in\tau(\phi)}} \widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-1}(\mu,\phi)\widehat{\mu}^{\sigma}_{-1}(\phi,\lambda) + \\ & - \sum_{\mu|\kappa\in\tau(\mu)} \big[\widehat{\mu}^{\sigma}_{-1}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-2}(\mu,\lambda) + \widehat{\mu}^{\sigma}_{-2}(\gamma,\mu)\widehat{\mu}^{\sigma}_{-1}(\mu,\lambda)\big] \\ & - \begin{cases} \widehat{\mu}^{\sigma}_{-1}(\gamma,\lambda^\kappa) & t_\lambda(\kappa) = 3Ci \text{ or } \texttt{3i} \\ 0 & \textit{else} \end{cases} \\ & + \begin{cases} \widehat{\mu}^{\sigma}_{-1}(\gamma_\kappa,\lambda) & t_\gamma(\kappa) = 3Cr \text{ or } \texttt{3r} \\ 0 & \text{else} \end{cases} \end{aligned} \tag{13.2.5}$$

As in the length 2 case, all terms are zero unless $\gamma \xrightarrow{\kappa} \lambda$ or $\gamma < \lambda$, except possibly the last two.

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