NOTES ON CELLS OF HARISH-CHANDRA MODULES AND SPECIAL UNIPOTENT REPRESENTATIONS

PETER E. TRAPA

Let $G_{\mathbb{R}}$ denote the real points of a connected complex reductive algebraic group defined over $\mathbb{R}$). (For the basic results on cells below, this class of groups is unnecessarily restrictive. But we shall need the assumption when we discuss unipotent representations below.) Let $\mathfrak{g}_{\mathbb{R}}$ denote the Lie algebra of $G_{\mathbb{R}}$ and write $\mathfrak{g}$ for the complexification of $\mathfrak{g}_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup in G, and write K for its complexification. Let HC_λ be the category of Harish-Chandra modules with infinitesimal character λ.

Cells are designed to capture some of the information of tensoring Harish-Chandra modules with finite-dimensional modules. More precisely, given two objection X and Y in HC_λ, write $X > Y$ if there exists a finite-dimensional representation of G appearing the tensor algebra $T(\mathfrak{g})$ such that Y appears a subquotient of $X \otimes F$. Write $X \sim Y$ if both $X > Y$ and $Y > X$. Then $\sim$ is an equivalence relation.

Definition 0.1. Equivalence classes for the relation $\sim$ are called *cells*. Given an irreducible object X in HC_λ, we write $\mathrm{cell}(X)$ for the cell containing X. Clearly the set of cells form a partition of the irreducible objects in HC_λ.

Notice that we could have defined $X >' Y$ if Y appeared in $X \otimes \mathfrak{g}$ (the tensor product of the X with the adjoint representation), and $X \sim' Y$ if both $X >' Y$ and $Y >' X$. Then $\sim$ is the relation obtained by taking the transitive closure of $\sim'$. This is useful to keep in mind, since the label of the edge joining vertices X and Y in the W-graph is the multiplicity of Y in a composition series for $X \otimes \mathfrak{g}$. At infinitesimal character ρ, this information is given in the `wgraph` command in `atlas`. (The output of `atlas` is enough to supply this information for any infinitesimal character λ, but this requires more work.)

A companion definition is that of a cone.

Definition 0.2. Given an irreducible object X in HC_λ, the *cone* over X is defined to be

$$\mathrm{cone}(X) = \{Y \mid X > Y\}.$$

Obviously $\mathrm{cell}(X) \subset \mathrm{cone}(X)$.

Example 0.3. Suppose $G_{\mathbb{R}}$ is a connected reductive Lie complex group and consider HC_ρ, the category of Harish-Chandra modules with the same infinitesimal character as the trivial representation. Then the irreducible objects in HC_ρ are parameterized by W, the Weyl group of $\mathfrak{g}_{\mathbb{R}}$. Thus cells may be viewed as subsets of W. In this context, such cells are usually called double (or two-sided) cells. There is a further partitioning of such double cells into left (or right) cells obtained as follows.

Since the complexified Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_{\mathbb{R}} \oplus \mathfrak{g}_{\mathbb{R}}$, any finite-dimensional representation of $\mathfrak{g}$ is an external tensor product of the form $F_\ell \boxtimes F_r$. In this setting, we could modify the definitions above by allowing tensoring with only finite-dimensional representations of the form

$$F = F_\ell \boxtimes 1.$$

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(In fact, one can take any F_ℓ here, but those which do not appear in the tensor powers of the adjoint representation do not contribute to the definition.) The analogous definitions lead to equivalence classes called left cells. In the same way, one may define right cells. □

Given two irreducible Harish-Chandra modules in the same cell, it is natural to ask what representation-theoretic properties they share. One of the most useful common properties of X and Y is their associated variety, which we now explain.

Definition 0.4 (Vogan)**.** Let X be a finitely-generated $(\mathfrak{g}, K)$-module. Then there always exists a good increasing K-invariant filtration of X compatible with the graded action of the enveloping algebra of $\mathfrak{g}$. If we denote the filtration by (X^j), we have, in particular, that

$$\mathrm{U}^p(\mathfrak{g}) \cdot X^j \subset X^{p+j}$$

with equality if j is large enough. This, together with the remaining properties of a good filtration, are designed so that the associated graded module $\mathrm{gr}(X)$ is in fact a finitely-generated module for the associated graded algebra of the enveloping algebra of $\mathfrak{g}$, namely the symmetric algebra $S(\mathfrak{g})$. Since the filtration was assumed to be K invariant, the $S(\mathfrak{g})$ action on $\mathrm{gr}(X)$ descends to a (finitely-generated) action of $S(\mathfrak{g}/\mathfrak{k})$ on $\mathrm{gr}(X)$. In this way, $\mathrm{gr}(X)$ becomes a finitely generated $(S(\mathfrak{g}/\mathfrak{k}), K)$-modules. Then the *associated variety* of X, denoted $\mathrm{AV}(X)$, is defined to be the support of the $\mathrm{gr}(X)$ as an $S(\mathfrak{g}/\mathfrak{k})$ module.

Alternatively, by a version of Serre's theorem, we can think of $\mathrm{gr}X$ as a K-equivariant coherent sheaf on $(\mathfrak{g}/\mathfrak{k})^*$. Then $\mathrm{AV}(X)$ is simply the support of this sheaf.

In either case, we have that $\mathrm{AV}(X)$ is a closed subvariety of $(\mathfrak{g}/\mathfrak{k})^*$. Because X is also a module for K, $\mathrm{AV}(X)$ is actually a K-invariant subset of $(\mathfrak{g}/\mathfrak{k})^*$. Since X was assumed to be finitely-generated, it is annihilated by an ideal of finite-codimension in the center of the enveloping algebra of $\mathfrak{g}$. By Kostant's theory of harmonics, this in turn implies that $\mathrm{AV}(X)$ consists of nilpotent elements. Thus $\mathrm{AV}(X)$ is a closed K invariant subvariety,

$$\mathrm{AV}(X) \subset \mathcal{N}(\mathfrak{g}/\mathfrak{k})^* := \mathcal{N}(\mathfrak{g}^*) \cap (\mathfrak{g}/\mathfrak{k})^*.$$

By Kostant-Rallis, there are finitely many K orbits on $\mathcal{N}(\mathfrak{g}/\mathfrak{k})$. So, finally, we may write

$$AV(X) = \overline{\mathcal{O}_1^K} \cup \cdots \cup \overline{\mathcal{O}_j^K},$$

for orbits $\mathcal{O}_i^K$ of K on $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$. □

Given an irreducible Harish-Chandra module X, there is another way to attach a nilpotent orbit to X. Let

$$I = \mathrm{Ann}_{U(\mathfrak{g})} X,$$

a primitive ideal of $\mathrm{U}(\mathfrak{g})$. Then we may consider the associated graded ideal $\mathrm{gr}(I)$ in $\mathrm{gr}(\mathrm{U}(\mathfrak{g})) = S(\mathfrak{g})$, and the subvariety of $\mathfrak{g}^*$ it defines,

$$\mathrm{AV}(\mathrm{Ann}(X)) := \mathcal{V}(\mathrm{gr} I) \subseteq \mathfrak{g}^*.$$

Thus $\mathrm{AV}(\mathrm{Ann}(X))$ is a closed subvariety of $\mathfrak{g}^*$ invariant under the action of $G := \mathrm{Ad}(\mathfrak{g})$. Kostant's results on the nilpotent cone once again allow one to deduce that $\mathrm{AV}(\mathrm{Ann}(X))$ is in fact contained in $\mathcal{N}(\mathfrak{g}^*)$, and hence is a union of a (finite) number of closures of nilpotent coadjoint orbits. In fact, it's just a single one.

Theorem 0.5 (Joseph, Borho-Brylinski)**.** *Let I denote a primitive ideal in* $\mathrm{U}(\mathfrak{g})$*. Then there exists a unique nilpotent coadjoint orbit $\mathcal{O}$ such that*

$$\mathrm{AV}(I) = \overline{\mathcal{O}}.$$

It's worth remarking that

$$\text{GKdim}(X) = \frac{1}{2}\dim(\text{AV}(\text{Ann}(X)),$$

where GKdim(X) denotes the Gelfand-Kirillov dimension of X.

The next result relates AV(X) and AV(Ann(X)).

Theorem 0.6 (Vogan)**.** *Suppose X is an irreducible Harish-Chandra modules for $G_{\mathbb{R}}$. Write*

$$\text{AV}(X) = \overline{\mathcal{O}_1^K} \cup \cdots \cup \overline{\mathcal{O}_j^K},$$

and

$$\text{AV}(\text{Ann}X) = \overline{\mathcal{O}}.$$

Then each $\mathcal{O}_i^K$ is a Lagrangian submanifold of the canonical symplectic structure on $\mathcal{O}$. In particular, for each i, we have

$$G \cdot \mathcal{O}_i^K = \mathcal{O},$$

and

$$\text{GKdim}(X) = \dim(\mathcal{O}_i^K).$$

The next result shows that all these invariants are constant on cells.

Theorem 0.7. *If X and Y are two irreducible Harish-Chandra modules for $G_{\mathbb{R}}$ that belong to the same cell, then*

$$\text{AV}(X) = \text{AV}(Y).$$

In particular,

$$\text{AV}(\text{Ann}X) = \text{AV}(\text{Ann}Y).$$

The converse to either condition is false (even if one allows tensoring with finite dimensional representations not in the tensor algebra of $\mathfrak{g}$).

It's natural to ask what orbits arise through the associated variety constructions above. Here are the answers.

Theorem 0.8 (Barbasch-Vogan)**.** *Suppose I is a primitive ideal that contains a codimension one ideal in the center $\text{Z}(\mathfrak{g})$ of the enveloping algebra corresponding (via the Harish-Chandra isomorphism) to an integral weight. Then AV(I) is the closure of an orbit which is special in the sense of Lusztig. Conversely, every special obit arises in this way. More precisely, consider the set of primitive ideals containing the augmentation ideal in $\text{Z}(\mathfrak{g})$. Then each special orbit arises as the dense orbit in the associated variety of such a primitive ideal.*

Theorem 0.9 (Barbasch-Vogan)**.** *Recall that $G_{\mathbb{R}}$ is the real points of an connected reductive algebraic group G defined over $\mathbb{R}$. Let τ denote the outer automorphism of G corresponding to the inner class containing $G_{\mathbb{R}}$. Suppose X is an irreducible Harish-Chandra module for $G_{\mathbb{R}}$ and $\mathcal{O}^K$ is a nilpotent K orbit which is dense in an irreducible component of AV(X). Then $\tau(\mathcal{O}^K) = \mathcal{O}^K$.*

Conversely suppose $\mathcal{O}^K$ is a nilpotent K orbit on $(\mathfrak{g}/\mathfrak{k})^*$ such that $\tau(\mathcal{O}^K) = \mathcal{O}^K$. Then there is an irreducible Harish-Chandra module X for some real form in the inner class containing $G_{\mathbb{R}}$ such that $\mathcal{O}^K$ is dense in an irreducible component of AV(X).

For instance, if $G_{\mathbb{R}}$ is split, then τ is trivial, and every orbit $\mathcal{O}^K$ appears as an orbit dense in an irreducible component of AV(X) for some Harish-Chandra module X for a group in the inner class containing $G_{\mathbb{R}}$. (In fact, X may be taken to be a Harish-Chandra for the split form $G_{\mathbb{R}}$ itself.)

Cells are obviously intrinsically interesting, and there really is no need to motivate their study. But nonetheless a very important motivation exists, namely the precise Barbasch-Vogan formulation of some vague conjectures of Arthur. We recall that theory now.

Fix a nilpotent adjoint orbit $\mathcal{O}^\vee$ for $\mathfrak{g}^\vee$, the Langlands dual of $\mathfrak{g}$. Fix a Jacobsen-Morozov triple $\{e^\vee, h^\vee, f^\vee\}$ for $\mathcal{O}^\vee$, and set

$$\chi(\mathcal{O}^\vee) = \frac{1}{2}\mathfrak{h}^\vee.$$

Then $\chi(\mathcal{O}^\vee)$ is an element of some Cartan subalgebra $\mathfrak{h}^\vee$ of $\mathfrak{g}^\vee$. There is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}^\vee$ canonically identifies with $\mathfrak{h}^*$. Hence we may view

$$\chi(\mathcal{O}^\vee) \in \mathfrak{h}^*.$$

There were many choices made in the definition of $\chi(\mathcal{O}^\vee)$. But nonetheless the infinitesimal character corresponding to $\chi(\mathcal{O}^\vee)$ *is* well-defined; i.e. $\chi(\mathcal{O}^\vee)$ specifies a well-defined maximal ideal in $\mathrm{Z}(\mathfrak{g})$. We call this the unipotent infinitisimal character attached to $\mathcal{O}^\vee$.

By an old result of Dixmier, ther exists a unique maximal primitive ideal in $\mathrm{U}(\mathfrak{g})$ with infinitesimal character $\chi(\mathcal{O}^\vee)$. Denote it by $I(\mathcal{O}^\vee)$, and let $d(\mathcal{O}^\vee)$ denote the dense nilpotent coadjoint orbit for $\mathfrak{g}$ inside $\mathrm{AV}(I(\mathcal{O}^\vee)$. The orbit $d(\mathcal{O}^\vee)$ is called the Spaltenstein dual of $\mathcal{O}^\vee$ (after Spaltenstein who first defined it in a different way). Notice that Theorem 0.8 implies $d(\mathcal{O})$ is special when $\mathcal{O}^\vee$ is even (or, equivalently, $\chi(\mathcal{O}^\vee)$ is integral). Conversely, it turns out that *every* special nilpotent coadjoint orbit arises this way.

Define

$$\mathrm{Unip}(\mathcal{O}^\vee) = \{X \in \mathrm{irrHC}_{\chi(\mathcal{O}^\vee)} \mid \mathrm{Ann}(X) = I(\mathcal{O}^\vee)\}.$$

This is the set of special unipotent representations for $G_\mathbb{R}$ attached to $\mathcal{O}^\vee$. Notice that such a representation cannot be too small (since $\chi(\mathcal{O}^\vee))$ is singular) but can't be too large (since the associated variety of its annihilator is specified). In fact, the definition shows that special unipotent representations are characterized as existing on the interface of these two competing restrictions.

Conjecture 0.10 (Barbasch-Vogan, Arthur)**.** *The set* $\mathrm{Unip}(\mathcal{O}^\vee)$ *consists of unitary representations.*

Recall that the output of `atlas` gives information only about the translation family containing the trivial infinitesimal character. But the infinitesimal character $\chi(\mathcal{O}^\vee)$ is usually singular. (For instance, if $\mathcal{O}^\vee$ is even, then $\chi(\mathcal{O}^\vee)$ is regular only if $\mathcal{O}^\vee$ is principal, in which case $\mathrm{Unip}(\mathcal{O}^\vee)$ consists only of the trivial representation). The theory of cells allows one to pass from `atlas` output to statements about $\mathrm{Unip}(\mathcal{O}^\vee)$ if $\mathcal{O}^\vee$ is even. We now explain how.

Fix an even orbit $\mathcal{O}^\vee$, and set $\mathcal{O} = d(\mathcal{O}^\vee)$. Fix a cell C of irreducible modules in $\mathrm{HC}_{\chi(\mathcal{O}^\vee)+2\rho}$ such that if $X \in C$, $\mathcal{O}$ is dense in $\mathrm{AV}(\mathrm{Ann}(X))$. Define a translation functor

$$\psi \;:\; \mathrm{HC}_{\chi(\mathcal{O}^\vee)+2\rho} \longrightarrow \mathrm{HC}_{\chi(\mathcal{O}^\vee)}$$

which "pushes" to the walls defined by $\chi(\mathcal{O}^\vee)$ (and which "crosses" no walls). Then ψ takes irreducible modules to either irreducible modules or zero.

Proposition 0.11. *In the setting above,*

$$\{\psi(X) \mid X \in C\} \subset \mathrm{Unip}(\mathcal{O}^\vee).$$

Morever, as one varies the cell C*, every element of* $\mathrm{Unip}(\mathcal{O}^\vee)$ *arises in this way.*

Now `atlas` computes cells as well as the Langlands parameters of representations of $G_\mathbb{R}$ with regular integral infinitesimal character. The effect of ψ on these parameters is easy to write down. Thus one concludes:

Theorem 0.12. *The Langlands parameters of the set of special unipotent representations of* $G_\mathbb{R}$ *with integral infinitesimal character are effectively computable from the output of* `atlas` .

We conclude this lecture with a discussion of a few concrete problelms suggested by the theory sketched above.

Problem: W-module structure. Fix a cell C of Harish-Chandra modules with regular integral infinitesimal character. Then the coherent continuation representation gives the free $\mathbb{Z}$ module $\mathbb{Z}[C]$ with basis index by C the structure of a canonical W-module. This module is based in the sense that it is given to us with a canonical basis The problem is to determine what based W-modules arise in this way.

For instance, if $G_{\mathbb{R}}$ is classical Barbasch-Vogan (in type A) and McGovern (in other types) proved that each cell is isomophic (as a based W-module) to a complex left cell. The numerology in E8 seems to support the analogous statement there. Such a statement fails for easy reasons in complex classical groups outside of type A. Is there a reasonable replacement perhaps in the spirit of Theorem 0.9 above?

Problem: Primitive ideals. Fix a cell C of Harish-Chandra modules with trivial infinitesimal character. Define a relation on element $X, Y \in C$ via $X \equiv Y$ iff $\mathrm{Ann}(X) = \mathrm{Ann}(Y)$. The problem is to compute this relation.

As discussed in Vogan's 1977 paper which introduced the generalized τ-invariant, the W-graph of a cell (obtained via the command `wcells`) provides easy necessary conditions. That paper proves they (or rather a tiny subset of them) are sufficient in type A. Garfinkle proved a similar statement for other classical types. Binegar has verified it for E8. Can one give a conceptual proof?

Problem: Associated Varieties. Fix a cell C and (any) $X \in C$. The problem is to compute $\mathrm{AV}(\mathrm{Ann}(X))$ and $\mathrm{AV}(X)$. Part of the problem is defining the word "compute" since it is not obvious how to parametrize nilpotent orbits in $\mathfrak{g}^*$ and $(\mathfrak{g}/frk)^*$. Since the associated variety of a cohomologically induced representation is easy to compute, one can compute lots of associated varieties. Are there are a manageable set of "basic" cases which, when combined with cohomological induction, suffice to compute associated varieties in general?

Problem: Enumerating all unitary representations with integral infinitesimal character. Let $\widehat{G}_{\mathbb{R}}^{\text{int,unit}}$ denote the set of unitary representations of $G_{\mathbb{R}}$ with integral infinitesimal character. We describe a conjecture of Vogan which describes $\widehat{G}_{\mathbb{R}}^{\text{int,unit}}$.

First we define a set of unitary representations $\widehat{G}_{\mathbb{R}}^{\text{int,conj}}$ which will be explicitly enumerable from the output of `atlas`. The conjecture will then be $\widehat{G}_{\mathbb{R}}^{\text{int,conj}} = \widehat{G}_{\mathbb{R}}^{\text{int,unit}}$.

Fix an irreducible Harish-Chandra module X for $G_{\mathbb{R}}$ with integral infinitesimal character. Suppose that there exists

(1) a θ-stable parabolic (in the sense of Vogan) $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$; and
(2) a special unipotent (in the sense of Barbasch-Vogan, as described above) $(\mathfrak{l}, L \cap K)$-module Z such that the infinitesimal character is in the *weakly fair range* for $\mathfrak{q}$ (in the sense of Vogan).

such that

$$X \text{ is a constituent of the cohomologically inducted module } \mathcal{R}_{\mathfrak{q}}(X).$$

Then a theorem of Vogan implies X is unitary. We let $\widehat{G}_{\mathbb{R}}^{\text{int,conj}}$ denote the set of all such X's obtained in this way. Thus

$$\widehat{G}_{\mathbb{R}}^{\text{int,conj}} \subset \widehat{G}_{\mathbb{R}}^{\text{int,unit}}.$$

Note that the cohomologically induced module $\mathcal{R}_{\mathfrak{q}}(Z)$ above may be reducible. Note also, however, that the Langlands parameters of its irreducible constituents are effectively computable from those of Z and from an explicit knowledge of the coherent continuation representation in the basis of irreducible modules. Theorem 0.12 shows that the Langlands parameters of special unipotent Z with integral infinitesimal character are known from the output of `atlas`. Moreover, `atlas` also computes the coherent continuation representation on irreducibles in the `wgraph command`. The conclusion is that $\widehat{G}_{\mathbb{R}}^{\mathrm{int,conj}}$ is effectively computable from the output of `atlas`.

Conjecture 0.13 (Vogan)**.** *In the setting above, $\widehat{G}_{\mathbb{R}}^{\mathrm{int,conj}}$ exhausts the set of unitary representations of $G_{\mathbb{R}}$ with integral infinitesimal character:*

$$\widehat{G}_{\mathbb{R}}^{\mathrm{int,conj}} = \widehat{G}_{\mathbb{R}}^{\mathrm{int,unit}}.$$

As an example, consider $G_{\mathbb{R}} = \mathrm{U}(p,q)$. Then the conjecture says that every unitary representations with integral infinitesimal character should be a weakly fair $A_{\mathfrak{q}}(\lambda)$ module (which, in this case, are all irreducible when nonzero). Even this special case is still open, although Barbasch has proved that any *spherical* unitary representation with integral infinitesimal character is indeed unitary.