Atlas Meeting V, Palo Alto, July 16-20, 2007

David Vogan: More cells [1]

Monday 10:45

Peter explained that to an irreducible HC module X one can attach the variety AV(X) which is a finite union of $K_{\mathbb{C}}$-orbits in

$$\overline{\mathscr{O}} \cap (\mathfrak{g}/\mathfrak{k})^*$$

where $\overline{\mathscr{O}} = \mathrm{AV}(\mathrm{Ann}\,\mathrm{X})$. We want to compute AV(X) because it is a weak form of lots of more classical invariants of X. For example,

– computation of $\mathrm{X}|_{\mathrm{K}}$

– understanding whether X comes from a nilpotent orbit by the "orbit method" (whathever it means).

Pause for a second. Recall there is something called Gelfand-Kirillov dimension of X, denoted Dim(X); it is the unique integer $d \geqslant 0$ so that if $(\mathrm{X}_k)_{k\geqslant 0}$ is a good filtration of X,

$$\dim_{\mathbb{C}} \mathrm{X}^k \sim a_{\mathrm{X}} k^d + \mathrm{O}(k^{d-1}) \qquad \text{as} \quad k \longrightarrow \infty$$

In particular, $\mathrm{Dim}(\mathrm{X}) = 0$ iff X is finite dimensional. So Dim(X) is a measure of how much infinite dimensional X is. From Peter's talk one may deduce that

$$\mathrm{Dim}(\mathrm{X}) = \dim_{\mathbb{C}} \mathrm{AV}(\mathrm{X}) = \frac{1}{2} \dim_{\mathbb{C}} \mathrm{AV}(\mathrm{Ann}\,\mathrm{X}).$$

THEME: (Philosophy of Kirillov, Kostant) Interesting unitary representations of G arise from coadjoint orbits of G on $i\mathfrak{g}_0^*$ (where $\mathfrak{g}_0 = \mathrm{Lie}(\mathrm{G})$).

Still mysterious unitary representations of G arise from *nilpotent* coadjoint orbits of G on $i\mathfrak{g}_0^*$.

WANT a map from real nilpotent coadjoint orbits (+ a little more data) to unitary representations of G.

What light can Atlas shed?

***Theorem 1* (Kostant-Sekiguchi)**. — *There is a natural bijection between nilpotent orbits of* G *on* $i\mathfrak{g}_0^*$ *and nilpotent orbits of* $\mathrm{K}_{\mathbb{C}}$ *on* $(\mathfrak{g}/\mathfrak{k})^*$.

WANT a map from nilpotent orbits on $\mathrm{K}_{\mathbb{C}}$ on $(\mathfrak{g}/\mathfrak{k})^*$ to unitary representations of G.

What we have is a map associating to an irreducible HC module X several nilpotents orbits of $\mathrm{K}_{\mathbb{C}}$ on $(\mathfrak{g}/\mathfrak{k})^*$. This map is many many to a few ones.

One would like to "invert" the AV correspondance.

Plan: given a $\mathrm{K}_{\mathbb{C}}$-orbit on $(\mathfrak{g}/\mathfrak{k})^*$:

– list all HC X such that $\mathrm{AV}(\mathrm{X}) = \overline{\mathscr{O}}$

– decide which ought to be the "orbit method" of $\mathscr{O}$

[1] notes by Patrick Polo

Remark: it is not clear which infinitesimal characters one should impose, so one cannot restrict from the beginning to a given infinitesimal character.

Theorem 2. — *Suppose* $\mathscr{O}$ *is a nilpotent* $\mathrm{K}_{\mathbb{C}}$*-orbit on* $(\mathfrak{g}/\mathfrak{k})^*$*. Assume that the boundary* $\partial\overline{\mathscr{O}}$ *has complex dimension* $\geqslant 2$ *in* $\overline{\mathscr{O}}$*. Then if* $\overline{\mathscr{O}}$ *is a component of* $\mathrm{AV(X)}$ *then* $\overline{\mathscr{O}} = \mathrm{AV(X)}$*.*

Light shed by Atlas: want Atlas to solve first question, that is, list all X with $\mathrm{AV(X)} = \overline{\mathscr{O}}$. This is a finite problem: by translation principles, one is reduced to look at a finite number of facets.

PROBLEMS Need parametrization of nilpotent $\mathrm{K}_{\mathbb{C}}$-orbit on $(\mathfrak{g}/\mathfrak{k})^*$ that Atlas understands and can compute with.

In the rest of the talk, will concentrate on the complex case, that is, want to deal with

$$\overline{\mathrm{Ad}(\mathfrak{g})\cdot \mathrm{AV(X)}} = \mathrm{AV(Ann\,X)}.$$

Let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{h}$ a Cartan subalgebra inside a Borel subalgebra $\mathfrak{b}$, W the Weyl group, S the set of simple reflections defined by $\mathfrak{b}$.

Definition 1. — A linear form $\lambda \in \mathfrak{g}^*$ is called *nilpotent* if $\ker\lambda$ contains a Borel subalgebra. These form a closed subvariety $\mathscr{N}(\mathfrak{g}^*)$ of $\mathfrak{g}^*$

Philosophy: as AV(X) lies inside $\mathfrak{g}^*$, we need to understand nilpotent elements in $\mathfrak{g}^*$ (not in $\mathfrak{g}$). It is a bad idea to use an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$.

G acts on $\mathscr{N}(\mathfrak{g}^*)$ with finitely many orbits. Want to parametrize these orbits. There are several possibilities: Jacobson-Morozov, Dynkin lists, Bala-Carter. These are case-by-case lists. The question is that we would like to describe the answer without lists (e.g. a rule telling where to put 0 or 1 or 2's on the Dynkin diagram).

Remark: For E_8 there are 70 nilpotent orbits, out of a total of possible $3^8 = 81^2 = 6561$ choices .

What David learned from Lusztig is that there is a natural inclusion

$$\mathscr{N}(\mathfrak{g}^*)/\,\mathrm{Ad}(\mathfrak{g}) \hookrightarrow \widehat{\mathrm{W}}, \qquad \mathscr{O} \mapsto \tau(\mathscr{O})$$

which is an extension of the Springer's correspondence. In Lusztig's point of view, this is the right way to parametrize nilpotent orbits: identify the image in $\widehat{\mathrm{W}}$.

Theorem 3. — *If* $\mathscr{C}$ *is a cell for a real form of* G *then* $\sigma(\mathscr{C})$ (= *the* W *representation carried by* $\mathscr{C}$) *contains exactly one special representation* τ *of* W (*probably with multiplicity one, not proved in general yet*)*, and*

$$\tau = \tau(\mathscr{O}(\mathscr{C})).$$

In a sense, this computes the orbit attached to $\mathscr{C}$: Compute the W-character of $\mathbb{Z}\mathscr{C}$, decompose it into irreducibles (using character table of W), find the special representation (using tables of special representations, e.g. in Lusztig's book), and the nilpotent orbit giving this special representation (using Carter's book"Finite groups of Lie type", or book by Collingwood-McGovern).

This could be done by Atlas, but there is (perhaps) a way to simplify this and/or complicate it to get $\mathrm{AV}(\mathscr{C}) = \overline{\mathscr{O}} \cap \mathfrak{k}^{\perp}$ (recall that $\mathfrak{k}^{\perp}$ denotes the orthogonal of $\mathfrak{k}$ in $\mathfrak{g}^{*}$).

Remark: the wcells command gives a matrix for each simple reflection s; this corresponds to wall-crossing action on simples.

Remark: the set of special representations is a subset of the representations arising from Springer's correspondence. In fact, they correspond to special nilpotent orbits (but the latter are defined in terms of special representations...). Lusztig's definition is in terms of representations of finite Chevalley groups.