Functors for Matching Kazhdan-Lusztig polynomials for $\mathrm{GL}(n, F)$

Peter E. Trapa
(joint with Dan Ciubotaru)

$$F_\lambda \ : \ X \mapsto \mathrm{H}_0(X \otimes F)_{[\lambda + N\rho]}$$

KLV polynomials

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$K_{\mathbb{R}}$ maximal compact subgroup, $K = (K_{\mathbb{R}})_{\mathbb{C}}$
$\mathfrak{B}$ variety of Borel subalgebras in $\mathfrak{g}$.
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Given $\mathcal{L}, \mathcal{L}' \in \mathrm{Loc}_K(\mathfrak{B})$, we can define the Kazhdan-Lusztig-Vogan polynomial

$$p_{\mathcal{L}\mathcal{L}'}(q).$$

Notation for special case

If $\mathcal{L}$ and $\mathcal{L}'$ are the constant sheaves on $Q, Q' \in K\backslash\mathfrak{B}$, instead write

$$p_{Q,Q'} = p_{\mathcal{L},\mathcal{L}'}$$

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$$\mathrm{Loc}_K(\mathfrak{B}) \leftrightarrow W.$$

If $Q(x)$ and $Q(y)$ are two orbits parametrized by x and y, then we write

$$p_{x,y} = p_{Q(x),Q(y)}.$$

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In fact

$$\mathrm{Loc}_K(\mathfrak{B}) \;\leftrightarrow\; (\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))\backslash\mathfrak{B}.$$

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Obviously this is more impressive if $\mathcal{S}^1$ is large.

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- Indicate briefly a general setting to search for such matchings. (Works in all simple adjoint classical cases, for instance, but breaks in interesting ways in a handful of exceptional ones.)
- Handoff to Dan: a functor "implementing" the matching.

Intermediate Combinatorial Set

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We assume this is the case and rewrite

$$\lambda = (\overbrace{\lambda_1, \ldots, \lambda_1}^{n_1} > \cdots\cdots\cdots > \overbrace{\lambda_k, \ldots, \lambda_k}^{n_k}).$$

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$$W(\lambda) = S_{n_1} \times \cdots\cdots\cdots \times S_{n_k}.$$

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For instance, if $\lambda = (5, 5, 4, 4, 4, 4, 3, 3, 3, 2, 2, 1)$, consider:

5	4	3	2	1
•	•	•	•	•
•	•	•	•	
	•	•		
	•			

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Consider a directed graph such that on this vertex set such that

- The only allowable edges are of the form $x_i \to x_{i+1}$ with x_i in the λ_i pile and x_{i+1} in the λ_{i+1} pile; and
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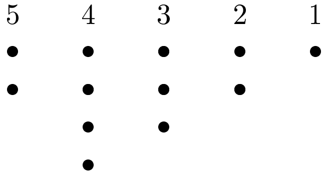

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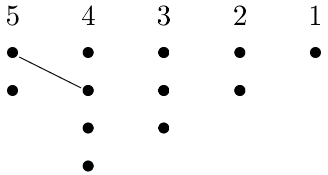

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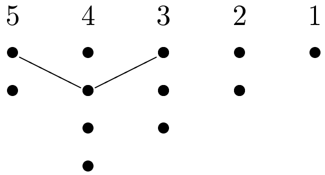

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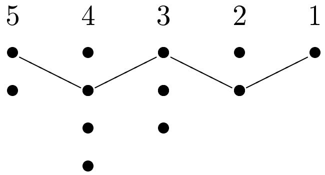

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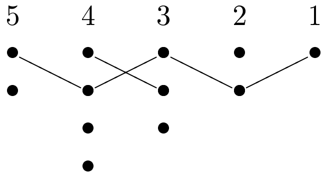

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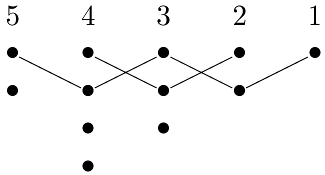

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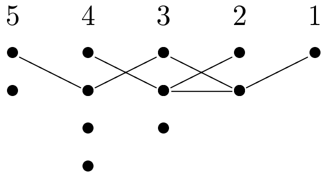

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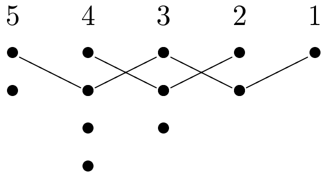

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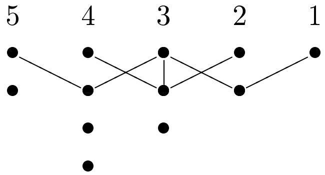

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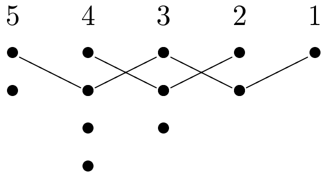

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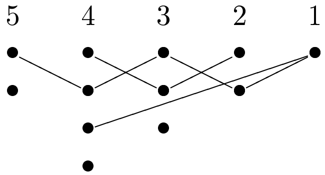

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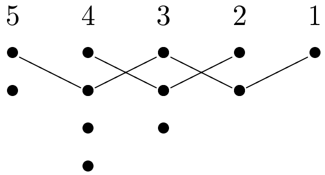

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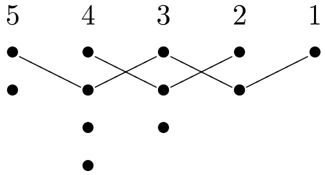

Clearly $W(\lambda) \simeq S_{n_1} \times \cdots \times S_{n_k}$ acts on such graphs.

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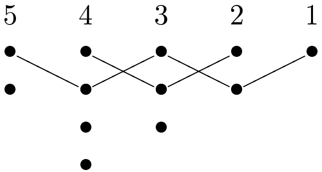

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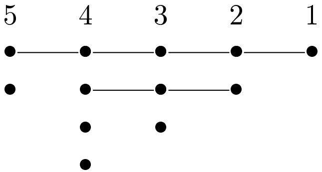

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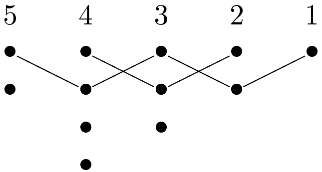

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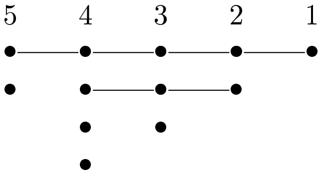

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The set of equivalence classes, denoted $\mathcal{MS}(\lambda)$, are called λ-multisegments.

How do multisegments naturally arise?

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Except this doesn't make sense.

Try again: How do multisegments naturally arise?

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There is a beautiful generalization to all types (Kawanaka, Lusztig, Vinberg).

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$$G^\vee(\lambda) \simeq$$
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Example of parametrization

Take $\lambda = (5,5,4,4,4,4,3,3,3,2,2,1)$. So

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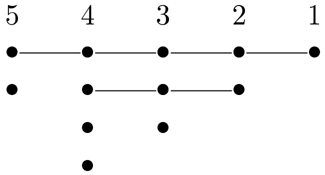

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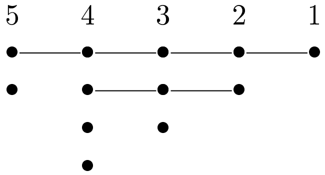

5	4	3	2	1
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a_2	b_2	c_2	d_2	
	b_3	c_3		
	b_4			

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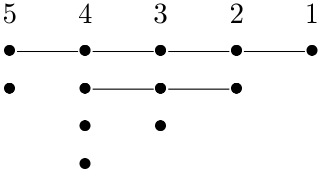

$$\begin{array}{ccccccccc} 5 & & 4 & & 3 & & 2 & & 1 \\ a_1 & \longmapsto & b_1 & \longmapsto & c_1 & \longmapsto & d_1 & \longmapsto & e_1 \\ a_2 & & b_2 & \longmapsto & c_2 & \longmapsto & d_2 & & \\ & & b_3 & & c_3 & & & & \\ & & b_4 & & & & & & \end{array}$$

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Since $\mathcal{MS}(\lambda)$ parametrizes orbits of $G^\vee(\lambda)$ on $\mathfrak{g}_1^\vee(\lambda)$, given $\mathbf{s}, \mathbf{s}' \in \mathcal{MS}(\lambda)$ we can consider

$$p_{\mathbf{s},\mathbf{s}'}(q) := p_{\mathcal{L},\mathcal{L}'}(q)$$

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In general, Lusztig (2006) has given an algorithm to compute these polynomials.

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such that all Kazhdan-Lusztig-Vogan polynomials match:

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In particular, this gives a matching of KLV polynomials for $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{U}(p,q)$.

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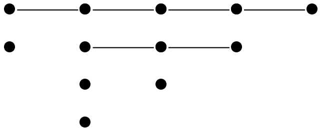

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$$\Phi_1(\mathbf{s}) := \text{longest element in } \sigma W(\lambda).$$

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Then there is a bijection

$$\Sigma_\pm(n) \leftrightarrow \coprod_{p+q=n} (\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C}))\backslash\mathfrak{B}_n.$$

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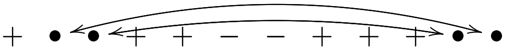

$$+ \;\bullet\; \bullet\; +\; +\; -\; -\; +\; +\; +\; \bullet\; \bullet \quad \in \quad \Sigma_{\pm}(12).$$

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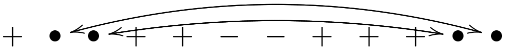

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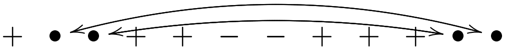

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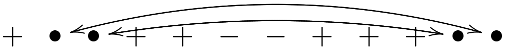

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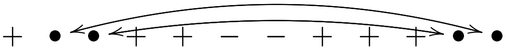

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$$+\,1\,2\,+\,+\,-\,-\,+\,+\,+\,2\,1$$

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3. It's easy to translate the information from the `atlas` command `kgb` in this parametrization.

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Set $\alpha = e_1 - e_2$, $\beta = e_2 - e_3$, and consider $(\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}))\backslash\mathfrak{B}_3$.

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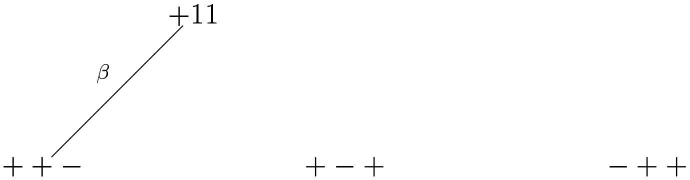

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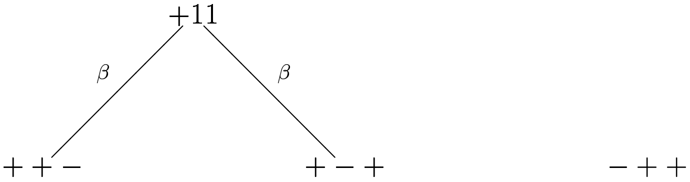

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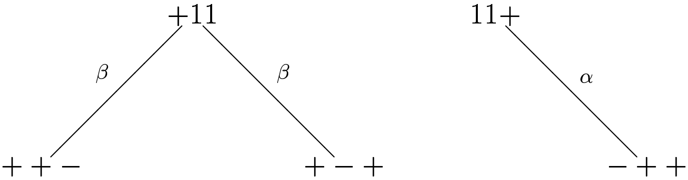

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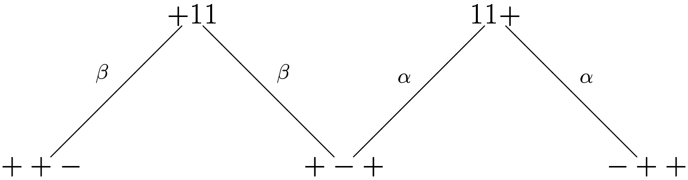

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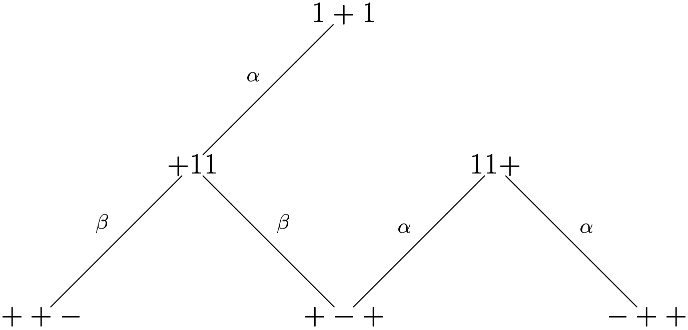

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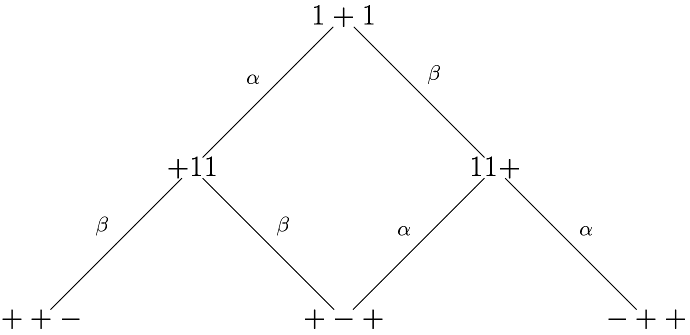
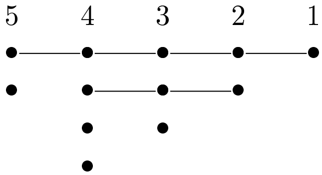


Defining $\Phi_2 \ : \ \mathcal{MS}(\lambda) \longrightarrow \Sigma_{\pm}(n)$

Consider our running example:



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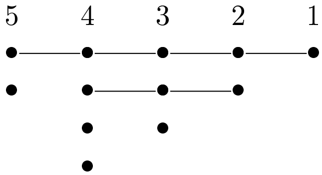

5	4	3	2	1
+	−	+	−	+
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	−	+		
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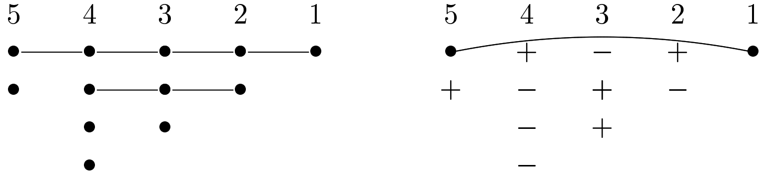

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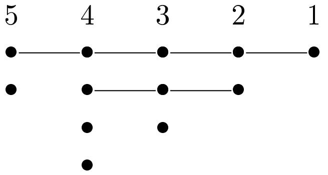

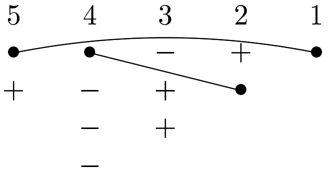

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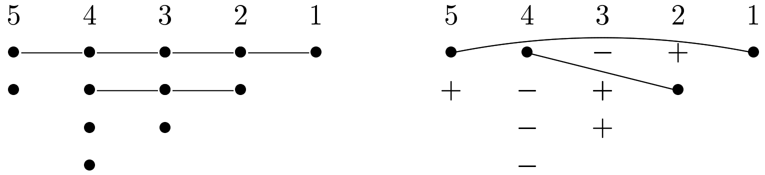
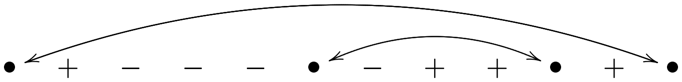


Now "flatten".

Flatten...



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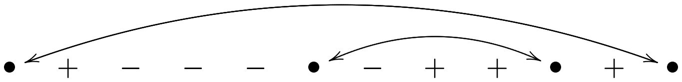

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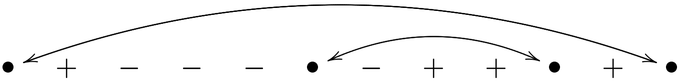

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Remedy: take largest dimensional orbit obtained this way.

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Theorem (Zelevinsky, CT)

All Kazhdan-Lusztig-Vogan polynomials match:

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The upshot

We have defined injections

$$\Phi_1 \; : \; \mathcal{MS}(\lambda) \longrightarrow S_n$$

and

$$\Phi_2 \; : \; \mathcal{MS}(\lambda) \longrightarrow \coprod_{p+q=n} (\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C}))\backslash\mathfrak{B}_n.$$

Theorem (Zelevinsky, CT)

All Kazhdan-Lusztig-Vogan polynomials match:

$$p_{\mathbf{s},\mathbf{s}'} = p_{\Phi_1(\mathbf{s}),\Phi_1(\mathbf{s}')} = p_{\Phi_2(\mathbf{s}),\Phi_2(\mathbf{s}')}.$$

In particular, this gives a matching of KLV polynomials for $\mathrm{GL}(n,\mathbb{C})$ *and* $\mathrm{U}(p,q)$.

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$$\bigoplus_i K\mathcal{HC}(G_i)_\lambda \overset{\text{dual}}{\leftrightarrow} \text{geometry of } \mathcal{G}^\vee(\mathbb{C}) \text{ orbits on } X_{\mathbb{R},\lambda}.$$

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In the case of $\mathcal{G} = \mathcal{GL}(n)$, this unravels (on the level of orbits) to give the map

$$\Phi_2 \ : \ \mathcal{MS}(\lambda) \longrightarrow \coprod_{p+q=n} (\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C}))\backslash\mathfrak{B}_n.$$

Generalizations?

If $\mathcal{G}$ is simple, adjoint, and classical, one may unravel the natural map $X_{F,\lambda} \longrightarrow X_{\mathbb{R},\lambda}$ in much the same way.

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Find an anologous matching of KLV polynomials for $\mathcal{UN}(\mathcal{G}/F)$ and $\mathcal{HC}(\mathcal{G}/\mathbb{R})$ (and a weaker one for $\mathcal{HC}(\mathcal{G}/\mathbb{C})$).

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Are there *functors* explaining these relationships?

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we anticipate functors

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$$\mathcal{HC}(\mathrm{GL}(n,\mathbb{R}))_\lambda \longrightarrow \mathcal{UN}(\mathrm{GL}(n,\mathbb{Q}_p))$$

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should take standard modules to standard modules (or zero) and irreducibles to irreducibles (or zero).

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See Dan Ciubotaru's talk.