COMPUTING REAL WEYL GROUPS

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Let $\mathbb{G}$ be a complex connected reductive algebraic group defined over $\mathbb{R}$. Let $\mathbb{H}$ denote a maximal algebraic torus in $\mathbb{G}$. Write G for the real points of G and H for the real points of H. In applications to the representation theory of G, the real Weyl group

$$W(G,H) := N_G(H)/H$$

plays a fundamental role. Meanwhile, it is the complex Weyl group

$$W(\mathbb{G},\mathbb{H}) := N_{\mathbb{G}}(\mathbb{H})/\mathbb{H}$$

which is more easily accessible. The point of these notes is to explain how to describe $W(G,H)$ in an algorithmic way; that is, in the way (roughly) that `atlas` implements internally. There are perhaps shortcomings of this approach which are illustrated in an example at the end.

Since $N_{\mathbb{G}}(\mathbb{H})$ is clearly defined over $\mathbb{R}$, there is the group of real points:

$$W(\mathbb{G},\mathbb{H})(\mathbb{R}) := [N_{\mathbb{G}}(\mathbb{H})/\mathbb{H}]\,(\mathbb{R})$$

which is intermediate in the sense that

$$W(G,H) \subset W(\mathbb{G},\mathbb{H})(\mathbb{R}) \subset W(\mathbb{G},\mathbb{H}).$$

The inclusions are all obvious.

Let $R = R(\mathbb{G},\mathbb{H})$ denote the root lattice of $\mathbb{H}$ in $\mathbb{G}$ and let $R^\vee = R^\vee(\mathbb{G},\mathbb{H})$ denote the coroot lattice. Given $\alpha \in R$, we obtain a reflection s_α of the euclidean space $R \otimes_{\mathbb{Z}} \mathbb{R}$ which preserves R. The group $W(\mathbb{G},\mathbb{H})$ is generated by the reflections s_α.

Now let θ be a Cartan involution for G. This means, in particular, that θ is an algebraic involution of $\mathbb{G}$ that preserves G so that the fixed points of θ in G are a maximal compact subgroup of G. We can and do choose θ so that it preserves $\mathbb{H}$. Then θ acts on $X^*(\mathbb{H})$ and $X_*(\mathbb{H})$ and preserves both R and $R^\vee$. Then we have the usual taxonomy of elements $\alpha \in R$:

- (C) α is complex if $\theta\alpha \notin \{\pm\alpha\}$;
- (r) α is real if $\theta\alpha = -\alpha$ (i.e. $\alpha(H) \subset \mathbb{R}$);
- (i) α is imaginary if $\theta(\alpha) = \alpha$ (i.e. $\alpha(H) \subset i\mathbb{R}$); moreover:
  - (n) α is noncompact if θ acts by -1 on the α root space inside $\mathfrak{g} := \mathrm{Lie}(\mathbb{G})$; and
  - (c) α is compact if θ acts by $+1$ on the α root space inside $\mathfrak{g} := \mathrm{Lie}(\mathbb{G})$.

These are the only possibilities that can arise. Note that (C), (r), and (i) depend only on θ restricted to H while (n) and (c) depend on the action of θ on all of G.

The first easy point to note is that

$$W(\mathbb{G},\mathbb{H})(\mathbb{R}) = W(\mathbb{G},\mathbb{H})^\theta. \tag{0.1}$$

In `atlas` θ and the elements of $W(\mathbb{G},\mathbb{H})$ are automorphisms of the lattice $X^*(\mathbb{H})$. That is, they are integer matrices of size equal to the rank of $\mathbb{G}$.

There is an interesting aside here. One can ask the following question: which θ's appear? To answer it, begin by fixed a based root datum $(X^*, R^+, X_*, (R^\vee)^+)$. The "inner class" command in `atlas` fixes an involutive automorphism τ of the based root datum. This is an automorphism τ of X^* (i.e. a square integer matrix) of order one or two such that τ preserves R^+ and its transpose

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preserves $(R^\vee)^+$. This is an "inner class of real forms of G". (If G is split, for instance, τ is the identity.) Consider the semidirect product

$$W(\mathbb{G},\mathbb{H}) \rtimes \{1,\tau\}$$

where τ acts by permuting the simple reflections that generate the complex Weyl group. A *twisted involution* is an element of order two in the coset $\tau \cdot W(\mathbb{G},\mathbb{H})$. (This is the data that appears in the last column of the `kgb` command: the restriction of θ to the nilradical of a representative of the orbit containing a θ-stable Cartan is an involution.)

Returning to the description of real Weyl groups, (0.1) indicates that we first need to determine what elements commute with θ. Clearly every reflection in a real or imaginary root does — in fact, these are the only reflections that do — but there are more elements that do. To get started, let R_{re} denote the set of real roots and let R_{im} denote the set of imaginary roots. Clearly $W(R_{\mathrm{re}})$ and $W(R_{\mathrm{im}})$ commute, so we have

$$W(R_{\mathrm{re}}) \times W(R_{\mathrm{im}}) \subset W(\mathbb{G},\mathbb{H})^\theta.$$

Fix choices R^+_{re} and R^+_{im} and define

$$2\rho^\vee_{\mathrm{re}} = \sum_{\alpha\in R^+_{\mathrm{re}}} \alpha^\vee$$
$$2\rho^\vee_{\mathrm{re}} = \sum_{\alpha\in R^+_{\mathrm{im}}} \alpha^\vee,$$

the half-sums of real and imaginary positive coroots. Let

$$R_{\mathrm{cx}} = \{\alpha \in R \mid \alpha(\rho^\vee_{\mathrm{re}}) = \alpha(\rho^\vee_{\mathrm{im}}) = 0\},$$

the set of roots orthogonal to both $\rho^\vee_{\mathrm{re}}$ and $\rho^\vee_{\mathrm{im}}$. This is a root system (as is easy to check). It is θ stable since both $\rho^\vee_{\mathrm{re}}$ and $\rho^\vee_{\mathrm{im}}$ are θ stable. A nice reference for the following result is Lemma 3.1 of IC4.

Lemma 0.2. *The root system R_{cx} decomposes into an orthogonal disjoint union,*

$$R_{\mathrm{cx}} = R_L \ \cup\ R_R$$

of subroot systems in such a way that θ is an isomorphism $R_L \to R_R$ of root systems. Thus

$$W(R_{\mathrm{cx}}) = W(R_L) \times W(R_R)$$

and

$$W(R_{\mathrm{cx}})^\theta = \{(w,\theta w) \mid w \in W(R_L)\} \simeq W(R_L) \simeq W(R_R).$$

In particular, $W(R_{\mathrm{cx}})^\theta$ is generated by the products of commuting reflections $s_\alpha s_{\theta\alpha}$ for $\alpha \in W(R_L)$ (or $W(R_R)$).

The next result is Proposition 3.12 of IC4.

Proposition 0.3. *The subgroup $W(R_{\mathrm{re}}) \times W(R_{\mathrm{im}})$ of $W(\mathbb{G},\mathbb{H})^\theta$ is normal. Moreover*

$$W(\mathbb{G},\mathbb{H})^\theta \simeq (W(R_{\mathrm{re}}) \times W(R_{\mathrm{im}})) \rtimes W(R_{\mathrm{cx}}).$$

The `atlas` software gives each of these terms in the command `realweyl`. In fact it gives more information (the finite 2-group "A") which we now describe.

Set

$$W_{\mathbb{R}}(R_{\mathrm{im}}) = W(R_{\mathrm{im}}) \cap W(G,H).$$

Set

$$W_{\mathrm{gr}}(R_{\mathrm{im}}) = \{w \in W(R_{\mathrm{im}}) \mid w \text{ preserves compact and noncompact roots}\};$$

equivalently, $W_{\mathrm{gr}}(R_{\mathrm{im}})$ is the normalizer in $W(R_{\mathrm{im}})$ of the subroot system $R_{\mathrm{cpt,im}}$ of compact imaginary roots. (The subscript gr is meant to stand for "grading". In IC4 the less-descriptive subscript 2 is used.) Then it is easy to verify that

$$W(R_{\mathrm{cpt,im}}) \subset W_{\mathbb{R}}(R_{\mathrm{im}}) \subset W_{\mathrm{gr}}(R_{\mathrm{im}}).$$

Here is the general description of real Weyl groups.

Theorem 0.4. *The real Weyl group $W(G,H)$ admits the following description*

$$W(G,H) = [W_{\mathbb{R}}(R_{\mathrm{im}}) \times W(R_{\mathrm{re}})] \rtimes W(R_{\mathrm{cx}})^{\theta}.$$

In general,

$$W_{\mathrm{gr}}(R_{\mathrm{im}}) = W(R_{\mathrm{cpt,im}}) \rtimes B$$

where B is a product of $\mathbb{Z}/2$'s. It is also true that

$$W_{\mathbb{R}}(R_{\mathrm{im}}) = W(R_{\mathrm{cpt,im}}) \rtimes A,$$

where A is a subgroup of B, and hence also a product of $\mathbb{Z}/2$'s. The rank of this group (and its generators in $W(\mathbb{G},\mathbb{H})$) are specified in the `realweyl` command, thus giving a complete description of $W(G,H)$.

Let's look at an example in split E8. Use the following coordinates for the 240 roots inside $\mathbb{R}^8$:

$$\pm e_i \pm e_j \qquad 1 \le i \ne j \le 8$$

$$\frac{1}{2}(\varepsilon_1,\ldots,\varepsilon_8) \qquad \varepsilon \in \{\pm 1\} \qquad \prod_{i=1}^{8} = 1.$$

Suppose θ acts via

$$\theta(x_1,\ldots,x_8) = (-x_1,\ldots,-x_4,x_5,\ldots x_8).$$

Then

$$R_{\mathrm{re}} = \{\pm e_i \pm e_j \mid 1 \le i,j \le 4\} \simeq \mathrm{D4}.$$
$$R_{\mathrm{im}} = \{\pm e_i \pm e_j \mid 5 \le i,j \le 8\} \simeq \mathrm{D4}.$$

For the "standard" choices, we have

$$2\rho^{\vee}_{\mathrm{re}} = (6,4,2,0,0,0,0,0),$$

and

$$2\rho^{\vee}_{\mathrm{im}} = (0,0,0,0,6,4,2,0).$$

So

$$R_{\mathrm{cx}} = \{0,0,0,\pm 1,0,0,0,\pm 1)\} \cup \{\frac{1}{2}(\varepsilon_1,-\varepsilon_1,-\varepsilon_1,\varepsilon_2,\varepsilon_3,-\varepsilon_3,-\varepsilon_3,\varepsilon_4)\} \simeq \mathrm{A2}\times\mathrm{A2},$$

and the conclusion of Lemma 0.2 is nicely illustrated,

$$W(R_{\mathrm{cx}})^{\theta} \simeq W(\mathrm{A2}).$$

Notice that R_{cx} has 12 elements while there are $240-24-24=192$ complex roots in R.

From Proposition 0.3, we conclude

$$\begin{aligned} W(\mathbb{G},\mathbb{H})^{\theta} &\simeq (W(R_{\mathrm{re}}) \times W(R_{\mathrm{im}})) \rtimes W(R_{\mathrm{cx}}) \\ &\simeq (W(\mathrm{D4}) \times W(\mathrm{D4})) \rtimes W(\mathrm{A2}), \end{aligned}$$

where $W(A_2) \simeq S_3$ acts by the diagonal triality action.

Next note that

$$R_{\mathrm{cpt,im}} = \{\pm e_i \pm e_j \mid i,j \in \{1,2\} \text{ or } i,j \in \{3,4\}\}.$$

Consider the chain of inclusions

$$W(R_{\mathrm{cpt,im}}) \subset W_{\mathbb{R}}(R_{\mathrm{im}}) \subset W_{\mathrm{gr}}(R_{\mathrm{im}}) \subset W(R_{\mathrm{im}}).$$

The ends are easy to fill in,

$$(\mathbb{Z}/2)^4 \subset W_{\mathbb{R}}(R_{\mathrm{im}}) \subset W_{\mathrm{gr}}(R_{\mathrm{im}}) \subset W(D_4) \simeq S_4 \rtimes (\mathbb{Z}/2)^3.$$

Recall that `atlas` thinks of $W_{\mathrm{gr}}(R_{\mathrm{im}})$ as $W(R_{\mathrm{cpt,im}}) \rtimes B$ where B is two-group. In this example, it's easy to check that

$$W_{\mathrm{gr}}(R_{\mathrm{im}}) \simeq (\mathbb{Z}/2)^4 \rtimes V_4,$$

where $V_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$ is the Klein group. (Using the notation V_4 eliminates possible confusion with the other $\mathbb{Z}/2$'s floating around.) What is harder to see is that in this example the inclusion $W_{\mathbb{R}}(R_{\mathrm{im}}) \subset W_{\mathrm{gr}}(R_{\mathrm{im}})$ is an equality,

$$W_{\mathbb{R}}(R_{\mathrm{im}}) \simeq (\mathbb{Z}/2)^4 \rtimes V_4.$$

Thus we conclude

$$W(\mathrm{E8}, H) \simeq \big(W(\mathrm{D4}) \times [W((\mathrm{A1})^4) \rtimes V_4]\big) \rtimes W(\mathrm{A2}). \tag{0.5}$$

In fact, here is the exact output from `atlas` . (The Cartan subgroup in question is labeled 5.)

```
  This is the Atlas of Reductive Lie Groups Software Package version 0.2.4.
empty:  type
Lie type:  E8
enter inner class(es):  s
main:  realform
(weak) real forms are:
0:  e8
1:  e8(e7.su(2))
2:  e8(R)
enter your choice:  2
real:  realweyl
cartan class (one of 0,1,2,3,4,5,6,7,8,9):  5
Name an output file (hit return for stdout):
real weyl group is WĈ.((A.W_ic) x WR̂), where:
WĈ is isomorphic to a Weyl group of type A2
A is an elementary abelian 2-group of rank 2
W_ic is a Weyl group of type A1.A1.A1.A1
WR̂ is a Weyl group of type D4
generators for WĈ:
24354654376542
124235423167876542
generators for A:
24316542345765423143542876
1343167876
generators for W_ic:
5678765
254316542345676542314354265
1345431
134265431765423456787654231435426543176
generators for WR̂:
3
45654
7
2431542345678765423143542
```

There is one lingering question: is this really the most palatable format for outputing the description of $W(G,H)$? The answer is perhaps no. To understand why, note that when working with real Weyl groups (and in fact many other representation theory questions, like the ones treated in IC4), one often reduces to computations on the fundamental Cartan. In the case of real Weyl groups, here is how to do that.

Let R_f denote the roots that are perpendicular to $\rho^\vee_{\mathrm{re}}$. Then R_f is a θ-stable roots system (for the same reason R_{cx} was) and R_f has no real roots. In fact R_f corresponds to a Levi factor L of a real parabolic subgroup of G for which H is a fundamental Cartan. The important facts are

$$W(\mathbb{G},\mathbb{H})^\theta \simeq W(R_{\mathrm{re}}) \rtimes W(\mathbb{L},\mathbb{H})^\theta,$$

and

$$W(G,H) \simeq W(R_{\mathrm{re}}) \rtimes W(\mathbb{L},\mathbb{H})^\theta.$$

This reduces the computation of Weyl groups of real Cartan subalgebras to the corresponding computation for fundamental Cartan subalgebras. Furthermore, if we write $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ as usual, then even though $(R_T)|_{\mathfrak{t}}$ is only, in general, a nonreduced root system, we still have

$$W(\mathbb{L},\mathbb{H})^\theta \simeq W((R_T)|_{\mathfrak{t}}).$$

This is computationally quite nice in practice.

To conclude, consider the reduction to the fundamental case in the E8 example. Then $R_T \simeq$ E6 and L is split of type E6. The fundamental Cartan for L has a four-dimensional compact part and in fact

$$W((R_T)|_{\mathfrak{t}}) \simeq \mathrm{F4}.$$

Thus $W(\mathbb{L},\mathbb{H})^\theta \simeq W(\mathrm{F4})$. Moreover in the chain of inclusions

$$W(\text{roots of } \mathfrak{l}\cap\mathfrak{k},\mathfrak{t}) \subset W(L,H) \subset W_{\mathrm{gr}}((R_T)|_{\mathfrak{t}}),$$

both ends are easily seen to be the Weyl group of type C4. So, in fact,

$$W(L,H) \simeq W(\mathrm{C4}),$$

and so we conclude that

$$W(\mathrm{E8},H) \simeq W(\mathrm{D4}) \rtimes W(\mathrm{C4}). \tag{0.6}$$

Compare this with the conclusion of (0.5),

$$W(\mathrm{E8},H) \simeq \big(W(\mathrm{D4}) \times [W((\mathrm{A1})^4) \rtimes V_4]\big) \rtimes W(\mathrm{A2}). \tag{0.7}$$

Since $S_3 \rtimes V_4 \simeq S_4$, we see that the last part of previous equation really is $W(\mathrm{C4})$, and the descriptions match. But there is a good case to be made that the description of (0.6) is better than that of (0.7), the output from `atlas` .

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