On the double cover of split F_4 and its petite K-types

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joint work with Dan Barbasch

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Plan of the talk

- the double cover of split F_4
- the big unitarity problem (*find all unitary parameters*)
- the petit unitarity problem (*find some not-unitary parameters*)
- an informal definition of non-spherical petite K-types
- a formal definition of non-spherical petite K-types
- applications to the unitary dual of the double cover of split F_4

Plan of the talk

- $\boxed{\textbf{the double cover of split } F_4}$

- the big unitarity problem

- the petit unitarity problem

- an informal definition of non-spherical petite K-types

- a more technical definition of petite K-types

- applications to the unitary dual of the double cover of split F_4

The double cover of F_4

- G = the double cover of the split F_4 ($F_4 = G/\{\pm I\}$)
- $\pi\colon G \to F_4 = G/\{\pm I\}$, the projection
- $K = SP(1) \times SP(3)$
- Representations of K (classified by highest weight): $\mu = (a_1|a_2, a_3, a_4)$, with $a_1 \geq 0$ and $a_2 \geq a_3 \geq a_4 \geq 0$
- Genuine K-types ($-I$ does not act trivially): $\mu = (a_1|a_2, a_3, a_4)$, with $a_1 + a_2 + a_3 + a_4$ odd
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: Cartan decomposition of $\mathfrak{g}$
- $\mathfrak{a}$: maximal abelian subspace of $\mathfrak{p}$, $A = \exp(\mathfrak{a})$, $M = Z_K(\mathfrak{a})$
- $\Delta^+ = \{2\epsilon_j;\ \epsilon_i \pm \epsilon_j; \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4\}$, $\mathfrak{n} = \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $N = \exp(\mathfrak{n})$

Notations

For each root α, we can choose a Lie algebra homomorphism

$$\phi_\alpha \colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$$

such that

- $\boxed{Z_\alpha} = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ belongs to $\mathfrak{k} = \mathrm{Lie}(K)$
- $\boxed{\sigma_\alpha} = \exp(\frac{\pi}{2} Z_\alpha)$ belongs to $M' = N_K(\mathfrak{a})$, and
- $\boxed{m_\alpha} = \exp(\pi Z_\alpha)$ belongs to $M = Z_K(\mathfrak{a})$.

Metaplectic Roots

Exponentiating ϕ_α, we obtain group homomorphisms

$$\widetilde{\Phi}_\alpha \colon \widetilde{SL}(2,\mathbb{R}) \to G \qquad \Phi_\alpha \colon SL(2,\mathbb{R}) \to G/\pm I = F_4.$$

The root α is called metaplectic if $\widetilde{\Phi}_\alpha$ does not factor to $SL(2,\mathbb{R})$.

$\boxed{\text{only the long roots are metaplectic}}$

Consequences:

- If α is short, then m_α has order two and is central in M
- If α is long, then m_α has order four and $m_\alpha m_\beta = \pm m_\beta m_\alpha$
- If α is short, the eigenvalues of $d\mu(iZ_\alpha)$ are integers $\forall\, \mu \in \hat{K}$
- If α is long, the eigenvalues of $d\mu(iZ_\alpha)$ are integers if μ is not genuine, and half-integers if μ is genuine.

Fine K-types

Let μ be an irreducible representation of K. Then

- μ *has level* l if $|\gamma| \leq l$, for every eigenvalue γ of $d\mu(iZ_\alpha)$ and every root α
- μ *is fine* if μ has level 1 (or less)

There are 2 genuine fine K-types: $(1|000)$ and $(0|100)$
and 3 non-genuine fine K-types: $(2|000)$, $(1|100)$ and $(0|000)$.

The group M

The group $M = Z_K(\mathfrak{a})$ is a finite group of order 32. Because $\pi(M) = M/\{\pm I\}$ is abelian, the irreducible representations of M have dimension one or two.

There are 16 non-genuine linear characters, and 4 genuine two-dimensional irreducible representations.

The Weyl group acts on $\hat{M}$. The restrictions to M of a *fine* K-type is a single orbit, and every representation of M is contained in a unique fine K-type.

Definition: Fix $\delta \in \hat{M}$. A root α is *good* for δ if s_α stabilizes δ.

		orbit	dim.	$W(\delta)$	fine K-type
non-genuine	$\longrightarrow$	$\boxed{\delta_0}$	1	$W(F_4)$	$(0\|0,0,0)$
non-genuine	$\longrightarrow$	$\boxed{\delta_3}$	3×1	$W(C_4)$	$(2\|0,0,0)$
non-genuine	$\longrightarrow$	$\boxed{\delta_{12}}$	12×1	$W(B_3A_1)$	$(1\|1,0,0)$
genuine	$\longrightarrow$	$\boxed{\delta_2}$	2	$W(F_4)$	$(1\|0,0,0)$
genuine	$\longrightarrow$	$\boxed{\delta_6}$	3×2	$W(B_4)$	$(0\|1,0,0)$

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Langlands quotient

For every irreducible representation (δ, V^δ) of M, and every strictly dominant real character ν, we set

$\boxed{X_P(\delta,\nu)}$ = *the minimal principal series induced from* $\delta \otimes \nu$

$\boxed{\bar{X}_P(\delta,\nu)}$ =*the unique irreducible composition factor of* $X_P(\delta,\nu)$ *which contains the fine* K*-type* μ_δ *corresponding to* δ.

The Langlands quotient $\bar{X}_P(\delta,\nu)$ can be obtained as the quotient of $X_P(\delta,\nu)$ modulo the Kernel of an intertwining operator

$$A\colon X_P(\delta,\nu) \longrightarrow X_{\bar{P}}(\delta,\nu)$$

where $\bar{P}$ is the opposite parabolic.

The big unitarity problem

For every irreducible representation δ of M, compute the set of unitary parameters

$$\boxed{\{\nu \in \mathfrak{a} \cap \mathbb{R} \colon \nu \text{ is dominant and } \bar{X}_P(\delta, \nu) \text{ is unitary}\}}$$

To check the unitarity of $\bar{X}_P(\delta, \nu)$, we need to

1. *construct an invariant Hermitian form on $\bar{X}_P(\delta, \nu)$, if possible*
2. *verify whether this Hermitian form is positive definite.*

Invariant Hermitian forms on $\bar{X}_P(\delta, \nu)$

The long Weyl group element of F_4 ($\omega = -Id$) carries δ into δ and ν in $-\nu$. So we can use ω to construct an *Hermitian* intertwining operator

$$A(\omega, \delta, \nu) \colon X_P(\delta, \nu) \to X_P(\delta, -\nu).$$

This operator gives a *non degenerate* invariant Hermitian form on the Langlands quotient.[a]

$\bar{X}_P(\delta, \nu)$ is unitary if and only if $A(\omega, \delta, \nu)$ is positive semidefinite.

[a] Because $\bar{X}_P(\delta, \nu)$ contains only one copy of the fine K-type μ_δ corresponding to δ, we can normalize the operator by requiring that it acts trivially μ_δ. Then we obtain the *unique* non-degenerate invariant Hermitian form on $\bar{X}_P(\delta, \nu)$.

Remarks

The big unitarity problem is too hard:

Computing the signature of the operator $A(\omega, \delta, \nu)$ is extremely complicated, especially if the K-type is very big.
Moreover, we should check the signature on infinitely many K-types.

Instead, we look at the petit unitarity problem.

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the petit unitarity problem

- find finitely many K-types (called "**petite**") on which it is easy to compute the signature of the intertwining operator
- use these petite K-types to rule out big regions of not-unitarity.[a]

[a] The notion of *spherical* petite K-type is due to Vogan and Barbasch. We will present a generalization to the non-spherical case.

Spherical Petite K-Types

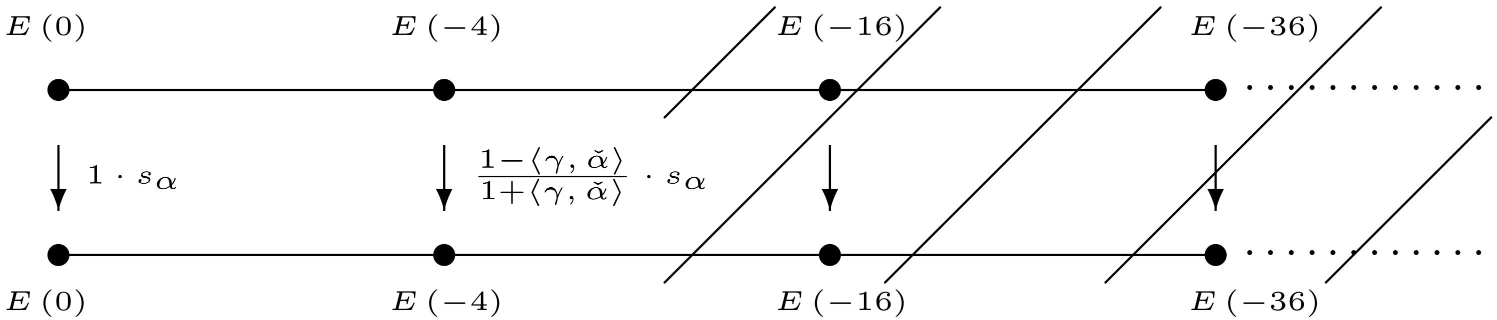
Let be μ a *spherical* K-type, i.e. assume that $Res_M(\mu)$ contains the trivial representation of M.

μ is called **petite** if it has level ≤ 3.

Remark: *if μ is a spherical petite K-type, then $d\mu(Z_\alpha^2)$ acts on the isotypic component of the trivial representation of M with eigenvalues 0 or -4. This condition makes the intertwining operator on μ "very special", and relatively easy to compute.*

intertwining operator on spherical petite K-types

The intertwining operator has a decomposition as a product of operators corresponding to simple reflections.
The factor corresponding to α acts by



Intertwining operator on spherical petite K-types

On a spherical petite K-type the intertwining operator behaves exactly like a p-adic operator.

Because the p-adic spherical unitary dual in known, this matching provides **non-unitarity certificates**.

We obtain an embedding of the real <u>spherical</u> unitary dual into the p-adic <u>spherical</u> unitary dual.

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non-spherical petite K-types: an informal definition

To every non-trivial representation δ of M, we associate a real linear group G_0 (depending on δ).

A K-type μ containing δ is called **"petite for δ"** if the non-spherical intertwining operator for G on μ matches a spherical intertwining operator for G_0 on some petite K_0-type μ_0.

The spherical unitary dual of G_0 is known, and is detected by a finite number of *relevant* K_0-types.

If we can match all the relevant K_0-types, then we obtain non-unitarity certificates for Langlands quotients of G:

$$\bar{X}^G(\delta,\nu) \text{ is unitary } \Rightarrow \bar{X}^{G_0}(\text{triv},\nu_0) \text{ is unitary.}$$

the linear group $G_0 = G_0(\delta)$

The Weyl group W of G acts on $\hat{M}$ by

$$([\sigma] \cdot \tau)(m) = \tau(\sigma^{-1} m \sigma).$$

Let $W(\delta) \subseteq W$ be the stabilizer of δ.

It turns out that $W(\delta)$ is the Weyl group of some root system Δ_0. Δ_0 has the same rank as Δ, and in general is not a sub-root system.

We define G_0 to be

- the real split group with root system $\boxed{\Delta_0}$ if δ is non-genuine
- the real split group with root system $\boxed{\check{\Delta}_0}$ if δ is genuine.

G_0 is always linear, and in general is not a subgroup of G.

	orbit-type	Δ_0	linear group $G_0(\delta)$
non-genuine $\longrightarrow$	δ_0	F_4	F_4
non-genuine $\longrightarrow$	δ_3	C_4	$SP(4)$
non-genuine $\longrightarrow$	δ_{12}	$B_3 A_1$	$SO(3,4) \times SL(2)$
genuine $\longrightarrow$	δ_2	F_4	$F_4\check{}$
genuine $\longrightarrow$	δ_6	B_4	$SP(4)$

If we have "enough" petite K-types for δ, then we can relate the unitarity of a Langlands quotient of G induced from δ to the unitarity of a Langlands quotient of $G_0(\delta)$ induced from the trivial.

the spherical K_0-type μ_0

Suppose that there exists a spherical K_0-type μ_0 s.t.

1. μ_0 has level at most 3
2. as $W(\delta)$-representations

$$\mathrm{Hom}_M(V^{\mu}, V^{\delta}) = \mathrm{Hom}_{M_0}(V^{\mu_0}, V^{\delta_0}).$$

Then μ is petite if and only if the intertwining operator for G on μ matches an intertwining operator for G_0 on μ_0.

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non-spherical petite K-types: a more technical definition

Let μ be a K-type containing δ. If μ is petite, the intertwining operator on μ should have certain properties $(\dots)$.

The intertwining operator acts on

$$\mathrm{Hom}_M(V^\mu, V^{\mu_\delta}) = \bigoplus_j \mathrm{Hom}_M(V^\mu, V^{\delta_j})$$

and depends on the eigenvalues of the $d\mu(Z_\alpha^2)$'s (α simple) on the isotypic component in μ of all the M-types δ_j in the W-orbit of δ.[a]

To define a petite K-type for δ, we essentially need to impose some restrictions on the eigenvalues of the various Z_α^2's.

[a] μ_δ is the unique fine K-type containing δ. Every M-type δ_j in the W-orbit of δ appears in μ_δ with multiplicity one: $\mathrm{Res}_M(\mu_\delta) = \bigoplus_j \delta_j$.

non-spherical petite K-types: a more technical definition

Let μ be a K-type containing δ. If μ is petite, the intertwining operator on μ should should have certain properties $(\ldots)$.

The intertwining operator acts on

$$\mathrm{Hom}_M(V^\mu, V^{\mu_\delta}) = \bigoplus_j \mathrm{Hom}_M(V^\mu, V^{\delta_j})$$

and depends on the eigenvalues of the various $d\mu(Z_\alpha^2)$'s on the isotypic component in μ of the W-orbit of δ.[a]

It is clear that the definition of petite K-type must be a restriction on these eigenvalues.

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[a] μ_δ is the unique fine K-type containing δ. Every M-type δ_j in the W-orbit of δ appears in μ_δ with multiplicity one: $\mathrm{Res}_M(\mu_\delta) = \bigoplus_j \delta_j$.

Technicalities

- The intertwining operator on μ has a factorization as a product of operators $R_\mu(s_\alpha, \gamma)$ corresponding to simple reflections.
- The action of a single factor $R_\mu(s_\alpha, \gamma)$ does not respect the decomposition

$$\mathrm{Hom}_M(V^\mu, V^{\mu_\delta}) = \bigoplus_j \mathrm{Hom}_M(V^\mu, V^{\delta_j})$$

  but preserves the decomposition of $\mathrm{Hom}_M(V^\mu, V^{\mu_\delta})$ in eigenspaces of $d\mu(Z_\alpha^2)$: $\boxed{\mathrm{Hom}_M(V^\mu, V^{\mu_\delta}) = \bigoplus_{n \in \mathbb{N}/2} E(-n^2).}$

- $R_\mu(s_\alpha, \gamma)$ acts on the $(-n^2)$-eigenspace of $d\mu(Z_\alpha^2)$ by

$$R_\mu(s_\alpha, \gamma)T(v) = \underbrace{c(\alpha, \gamma, n)}_{a\ scalar} \quad \underbrace{\mu_\delta(\sigma_\alpha)T(\mu(\sigma_\alpha)^{-1}v)}_{action\ of\ s_\alpha\ on\ \mathrm{Hom}_M(V^\mu, V^{\mu_\delta})}$$

example 1: $d\mu(Z_\alpha^2)$ has even eigenvalues

The operator $R_\mu(s_\alpha, \gamma)$ acts on $\left[\bigoplus_{n\in 2\mathbb{N}} E\left(-n^2\right)\right]$ by

$$
\begin{tikzcd}
E(0) \arrow[r, dash] \arrow[d, "1 \cdot s_\alpha"] & E(-4) \arrow[r, dash] \arrow[d, "\frac{1-x}{1+x} \cdot s_\alpha"] & E(-16) \arrow[r, dash] \arrow[d, "\frac{1-x}{1+x}\frac{3-x}{3+x} \cdot s_\alpha"] & E(-36) \arrow[r, dash] \arrow[d, "\frac{1-x}{1+x}\frac{3-x}{3+x}\frac{5-x}{5+x} \cdot s_\alpha"] & \cdots\cdots\cdots \\
E(0) \arrow[r, dash] & E(-4) \arrow[r, dash] & E(-16) \arrow[r, dash] & E(-36) \arrow[r, dash] & \cdots\cdots\cdots
\end{tikzcd}
$$

with $x = \langle \gamma, \check{\alpha} \rangle$.

example 2: $d\mu(Z_\alpha^2)$ has odd eigenvalues

The operator $R_\mu(s_\alpha, \gamma)$ acts on $\left[\bigoplus_{n\in 2\mathbb{N}+1} E\left(-n^2\right)\right]$ by

$$
\begin{tikzcd}
E(-1) \arrow[r, dash] \arrow[d, "1 \cdot s_\alpha"] & E(-9) \arrow[r, dash] \arrow[d, "\frac{2-x}{2+x} \cdot s_\alpha"] & E(-25) \arrow[r, dash] \arrow[d, "\frac{2-x}{2+x}\frac{4-x}{4+x} \cdot s_\alpha"] & E(-49) \arrow[r, dash] \arrow[d, "\frac{2-x}{2+x}\frac{4-x}{4+x}\frac{6-x}{6+x} \cdot s_\alpha"] & \cdots\cdots\cdots \\
E(-1) \arrow[r, dash] & E(-9) \arrow[r, dash] & E(-25) \arrow[r, dash] & E(-49) \arrow[r, dash] & \cdots\cdots\cdots
\end{tikzcd}
$$

with $x = \langle \gamma, \check{\alpha} \rangle$.

example 3: $d\mu(Z_\alpha^2)$ has half-integers eigenvalues

The operator $R_\mu(s_\alpha,\gamma)$ acts on $\left[\bigoplus_{n\in\mathbb{N}+\frac{1}{2}} E\left(-n^2\right)\right]$ by

$$\begin{tikzcd}
E(-\frac{1}{4}) \arrow[r, dash] \arrow[d, "1\cdot s_\alpha"] & E(-\frac{9}{4}) \arrow[r, dash] \arrow[d, "\frac{\frac{1}{2}-x}{\frac{1}{2}+x}\cdot s_\alpha"] & E(-\frac{25}{4}) \arrow[r, dash] \arrow[d, "\frac{\frac{3}{2}-x}{\frac{3}{2}+x}\cdot s_\alpha"] & E(-\frac{49}{4}) \arrow[r, dash] \arrow[d, "\frac{\frac{1}{2}-x}{\frac{1}{2}+x}\frac{\frac{5}{2}-x}{\frac{5}{2}+x}\cdot s_\alpha"] & \cdots\cdots\cdots \\
E(-\frac{1}{4}) \arrow[r, dash] & E(-\frac{9}{4}) \arrow[r, dash] & E(-\frac{25}{4}) \arrow[r, dash] & E(-\frac{49}{4}) \arrow[r, dash] & \cdots\cdots\cdots
\end{tikzcd}$$

with $x=\langle\gamma,\,\check{\alpha}\rangle$.

intertwining operator on non- spherical petite K-types

If μ is a petite K-type, every factor $R_\mu(s_{\alpha_i}, \gamma_i)$ of the intertwining operator must satisfy some conditions.

These conditions depend on whether the reflection s_{α_i} stabilizes a certain M-type δ_i in the orbit of δ.[a]

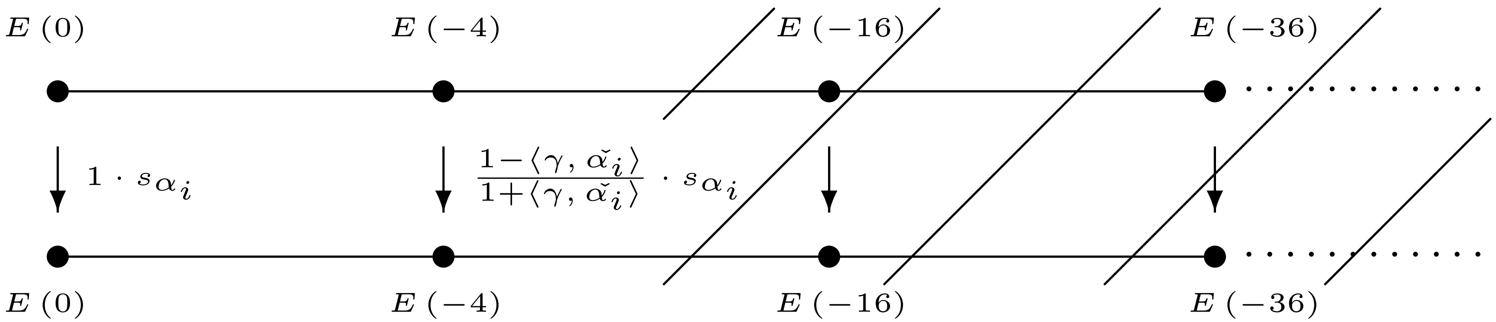
- If α_i stabilizes δ_i (i.e. it is good for δ_i), then $R_\mu(s_{\alpha_i}, \gamma_i)$ should behave as a factor of a petite spherical intertwining operator.
- If α_i does not stabilize δ_i (i.e. it is bad for δ_i), then $R_\mu(s_{\alpha_i}, \gamma_i)$ should be independent of the parameter γ_i.

This behavior is equivalent to some eigenvalues-restrictions.

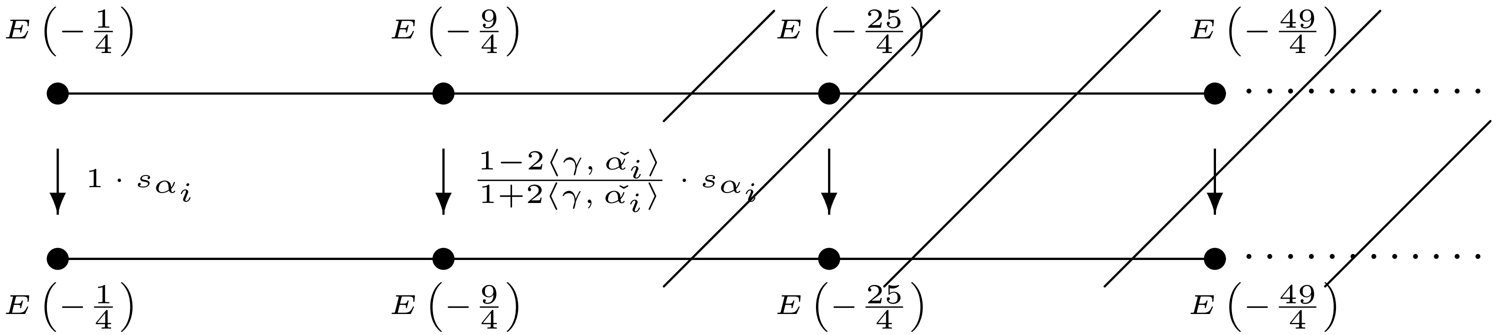
[a] If $\alpha_1, \alpha_2 \ldots \alpha_r$ are the simple reflections involved in the decomposition, we define inductively $\delta_1 = \delta, \delta_2 = s_{\alpha_1}(\delta_1)$, ..., $\delta_r = s_{\alpha_{r-1}}(\delta_{r-1})$.

restrictions for μ petite and α_i <u>good</u> for δ_i

Look at the eigenvalues of $d\mu(Z_{\alpha_i}^2)$ on the δ_i-isotypic in μ.
If the eigenvalues are of the form $-(2n)^2$, we only allow 0 and -4

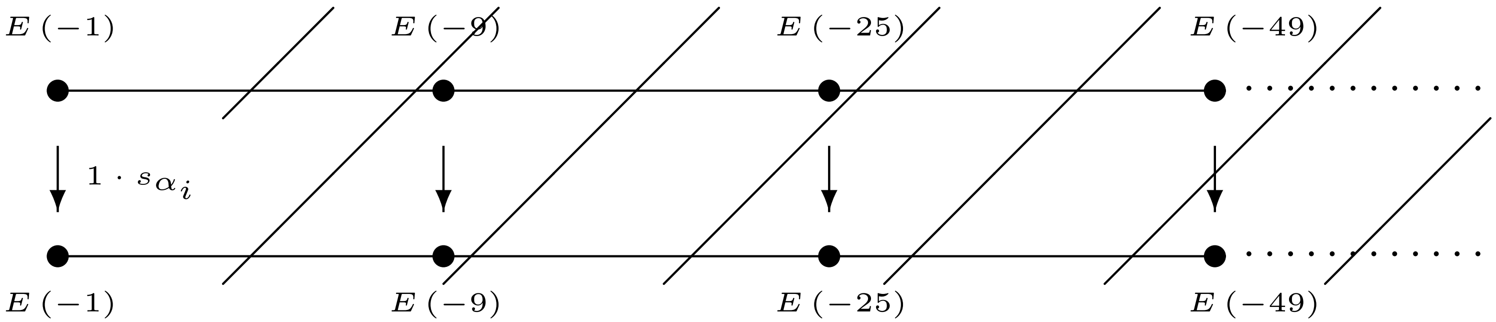


If the eigenvalues are of the form $-\left(\frac{2n+1}{2}\right)^2$, we only allow $-\frac{1}{4}$, $-\frac{9}{4}$

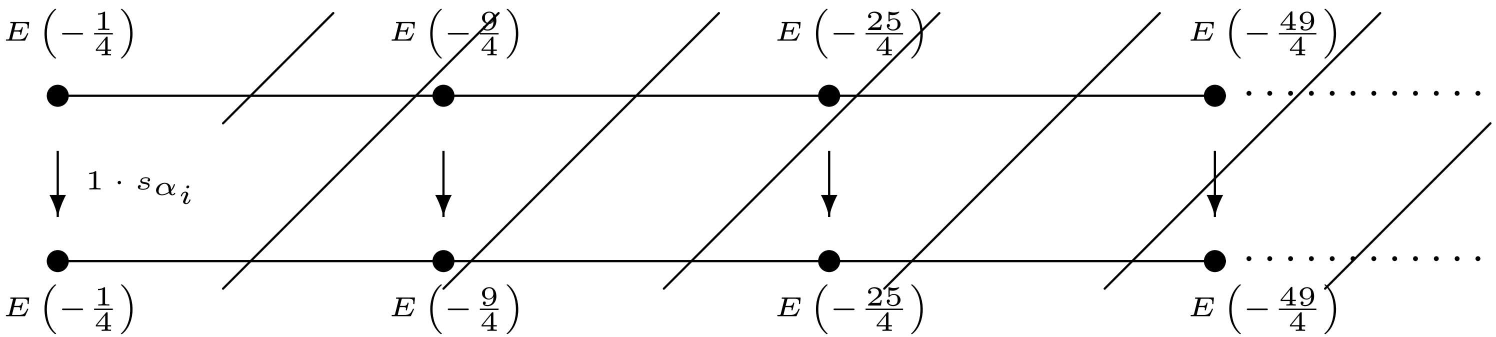


restrictions for μ petite and α_i <u>bad</u> for δ_i

Again, look at the eigenvalues of $d\mu(Z^2_{\alpha_i})$ on the δ_i-isotypic in μ.
If the eigenvalues are of the form $-(2n+1)^2$, we only allow -1



If the eigenvalues are of the form $-\left(\frac{2n+1}{2}\right)^2$, we only allow $-\frac{1}{4}$



The Main Theorem

Let μ be a petite K-type for δ, i.e. assume that μ satisfies the eigenvalues-conditions described above.

Suppose that there exists a spherical K_0-type μ_0 s.t.

1. *μ_0 has level at most 3*
2. *as $W(\delta)$-representations*

$$\mathrm{Hom}_M(V^\mu, V^\delta) = \mathrm{Hom}_{M_0}(V^{\mu_0}, V^{\delta_0}).$$

Then the intertwining operator for G on μ matches an intertwining operator for G_0 on μ_0.

A technical remark

Let μ be a petite K-type. The restrictions on the eigenvalues of $d\mu(Z^2_{\alpha_i})$ are "local" conditions: they are imposed on the isotypic of the various δ_i in μ, not "globally" on μ.

It follows that, if δ is non-trivial, we cannot identify a petite K-type for δ just by looking at its level.[a]
Most often, an explicit construction of the K-type is required.[b]

This is just one of the many complications that make the non-spherical case so much harder than the spherical one.

[a] If δ is trivial, every K-type of level at most 3 is petite. If δ is non-trivial, only about a half of the K-types of level 3 turns out to be petite.

[b] We have constructed all our petite K-types using *mathematica*.

genuine petite K-types and other K-types of level ≤ 3

K-type	mult. of δ_6
$(0\|1,0,0)$	1
$(2\|1,0,0)$	3
$(1\|2,0,0)$	4
$(1\|1,1,0)$	4
$(0\|1,1,1)$	1
$(2\|1,1,1)$	3
$(4\|1,0,0)$	5
$(3\|2,0,0)$	8
$(3\|1,1,0)$	8
$(0\|3,0,0)$	5
$(2\|3,0,0)$	8
$(0\|2,1,0)$	8
$(2\|2,1,0)$	5
$(1\|2,1,1)$	8

K-type	mult. of δ_2
$(1\|0,0,0)$	1
$(3\|0,0,0)$	2
$(1\|2,0,0)$	9
$(1\|1,1,0)$	2
$(0\|1,1,1)$	4
$(2\|1,1,1)$	12
$(5\|0,0,0)$	3
$(3\|2,0,0)$	18
$(3\|1,1,0)$	4
$(0\|3,0,0)$	4
$(2\|3,0,0)$	12
$(0\|2,1,0)$	8
$(2\|2,1,0)$	24
$(1\|2,1,1)$	10

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- **applications to the unitary dual**

Find a good definition of *petite K-types*

$\downarrow$

For each given δ, find *all* the petite K-types

$\downarrow$

For each μ petite, find the representation of the stabilizer of δ on $\mathrm{Hom}_M(V^{\mu}, V^{\delta})$. Guess μ_0

$\downarrow$

Verify that the intertwining operators match

$\delta_2,\ \delta_{12}\ \swarrow \qquad \searrow\ \delta_3,\ \delta_6$

If you can match all the relevant K_0-types, deduce the existence of an inclusion of unitary duals

Otherwise, compute the intert. operator on some non-petite K-types and see what happens

example 1: δ_2

δ_2 is an irreducible genuine representation of M.

The stabilizer of δ_2 is the entire Weyl group $W = W(F_4)$. In particular, every root of F_4 is good for δ_2. *This is an easy example!*

We ask whether it is possible to realize all the relevant $W(F4)$-types using petite K-types for δ_2.

The relevant $W(F_4)$-types are: 1_1, 2_1, 2_3, 4_2, 8_1 and 9_1.

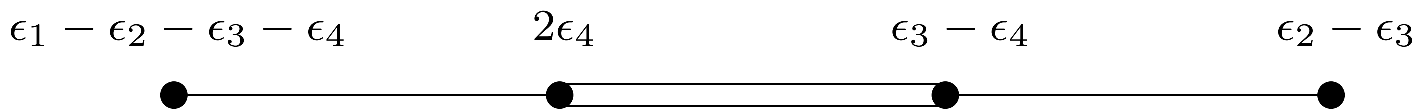
petite K-type	**mult. of δ_2**	**repres. of $W(F_4)$**
$(1\|0,0,0)$	1	1_1
$(3\|0,0,0)$	2	2_3
$(1\|2,0,0)$	9	9_1
$(1\|1,1,0)$	2	2_1
$(0\|1,1,1)$	4	4_2
$(0\|3,0,0)$	4	4_3
$(0\|2,1,0)$	8	8_1
$(1\|2,1,1)$	10	$1_2 + 9_2$

We match all of them! So there is an inclusion of unitary duals:

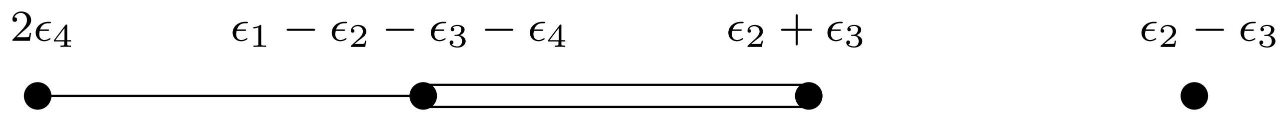
$$\bar{X}^G(\delta_2, \nu) \text{ unitary } \Rightarrow \bar{X}^G(\text{triv}, \nu) \text{ unitary.}$$

example 2: δ_{12}

Choose a set of simple roots for G (type F_4):



δ_{12} contains 12 one-dimensional representations of M. For each of them, the stabilizer is $W(B_3 \times A_1)$.
Let $\bar{\delta}_{12}$ be the character in δ_{12} that admits



as a basis for the good roots.

The following table shows that we can realize all the relevant $W(B_3)$-types and all the relevant $W(A_1)$-types using petite K-types for $\bar{\delta}_{12}$:

petite K-type	**mult. of δ_{12}**	**repres. of $W(B_3 \times A_1)$**
$(1\|1,0,0)$	1	$(3 \times 0) \times triv$
$(0\|1,1,0)$	1	$(3 \times 0) \times sign$
$(3\|1,0,0)$	2	$(21 \times 0) \times triv$
$(2\|1,1,0)$	3	$(2 \times 1) \times triv$
$(2\|2,0,0)$	3	$(1 \times 2) \times sign$
$(0\|2,0,0)$	1	$(0 \times 3) \times triv$

Because we can match all the relevant $W(B_3 \times A_1)$-types, there exists an inclusion of unitary duals:[a]

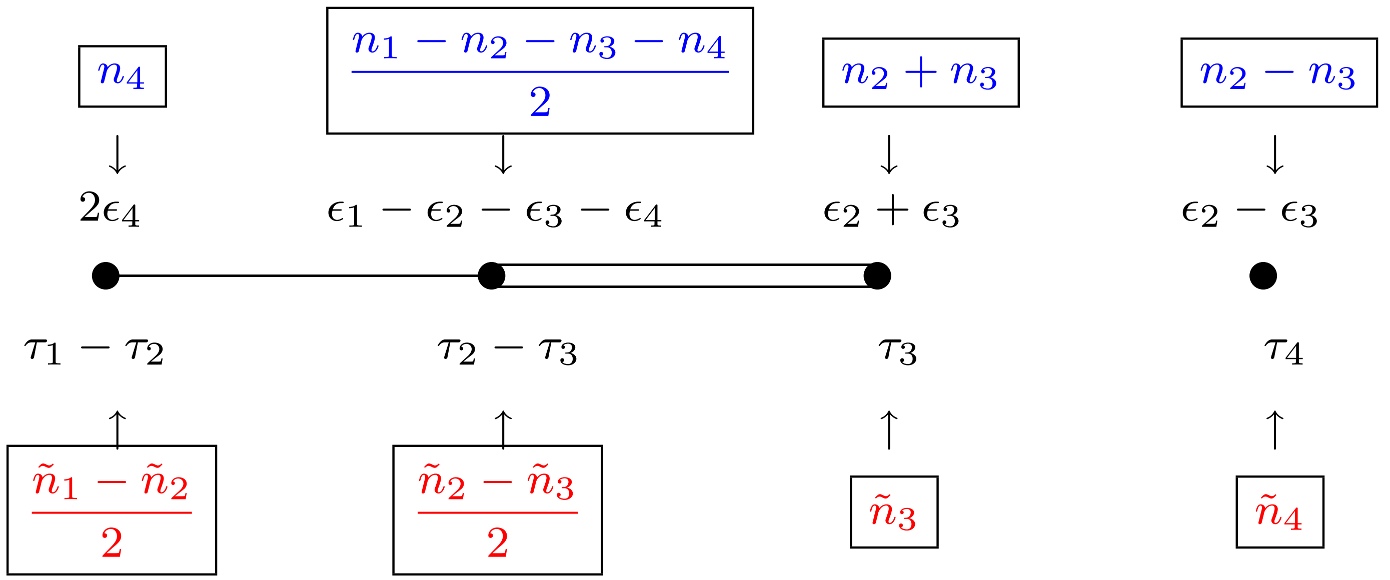
$$\boxed{\bar{X}^G(\delta_{12}, \gamma) \;\; \text{unitary}} \Rightarrow \boxed{\bar{X}^{SO(3,4)\times SL(2)}(\text{triv}, \gamma_0) \;\; \text{unitary}}$$

Notice that there is a shifting of parameters: if $\gamma{=}(n_1, n_2, n_3, n_4)$, then $\gamma_0 = (n_1 + n_4, n_1 - n_4, n_2 + n_3, n_2 - n_3)$.

[a] $SO(3,2) \times SL(2)$ is the real split group with root system $B_3 \times A_1$.

If γ=(n_1, n_2, n_3, n_4) is the parameter for F_4, let γ_0=$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4)$ be the corresponding parameter for $B3 \times A1$.

The inner product of γ with a basis for the good co-roots in F_4 should match the inner product of γ_0 with the simple co-roots in $B_3 \times A_1$:



example 3: δ_6

δ_6 contains three 2-dimensional irreducible representations of M. For each of them, the stabilizer of δ is $W(B4)$.

Let $\bar{\delta}_6$ the irreducible component of δ_6 that admits

as a basis for the good roots.

We would like to realize all the relevant $W(B_4)$-types using petite K-types for $\bar{\delta}_6$.

The following is a *complete* list of petite K-types for $\bar{\delta}_6$:

petite K-type	**mult. of $\bar{\delta}_6$**	**repres. of $W(B_4)$**
$(0\|1,0,0)$	1	4×0
$(2\|1,0,0)$	3	31×0
$(1\|2,0,0)$	4	1×3
$(1\|1,1,0)$	4	3×1
$(0\|1,1,1)$	1	0×4
$(2\|1,1,1)$	3	0×31

The relevant $W(B_4)$-types are:

$$4\times 0 \quad 31\times 0 \quad 3\times 1 \quad \boxed{2\times 2} \quad 1\times 3 \quad 0\times 4.$$

We cannot match 2×2**!!!**

The relevant $W(B_4)$-type 2×2 is missing. So we cannot deduce an inclusion of unitary duals.

We only get a weaker result:[a]

$$\boxed{\begin{array}{c}\text{set of unitary}\\ \text{parameters}\\ \text{for } (\bar{\delta}_6,\, G)\end{array}} \subseteq \boxed{\boxed{\begin{array}{c}\text{set of unitary}\\ \text{parameters}\\ \text{for } (\text{triv},\, SP(4))\end{array}} \cup \boxed{\begin{array}{c}\text{non-unitarity region}\\ \text{for } (\text{triv}, SP(4))\\ \text{ruled out by } 2 \times 2\end{array}}}$$

The region ruled out by 2×2 consists of all parameters of the form $\gamma_0 = (a + 1/2,\, a - 1/2,\, b,\, 1)$ with (a, b) in the *triangle* delimited by the lines $a = 1/2$, $b = 0$ and $a + b = 3/2$.

[a] Notice that the stabilizer of $\bar{\delta}_6$ is of type B_4 but we are taking $G_0 = SP(4)$. Indeed, $\bar{\delta}_6$ is genuine, so G_0 must be the split group with *co-roots* of type B_4.

example 4: δ_3

δ_3 contains three 1-dimensional irreducible representations of M. For each of them, the stabilizer of δ is $W(C4)$.

Let $\bar{\delta}_3$ the irreducible component of δ_3 that admits

as a basis for the good roots.

Next, we look at the *complete* list of petite K-types for $\bar{\delta}_3$, and we hope to realize all the relevant $W(C_4)$-types: 4×0 $\quad 0 \times 4$ $\quad 3 \times 1$ $\quad \boxed{1 \times 3}$ $\quad 2 \times 2$ $\quad 31 \times 0$.

petite K-type	**mult. of $\bar{\delta}_3$**	**repres. of $W(C_4)$**
$(2\|0,0,0)$	1	4×0
$(4\|0,0,0)$	1	0×4
$(0\|2,0,0)$	3	31×0
$(2\|2,0,0)$	6	2×2
$(2\|1,1,0)$	2	22×0
$(1\|3,0,0)$	4	111×1
$(1\|2,1,0)$	8	21×1
$(1\|1,1,1)$	4	3×1
$(0\|2,1,1)$	3	211×0
$(2\|2,1,1)$	7	$11 \times 11 + 1111 \times 0$

We cannot match 1×3**!!!**

The relevant $W(C_4)$-type 1×3 is missing. So we cannot deduce an inclusion of unitary duals.

Just like before, we only obtain a weaker result:

$$\boxed{\begin{array}{c}\text{set of unitary}\\ \text{parameters}\\ \text{for } (\bar{\delta}_3,\, G)\end{array}} \subseteq \boxed{\boxed{\begin{array}{c}\text{set of unitary}\\ \text{parameters}\\ \text{for (triv, } SP(4))\end{array}} \cup \boxed{\begin{array}{c}\text{non-unitarity region}\\ \text{for (triv,} SP(4))\\ \text{ruled out by } 1\times 3\end{array}}}$$

The region ruled out by 1×3 is the *line segment*

$$\gamma_0 = (3/2 + t,\, 1/2 + t,\, -1/2 + t,\, -3/2 + t)$$

with $1/2 \leq t \leq 3/2$.

work in progress

Understand if these “extra regions” contain any unitarity point.