CELLS IN WEYL GROUPS AND PRIMITIVE IDEALS

DAN BARBASCH

Abstract. These notes, taken by Dan Ciubotaru[1], are based on a talk given by Dan Barbasch at the AIM workshop "Atlas of Lie groups and representations IV", July 2006. The goal is to give a brief review of some basic facts in the theory of W-cells and primitive ideals.

1. The Hecke algebra of a Weyl group and cells

The main references for this section are [KL1] and [L], chapters 4 and 5.

1.1. Let (W, S) denote a Coxeter group (S is the set of simple reflections), and let $\ell : W \to \mathbb{N}$ be the length function.

Definition. *The algebra $\widetilde{\mathcal{H}}$ over $\mathbb{Z}[q]$ associated to the Coxeter group (W, S) is the vector space with basis $\{T_w\}_{w\in W}$ satisfying*

(1) $T_w \cdot T_{w'} = T_{ww'}$, *if $w, w' \in W$ with $\ell(ww') = \ell(w) + \ell(w')$.*
(2) $(T_s + 1) \cdot (T_s - q) = 0$, *for all $s \in S$.*

Every T_w is invertible, for example $T_s^{-1} = q^{-1}T_s + (q^{-1} - 1)$, for $s \in S$. Of course, when q is specialized to $q = 1$, $\mathcal{H} = \mathbb{Z}[W]$.

If $q = p^n$ for some prime p, the algebra $\widetilde{\mathcal{H}}$ is isomorphic to the intertwining algebra of the space of functions, $\{f : G(\mathbb{F}_q)/B(\mathbb{F}_q) \to \mathbb{C}\}$, on the flag variety of a finite group of Lie type. The algebra $\widetilde{\mathcal{H}} \otimes_{\mathbb{Z}[q]} \mathbb{C}$ is isomorphic to the group algebra $\mathbb{C}[W]$, but for this isomorphism one needs to consider $q^{1/2}$. Therefore, it is more convenient to work with an extension of $\widetilde{\mathcal{H}}$ instead.

Let $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ be the ring of Laurent polynomials in the variable $q^{1/2}$. Define the *Hecke algebra*

$$\mathcal{H} = \widetilde{\mathcal{H}} \otimes_{\mathbb{Z}[q]} \mathcal{A}. \tag{1.1.1}$$

1.2. We define the notion of *W-graph.*

This is a set of vertices X, with a set of edges Y and additional data:

(1) for every $x \in X$, a set $I_x \subset S$,
(2) for every edge $\{y, x\} \in Y$, a number $\mu(y, x) \in \mathbb{Z} \setminus \{0\}$,

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[1]The responsibility for all the errors in this exposition belongs to D.C.

subject to the requirements in definition 1.2 below.

Let E be the free $\mathcal{A}$-module with basis X. For every simple reflection $s \in S$, define a map $\tau_s : E \to E$ by

$$\tau_s = \begin{cases} -x, & \text{if } s \in I_x \\ qx \quad +q^{1/2} \displaystyle\sum_{\substack{y \in X \\ \{y,x\} \in Y \\ s \in I_y}} \mu(y,x)\ y, & \text{if } s \notin I_x \end{cases}. \tag{1.2.1}$$

Definition. *We call (X, Y) a W-graph if:*

(i) *$\tau_s : E \to E$ is an endomorphism of E, for all $s \in S$. (Note that this is equivalent with the requirement that the sum in (1.2.1) be finite for every $x \in X$.)*

(ii) *$\{\tau_s\}_{s \in S}$ satisfy the braid relations,i.e. if $s_1, s_2 \in S$ such that $ord_W(s_1, s_2) = m$, then*

$$\underbrace{\tau_{s_1}\tau_{s_2}\tau_{s_1}\dots}_{m} = \underbrace{\tau_{s_2}\tau_{s_1}\tau_{s_2}\dots}_{m}.$$

The two conditions in the definition of the W-graph are equivalent to the fact that there exists a unique representation $\phi : \mathcal{H} \to \mathrm{End}(E)$ such that $\phi(T_s) = \tau_s$.

1.3. Let $^- : \mathcal{A} \to \mathcal{A}$ denote the conjugation of $\mathcal{A}$ determined by $\overline{q^{1/2}} = q^{-1/2}$. Then define a conjugation in $\mathcal{H}$ by

$$\overline{\sum a_w T_w} = \sum \bar{a}_w T_{w^{-1}}^{-1}. \tag{1.3.1}$$

Set $q_w = q^{\ell(w)}$ and $\epsilon_w = (-1)^{\ell(w)}$, and let $\leq$ denote the Bruhat order on W.

Theorem ([KL1],1.1)**.** *For any $w \in W$, there exists a unique $C_w \in \mathcal{H}$ such that*

(1) $\overline{C}_w = C_w$,

(2) $C_w = \sum_{y \leq w} \epsilon_y \epsilon_w q_w^{1/2} q_y^{-1} \overline{P}_{y,w} T_y$,

where $P_{y,w}$ is a polynomial in q of degree at most $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ for $y < w$ and $P_{w,w} = 1$.

The polynomials $P_{y,w}$ are the *Kazhdan-Lusztig polynomials.* Note that the basis $\{T_w : w \in W\}$ of $\mathcal{H}$ is not fixed under $^-$, while the basis $\{C_w : w \in W\}$ is, and the matrix of polinomials $\{P_{y,w}\}$ can essentially be thought of as a change of basis matrix between these two bases.

It is a basic fact that all coefficients of the polynomials $P_{y,w}$ are *nonnegative* integers. There is no combinatorial proof yet of this fact. The proof in [KL2] uses deep methods from intersection cohomology. For an exposition, one can consult [Sp] (also the recent expository paper [Ri]).

There is also an inversion formula for the upper triangular matrix $\{P_{y,w}\}$. Assume W is finite, and let w_0 denote the longest element in W.

Proposition ([KL1],3.1)**.** *For all $y, w \in W$, $y \leq w$:*

$$\sum_{y \leq z \leq w} \epsilon_y \epsilon_z P_{y,z} P_{w_0 w, w_0 z} = \delta_{y,w}.$$

1.4. We recall the construction of the basic example of W-graphs.

Definition. *We say that $y \prec w$ if $y < w$ (in the Bruhat order), $\epsilon_y = -\epsilon_w$, and deg $P_{y,w} = \frac{1}{2}(\ell(w) - \ell(y) - 1)$. In this case define $\mu(y, w)$ to be the coefficient of $q^{\frac{1}{2}(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$. Also, set $\mu(w, y) = \mu(y, w)$.*

From proposition 1.3, it follows that $y \prec w$ if and only if $w_0 w \prec w_0 y$, and in this case $\mu(y, w) = \mu(w_0 w, w_0 y)$.

Let (W^0, S^0) denote the opposite group of the Coxeter group (W, S). Then $(W \times W^0, S \sqcup S^0)$ is also a Coxeter group.

For every $w \in W$, define

$$\mathcal{L}(w) = \{s \in S : sw < w\} \text{ and } \mathcal{R}(w) = \{s \in S : ws < w\}. \tag{1.4.1}$$

Let Γ_W denote the graph given by the following data:

$$\begin{aligned} &X = W; \\ &Y = \{\{y, w\} : y \prec w\}; \\ &I_w = \mathcal{L}(w) \sqcup \mathcal{R}(w)^0; \\ &\mu(y, w) \text{ as in definition 1.4.} \end{aligned} \tag{1.4.2}$$

Theorem ([KL1],1.3)**.** *The graph Γ_W defined in (1.4.2) is a $W \times W^0$-graph.*

Given a W-graph Γ, for every subset $S' \subset S$, and $W' = \langle S' \rangle \subset W$, Γ can also be viewed as a W'-graph. In particular, the $W \times W^0$-graph Γ_W is also a W-graph and a W^0-graph.

Define the relation $\leq_\Gamma$ as follows: $x \leq_\Gamma x'$ if there exists a sequence $x = x_0,\ x_1, \dots,\ x_n = x'$ such that $\{x_{i-1}, x_i\}$ is an edge in Γ_W, and $I_{x_{i-1}} \not\subset I_{x_i}$. Denote by $\sim_\Gamma$ the corresponding equivalence relation, *i.e.* $x \sim_\Gamma y$ if $x \leq_\Gamma y$ and $y \leq_\Gamma x$. On each equivalence class, Γ_W induces a W-graph, and therefore a representation of $\mathcal{H}$. The set of equivalence classes are ordered by $\leq_\Gamma$. In this way we can define:

(1) *double* cells, *i.e.* the equivalence classes of the $(W \times W^0, S \sqcup S^0)$-graph Γ_W.
(2) *left* cells, *i.e.* the equivalence classes of the (W, S)-graph Γ_W.
(3) *right* cells, *i.e.* the equivalence classes of the (W^0, S^0)-graph Γ_W.

The relations will be denoted by $\leq_L, \sim_L$ for left cells, respectively $\leq_R, \sim_R$ for right cells, and $\leq_{LR}, \sim_{LR}$ for double cells.

The bijection $w \longrightarrow w_0 w$ induces an order reversing involution on each one of the sets of left, right, and double cells. Note that each (left) W-cell is a representation of W (via the $\mathcal{H}$-module structure from definition 1.2). Denote it by $[X]$.

1.5. A very simple example is $W = W(A_2)$, with $S = \{s_1, s_2\}$. We compute the Kazhdan-Lusztig polynomials using *Atlas* with `klbasis` for $SL(3, \mathbb{C})$. (There are other programs which compute the classical KL-polynomials, *e.g.* F. DuCloux's *Coxeter* or the package *chevie* in GAP.) We find that $P_{y,w} = 1$ whenever $y < w$ (in the Bruhat order) and $P_{y,w} = 0$ otherwise. This implies that the W-graph looks exactly like the Bruhat order in this case with $\mu(y, w) = 1$ on all edges. There are four left W-cells (two of which are identical). The I_w's for the three distinct are: $\emptyset$, $\{1\} - \{2\}$, and $\{1, 2\}$. The corresponding representations of $W = S_3$ are the trivial, the reflection, respectively the sign.

A basic problem is to describe the decomposition of $[X]$ into W-representations for each W-cell X. A first remark is that

$$[w_0 X] \cong [X]^* \otimes sgn \text{ (as } W\text{-representations).} \tag{1.5.1}$$

Theorem ([KL1],1.4)**.** *If $W = S_n$ (the symmetric group), and X is a left cell of W, then the corresponding representation of $\mathcal{H}$ (over the quotient field of $\mathcal{A}$) on the W-graph of X is irreducible.*

In general though, the representation of W on the W-graph of a left cell is not irreducible. This can already be seen in $W(B_2)$, where there are four left cells, and two of them have 3 elements each.

1.6. There is a distinguished class of irreducible representations of the Weyl group W, called *special*. We recall (one of) their definition(s) next. Let σ be an irreducible W-representation. Then one associates to σ (by a construction due to Lusztig) a module $\sigma(q)$ of $\mathcal{H}$. It is known that there exists a unique polynomial $D_\sigma(q)$, called the *formal dimension* of $\sigma(q)$, which satisfies the equation

$$\sum_{w \in W} q_w^{-1} \mathrm{Tr}(T_w, \sigma(q))^2 = \dim(\sigma) \frac{\sum_{w \in W} q_w}{D_\sigma(q)}. \tag{1.6.1}$$

Let $a(\sigma)$ denote the smallest power of q in $D_\sigma(q)$. Also, let $b(\sigma)$ denote the smallest degree of the symmetric power of the reflection representation of W, in which σ appears as a subrepresentation. By a result of Lusztig, $a(\sigma) \le b(\sigma)$ for every $\sigma \in \widehat{W}$.

Definition. *A representation $\sigma \in \widehat{W}$ is called special if $a(\sigma) = b(\sigma)$.*

Lists of special representations for every Weyl group type, and relevant algorithms can be found in [L]. We remark that in type A, every W-representation is special. In type B_2, there are 5 representations, and 3 are special: the trivial, the reflection, and the sign.

In [KL1], the following properties are conjectured.

Conjecture ([KL1],1.7)**.**
(a) The W-representation $[X]$ corresponding to a left W-cell X contains a unique special W-representation.

(b) Every special representation appears in some $[X]$.
(c) Two left cells X, X' give rise to the same special W-representation if and only if there exists a double cell Y, such that $X \subset Y$ and $X' \subset Y$.

1.7. Part (c) of conjecture 1.6 says that if two double W-cells contain the same special representation, then they are the same. This is refined in [L], where $\widehat{W}$ is partitioned into families, each family containing exactly one special representation, and it is conjectured that these families correspond precisely to the double cells, *i.e.* :

Conjecture (Lusztig)**.** *The representations $\sigma, \sigma' \in \widehat{W}$ appear in the same double W-cell (we write $\sigma \sim_{LR} \sigma'$) if and only if they are in the same family.*

This conjecture was proven by Barbasch-Vogan (see section 2). The algorithms for classical Weyl groups and lists for exceptional groups are in [L], section 4. Let us record here the algorithm in the case W is of type B_n/C_n.

Every $\sigma \in \widehat{W}$ is parametrized by a pair of partitions $(a_1, \dots, a_m) \times (b_1, \dots, b_m)$, where $0 \le a_1 \le \dots \le a_m$, $0 \le b_1 \le \dots \le b_m$, and $\sum a_i + \sum b_j = n$. Set

$$\lambda_i = a_i + i - 1 \text{ and } \mu_j = b_j + j - 1, \tag{1.7.1}$$

and define the *symbol*

$$\Lambda = \begin{pmatrix} \lambda_1, \lambda_2, \dots, \lambda_{m+1} \\ \mu_1, \mu_2, \dots, \mu_m \end{pmatrix}. \tag{1.7.2}$$

Obviously, two symbols Λ and Λ' correspond to the same representation of W if $(\lambda'_1, \lambda'_2, \dots, \lambda'_{m+2}) = (0, \lambda_1 + 1, \dots, \lambda_{m+1} + 1$ and the similar condition for the μ-string are both satisfied. One partitions the set of symbols into equivalence classes relative to this, and denotes by $[\Lambda]$ the class of Λ and the corresponding representation of W.

Then the invariant $a([\Lambda])$ (as in section 1.6) can be computed explicitly from Λ (see (4.5.2) in [L]). It turns out that a representation $[\Lambda]$ is special if and only if

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \dots \le \mu_m \le \lambda_{m+1}. \tag{1.7.3}$$

Definition. *Two representations of W are in the same family, if their symbols have the same underlying set. It is clear that every family contains a unique special representation.*

For example in $\widehat{W(B_2)}$ we have:

Family	Symbols	Special
$\{2 \times 0\}$	$\left\{ \begin{pmatrix} 2 \\ - \end{pmatrix} \right\}$	2×0
$\{1 \times 1, 11 \times 0, 0 \times 2\}$	$\left\{ \begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix}, \begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix}, \begin{pmatrix} 0 & & 1 \\ & 2 & \end{pmatrix} \right\}$	1×1
$\{0 \times 11\}$	$\left\{ \begin{pmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \end{pmatrix} \right\}$	0×11

In section 2, we will present the results of [BV] relevant to these conjectures and the connection with the theory of primitive ideals. But before that, we will recall the Kazhdan-Lusztig conjecture regarding the multiplicities of composition factors in Verma modules.

1.8. Kazhdan-Lusztig conjecture. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, with $\mathfrak{h}$ a Cartan subalgebra, and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a Borel subalgebra. Then $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ and $W = W(\mathfrak{g}, \mathfrak{h})$, and let Δ^+ be the set positive roots with respect to $\mathfrak{b}$, and $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$.

For every $w \in W$, let

$$M_w = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{-w(\rho)-\rho} \tag{1.8.1}$$

be the Verma module of highest weight $-w(\rho) - \rho$, and let L_w be its unique irreducible quotient. Let ch M_w denote the character of M_w, and similarly denote ch L_w.

Theorem (Kazhdan-Lusztig conjecture)**.** *The following character identities hold:*

$$ch\ L_w = \sum_{y \leq w} \epsilon_y \epsilon_w P_{y,w}(1)\ ch\ M_y,$$

$$ch\ M_w = \sum_{y \leq w} P_{w_0 w, w_0 y}(1)\ ch\ L_y,$$

where $P_{y,w}(q)$ are the polynomials defined by theorem 1.3.

We should mention that this is actually a theorem (proven by Beilinson-Bernstein, Brylinski-Kashiwara).

2. Primitive Ideals

The main references for this section are [BV] and [Di].

2.1. Let $\mathfrak{g}$ denote a complex reductive Lie algebra, with Cartan subalgebra $\mathfrak{h}$. Let $\mathfrak{z}(\mathfrak{g})$ denote the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. The rest of the notation will be as in section 1.8. Recall the Harish-Chandra isomorphism

$$\omega : \mathfrak{z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})^W. \tag{2.1.1}$$

Via ω (and the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$), every weight $\lambda \in \mathfrak{h}^*$ defines a character of the center

$$\chi_\lambda : \mathfrak{z}(\mathfrak{g}) \longrightarrow \mathbb{C}, \tag{2.1.2}$$

which depends only on the W-conjugacy class of λ.

Definition. *A two-sided ideal $\mathcal{I}$ in $\mathcal{U}(\mathfrak{g})$ is called primitive if it is the annihilator in $\mathcal{U}(\mathfrak{g})$ of a (possibly infinite-dimensional) irreducible $\mathfrak{g}$-module.*

Set

$$\mathrm{Prim}_\lambda\, \mathcal{U}(\mathfrak{g}) = \{I \subset \mathcal{U}(\mathfrak{g}) \text{ two-sided primitive ideal: } I \cap \mathfrak{z}(\mathfrak{g}) = \ker \chi_\lambda\}. \tag{2.1.3}$$

The problem is to describe the set $\mathrm{Prim}_\lambda\, \mathcal{U}(\mathfrak{g})$.

For the purpose of this exposition, we will restrict to the case when $\lambda = -\rho$. (In [BV], the more general case of a regular weight λ, not necessarily integral, is considered, in which case one uses the *integral Weyl group* and *integral roots*).

For every $w \in W$, let $I(w)$ denote the annihilator of L_w (notation as in section 1.8). The starting point of the theory is the following fundamental theorem.

Theorem ([Du]). *The map $W \longrightarrow Prim_\rho\, \mathcal{U}(\mathfrak{g})$, given by $w \rightarrow I(w)$ is surjective.*

The results in [BV] provide a description of the fibers of this map. We will recall them next.

2.2. Assume the notation in section 1.8. For every $y, w \in W$, define

$$a_{y,w} = \epsilon_y \epsilon_w P_{y,w}(1) \text{ (if } y \not\leq w,\ a_{y,w} = 0); \tag{2.2.1}$$
$$a(w) = |\Delta^+| - \text{GKdim } L_w \geq 0.$$

(GKdim X denotes the Gelfand-Kirillov dimension of a $\mathcal{U}(\mathfrak{g})$-module X; see [V6] for definition and related results.)

Definition ([J1]). *Fix a dominant regular element $x \in \mathfrak{h}$. For every $w \in W$, define the polynomials $r_m(w) \in S(\mathfrak{h})$ by:*

$$r_m(w) = \sum_{y \in W} a_{y,w} (y^{-1}x)^m.$$

The polynomials $r_m(w)$ are zero for all $m < a(w)$. The polynomial $r(w) = r_{a(w)}(w)$ is called the *character polynomial* of L_w. If in the definition of $r(w)$ one chooses a different element x, the polynomial is the same up to multiplication by a positive constant. The polynomials $r(w)$ have some remarkable properties.

Theorem ([BV], II.2.6).
(a) $r(w)$ is a W-harmonic polynomial, and generates an irreducible representation $\sigma(w) \in \widehat{W}$ with respect to the action of W on $S(\mathfrak{h})$.
(b) $a(w) = b(\sigma(w))$ (notation as in section 1.6). Moreover, $\sigma(w)$ occurs with multiplicity one in $S^{a(w)}(\mathfrak{h})$.
(c) $I(w) = I(w')$ if and only if $r(w) = c \cdot r(w')$ for some constant $c > 0$. In other words, two elements $w, w' \in W$ are in the same fiber of Duflo's map if and only if the corresponding character polynomials differ only by a positive constant factor.

Moreover, as w runs over W, the polynomials $r(w)$ give distinguished bases for the representations $\sigma(w)$. We will make this more precise in section 3.3.

If $I \in \text{Prim}_\rho\, \mathcal{U}(\mathfrak{g})$, it must be of the form $I = I(w)$ for some $w \in W$. Set $\sigma(I) = \sigma(w) \in \widehat{W}$. This is called the *Goldie rank representation* for I ([J1]). Then one of the main results of [BV] is the following.

Theorem ([BV])**.** *For every $I \in Prim_\rho\, \mathcal{U}(\mathfrak{g})$, the Goldie rank representation $\sigma(I) \in \widehat{W}$ is a special representation; moreover, all special representations appear in this way. In particular,*

$$|Prim_\rho\, \mathcal{U}(\mathfrak{g})| = \sum_{\substack{\sigma \in \widehat{W} \\ \sigma \ special}} \dim \sigma.$$

2.3. Another important problem is to describe the order relation in $\mathrm{Prim}_\rho \mathcal{U}(\mathfrak{g})$. We will do this next, and also present the connection between these results and those concerning W-cells from section 1.

Definition. *Define a preorder relation on W by $w_1 \leq_L w_2$ if $I(w_1) \subset I(w_2)$. Also define $w_1 \leq_R w_2$ if $w_1^{-1} \leq_L w_2^{-1}$. The smallest preorder relation containing both $\leq_L$ and $\leq_R$ is denoted $\leq_{LR}$. Let $\sim_R$, $\sim_L$, respectively $\sim_{LR}$ denote the corresponding equivalence relations. The equivalence classes are called left, right respectively double cells, and are denoted by $\mathcal{C}_w^L$ etc.*

By results of Joseph, the group algebra $\mathbb{C}[W]$ can be identified with the $\mathbb{C}$-algebra with basis $\{M_w : w \in W\}$. In this identification, the regular representation of $W \times W$ is given by

$$(w_1, w_2) \cdot M_w = M_{w_1 w w_2^{-1}}. \tag{2.3.1}$$

Then $\mathbb{C}[W]$ has two distinguished bases, one given by $\{M_w\}$, and the other one by $\{L_w\}$. The change of basis matrix is $\{a_{y,w}\}$ (definition (2.2.1)) and it comes from the KL-conjecture.

Proposition ([J2],[V5])**.** *The following conditions are equivalent:*

(a) $w_1 \leq_L w_2$;

(b) There are sequences $\{z_0, z_1, \dots, z_n\} \subset W$ and $\{s_1, \dots, s_n\} \subset S$, such that $z_0 = w_1$, $z_n = w_2$, and for each $i = 1, \dots, n$,

$$(s_i, 1) \cdot L_{z_{i-1}} = \sum c_z^i L_z, \ \text{with } c_{z_i}^i \neq 0. \tag{2.3.2}$$

(c) There exists a finite dimensional representation F of $\mathfrak{g}$, such that $L_{w_2^{-1}}$ is a subquotient of $L_{w_1^{-1}} \otimes F$.

This theorem is based on the [V5]'s solution to a conjecture in [J3], and the relation proven in [J2] between composition series in Verma modules and composition series in principal series of complex semisimple groups. We will give some details in section 3.

From the theorem, it follows that each cell (left, right, double) $\mathcal{C}_w$ carries a natural representation of W, denoted $[\mathcal{C}_w]$, and

$$\bigoplus_{cells} [\mathcal{C}_w] \cong \mathbb{C}[W]. \tag{2.3.3}$$

The connection with section 1 comes via the KL-conjecture:

Corollary. *The ordering $\leq_L$ (and related notation) defined via primitive ideals is identical to the corresponding ordering defined via W-graphs.*

Then we can translate the results from W-cells (*e.g.* the involution induced by *sgn*) to the primitive ideals setting. One last result we present is the following.

Theorem ([BV])**.** *The conjecture 1.7 holds true. The special representation in each double cell is a Goldie rank representation.*

We just mention (see section 3 in [BV] for details) that, in the proof of these results, a very important role is played by induction and restriction to parabolics, and also by the *sgn* involution.

3. Complex case

Let G be a complex semisimple group, viewed as a real group, $B = HN$ be a Borel subgroup, with Cartan subgroup H, and let K denote a maximal compact subgroup such that $G = KB$. Set $T = K \cap B \subset H$, and $H = TA$. The complexified Lie algebra $\mathfrak{g}_c$ is isomorphic with the sum of two copies of $\mathfrak{g}$, $\mathfrak{g}_c \cong \mathfrak{g}_L \oplus \mathfrak{g}_R$.

3.1. Harish-Chandra modules.

Suppose that $\lambda_L, \lambda_R \in \mathfrak{h}^*$. Assume $\mu = \lambda_L - \lambda_R$ is in the weight lattice. Then μ defines a weight of T, and therefore a unique irreducible K-module $V(\mu)$ with highest weight μ.

Definition. *Let $\mu = \lambda_L - \lambda_R$ be in the weight lattice, and $\nu = \lambda_L + \lambda_R$. The standard module $X(\lambda_L, \lambda_R)$ is*

$$X(\lambda_L, \lambda_R) = Ind_B^G[\mathbb{C}_\mu \otimes \mathbb{C}_\nu]_{K-finite}, \tag{3.1.1}$$

where

(1) *$\mathbb{C}_\nu$ is the character of A determined by ν;*
(2) *$\mathbb{C}_\mu \otimes \mathbb{C}_\nu$ is defined to be trivial on N;*
(3) *the induction is normalized Harish-Chandra induction.*

The standard module $X(\lambda_L, \lambda_R)$ has finite composition series, in particular, a unique irreducible subquotient $\overline{X}(\lambda_L, \lambda_R)$ characterized by the fact that it contains the K-type $V(\mu)$ with multiplicity one.

The Langlands classification in this setting is formulated in the following theorem (due to Duflo).

Theorem (Langlands classification)**.** *Every irreducible $(\mathfrak{g}_c, K)$-module is of the form $\overline{X}(\lambda_L, \lambda_R)$. Moreover, $\overline{X}(\lambda_L, \lambda_R) \cong \overline{X}(\lambda'_L, \lambda'_R)$ if and only if there exists $w \in W$ such that $w\lambda_L = \lambda'_L$ and $w\lambda_R = \lambda'_R$.*

3.2. Annihilators in $\mathcal{U}(\mathfrak{g}_c)$. Using the isomorphism $\mathfrak{g}_c \cong \mathfrak{g}_L \oplus \mathfrak{g}_R$, the annihilator of an irreducible $(\mathfrak{g}_c, K)$-module in $\mathcal{U}(\mathfrak{g}_c)$ decomposes as

$$\mathrm{Ann}_{\mathcal{U}(\mathfrak{g}_c)}\overline{X}(\lambda_L, \lambda_R) = I_L \otimes \mathcal{U}(\mathfrak{g}_R) + \mathcal{U}(\mathfrak{g}_L) \otimes I_R, \tag{3.2.1}$$

for some primitive ideals I_R, I_L in $\mathcal{U}(\mathfrak{g})$.

We specialize to the case $\lambda_L = \lambda_R = \rho$. Let $\mathcal{X}(\rho)$ denote the set of all irreducible $(\mathfrak{g}_c, K)$-modules with *infinitesimal character* (ρ, ρ), *i.e.* a $(\mathfrak{g}_c, K)$-module $V \in \mathcal{X}(\rho)$ if the corresponding I_L, I_R from (3.2.1) are in $\mathrm{Prim}_\rho\, \mathcal{U}(\mathfrak{g})$.

It is known that the set $\mathcal{X}(\rho)$ is in bijection with W via

$$W \xrightarrow{\cong} \mathcal{X}, \quad w \longrightarrow \overline{X}(\rho, w\rho). \tag{3.2.2}$$

Denote by $I_L(w)$ and $I_R(w)$ the ideals appearing in the decomposition (3.2.1) of the annihilator of $\overline{X}(\rho, w\rho)$. If $\check{}\, : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ denotes the involution determined by $\mathfrak{g} \to \mathfrak{g},\ X \mapsto -X$, then $I_R(w) = \check{I}_L(w^{-1})$. By [J2], theorem 4.12,

$$I_L(w) = I(w) = \mathrm{Ann}\ L_w \text{ (notation as in section 2.1).} \tag{3.2.3}$$

The order relation on $\{I_L(w) : w \in W\}$ is described by the following (particular case of a) theorem of Vogan.

Theorem ([V5],3.2)**.** *$I_L(w_1) \subset I_L(w_2)$ if and only if there exists a finite dimensional representation F of $\mathfrak{g}$ such that $\overline{X}(\rho, w_2^{-1}\rho)$ is a subquotient of $\overline{X}(\rho, w_1^{-1}\rho) \otimes F$.*

Finally, the connection with the results in section 2.3, especially proposition 2.3, is given by theorem 5.4 in [J2]. In our particular case, it says

$$[M_w : L_y] = [X(\rho, w\rho) : \overline{X}(\rho, y\rho)], \tag{3.2.4}$$

where $[V : U]$ denotes the multiplicity of U in V.

In particular, the order relations for primitive ideals coming from the complex group are the same with those coming from Verma modules, and in turn, the same as those from W-graphs and W-cells.

3.3. The associated variety. Let $\{\mathcal{U}_n(\mathfrak{g})\}$ be the standard filtration of $\mathcal{U}(\mathfrak{g})$. The associated graded algebra

$$Gr(\mathcal{U}(\mathfrak{g})) = \bigoplus_{n=0}^{\infty} \mathcal{U}_n(\mathfrak{g})/\mathcal{U}_{n-1}(\mathfrak{g}) \tag{3.3.1}$$

is isomorphic to $S(\mathfrak{g})$.

If I is a primitive ideal, it inherits a filtration coming from $\{\mathcal{U}_n(\mathfrak{g})\}$, and we can form the associated graded $Gr(I) \subset S(\mathfrak{g})$. Then define the *associated variety* of I:

$$\upsilon(I) = \{f \in \mathfrak{g}^* : f(z) = 0, \text{ for all } z \in Gr(I)\}.$$

This is G-invariant, closed, and a subset of the nilpotent cone $\mathcal{N}^*$ of $\mathfrak{g}^*$.

Definition. *If π is an irreducible $(\mathfrak{g}_c, K)$-module, define the associated variety $\upsilon(\pi)$ by $\upsilon(\pi) = \upsilon(I_{\pi,L})$ (also equal to $\upsilon(I_{\pi,R})$).*

Theorem ([BB])**.** *In the complex case, $\upsilon(\pi)$ is the closure of a single nilpotent coadjoint orbit.*

Assume the notation from section 2. Let $\mathcal{O}$ denote a nilpotent coadjoint orbit in $\mathcal{N}^*$. The set $\mathrm{Prim}_\rho\,\mathcal{U}(\mathfrak{g})$ can be partitioned as

$$\mathrm{Prim}_\rho\,\mathcal{U}(\mathfrak{g}) = \sqcup_{\mathcal{O}\ \mathrm{nilpotent}}\mathrm{Prim}_\rho(\mathcal{U}(\mathfrak{g}), \mathcal{O}), \tag{3.3.2}$$

where a primitive ideal I is in $\mathrm{Prim}_\rho(\mathcal{U}(\mathfrak{g}), \mathcal{O})$ if (by definition) $\upsilon(I) = \overline{\mathcal{O}}$.

There is a related notion of *special* nilpotent orbits. In view of the definitions in sections 1.6, 1.7, the easiest way to define them is via *Springer's correspondence.* This can be viewed as a surjective (but not injective except in type A) map $\widehat{W} \longrightarrow \{\mathcal{O} : \text{nilpotent orbit}\}$. We call $\mathcal{O}$ special if it is the image of a special W-representation. The special Springer representation attached to $\mathcal{O}$ will be denoted $Sp(\mathcal{O})$.

Then, by [BV],

$$\mathrm{Prim}_\rho(\mathcal{U}(\mathfrak{g}), \mathcal{O}) \neq \emptyset \text{ if and only if } \mathcal{O} \text{ is special.} \tag{3.3.3}$$

Finally, we recall that in [J1], it is proved that the set

$$\{r(I) \text{ Goldie rank polynomial} : I \in \mathrm{Prim}_\rho(\mathcal{U}(\mathfrak{g}), \mathcal{O})\} \tag{3.3.4}$$

forms a basis for the special W-representation $Sp(\mathcal{O})$. For an exposition of relevant material and recent new results about the Goldie rank polynomial basis of special W-representations, see [Tr].

4. The real case

Let G be now a real group. Fix an infinitesimal character of a finite dimensional representation, say the trivial. Let $\mathcal{G}$ be the Grothendieck group generated by the $(\mathfrak{g}, K)$-modules with this infintesimal character. In [V1-4] and [LV], the generalizations of the results from complex groups are proven.

In this case, there are only *Harish-Chandra cells*, which are the analogue of the double cells. The W-action on $\mathcal{G}$ in terms of standard modules is "easy" to pin down. It is known explicitly for classical groups.

One can use the *Atlas* program to obtain relevant information. Explicitly the command *blockstabilizer* gives information about the Weyl group action on standard modules, while *wcell* gives information about the action on irreducibles. This information is sufficient to compute the cells, say using the software *Gap3/Chevie* to identify the W-representations.

Details to follow.

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(D. Barbasch) Department of Mathematics, Cornell University, Ithaca, NY 14853

E-mail address: barbasch@math.cornell.edu