Real Forms and Strong Real Forms
Quick Reference Guide
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These notes are intended as a reminder about my talks last year on Parameters for Real Groups (these have been updated this year, see the atlas site), and an introduction to the Atlas software and du Cloux's talks. See the references in my notes.

1 Real Forms

Let $\mathbb{G}$ be a reductive group. We have an exact sequence

$$1 \to \mathrm{Int}(\mathbb{G}) \to \mathrm{Aut}(\mathbb{G}) \to \mathrm{Out}(\mathbb{G}) \to 1. \tag{1.1}$$

Here $\mathrm{Int}(\mathbb{G}) \simeq \mathbb{G}_{ad} = \mathbb{G}/Z(\mathbb{G})$. If $\mathbb{G}$ is semisimple then $\mathrm{Out}(\mathbb{G})$ is isomorphic to the automorphism group of the Dynkin diagram of $\mathbb{G}$. For example if $\mathbb{G}$ is simple it is trivial unless it is of type A, D or E_6.

A real form is an element θ of $\mathrm{Aut}(\mathbb{G})$ of order 2; this is the Cartan involution of the real form. Two real forms are equivalent if they are conjugate by $\mathrm{Int}(\mathbb{G})$, and inner to each other if they have the same image in $\mathrm{Out}(\mathbb{G})$. The set of equivalence classes of real forms is partitioned into inner classes, parametrized by elements of $\mathrm{Out}(\mathbb{G})$ of order 2.

Tip: Two real forms are in the same inner class if and only if they have the "same" fundamental (i.e. most compact) Cartan subgroup. Inner forms with a compact Cartan subgroup correspond to the trivial outer automorphism.

Example 1.2 If $\mathbb{G} = SO(2n+1)$, i.e. of type B_n then $\mathrm{Out}(\mathbb{G})$ is trivial. The real forms are (up to equivalence) $SO(p,q)$ with $p+q = 2n+1$, and they are all in the same inner class. These groups all contain a compact Cartan subgroup.

If $G = SO(2n)$, i.e. of type D_n, then $\mathrm{Out}(\mathbb{G}) = \mathbb{Z}/2\mathbb{Z}$. The forms $SO(p,q)$ with p,q even are all in the trivial inner class (containing a compact Cartan subgroup) and the ones with p,q odd are all in the non-trivial inner class (no compact Cartan). The trivial inner class also includes $SO^*(2n)$.

Let $\mathbb{G} = SL(n)$. The forms $SU(p,q)$ with $p+q = n$ all contain a compact Cartan subgroup, and correspond to the trivial outer automorphism. The other class contains $SL(n,\mathbb{R})$, and also $SL(n/2,\mathbb{H})$ if n is even.

If $\mathbb{G} = Sp(2n)$ there is only one inner class, containing $Sp(2n,\mathbb{R})$ and $Sp(p,q)$ with $p+q = n$.

The exact sequence (1.1) is (non–canonically) split. An involution $\tau \in \mathrm{Out}(\mathbb{G})$ goes to the Cartan involution θ of the "most compact" form in this inner class. This is a distinguished real form for the inner class. (There is also a unique quasisplit inner form in the inner class. Our focus on Cartan involutions, and the fundamental Cartan, makes the most compact one important also.)

Example 1.3 If $\mathbb{G} = SO(2n+1)$ then the most compact inner form is $SO(2n+1,0)$. If $\mathbb{G} = SO(2n)$ the most compact one is $SO(2n,0)$ or $SO(2n-1,1)$. If $\mathbb{G} = SL(n)$ it is $SU(n,0)$ or $SL(n,\mathbb{R})$ (n odd) or $SL(n/2,\mathbb{H})$ (n even). For $Sp(2n)$ the compact one is $Sp(n,0)$.

The extended group $\mathbb{G}^\Gamma$ is the semidirect product of $\mathbb{G}$ and $\Gamma == \{1,\sigma\} \simeq \mathbb{Z}/2\mathbb{Z}$, with the action given by the most compact Cartan involution in the inner class.

Tip For the inner class corresponding to the $1 \in \mathrm{Out}(\mathbb{G})$, $\mathbb{G} = \mathbb{G} \times \mathbb{Z}/2\mathbb{Z}$, and you can pretty much ignore the extension. In du Cloux's notes the "twisted involution" is just an ordinary involution. I recommend you think about this case often. For example this is always the case if $\mathbb{G}$ has no outer automorphisms.

Remark 1.4 The group $\mathbb{G}^\Gamma$ is an L-group (but not on the dual side). That is, $\mathbb{G}^\Gamma$ is the L-group of the dual group $\mathbb{G}^\vee$, with respect to a certain inner class of $\mathbb{G}^\vee$.

What do we need to do to specify an inner class of real form, i.e. an element τ of $\mathrm{Out}(\mathbb{G})$ of order 2?

First assume $\mathbb{G}$ is semisimple. If a simple factor is fixed by τ, we have to specify either the compact inner class, or (if it exists) the other one. If two simple factors are switched by τ this corresponds to the complex group.

If $\mathbb{G}$ is reductive you also have to specify the real form of the center, which is a product of $\mathbb{R}^\times, S^1$ and $\mathbb{C}^\times$ terms.

2 Strong Real Forms

Fix an inner class, i.e. an element τ of $\mathrm{Out}(\mathbb{G})$ of order 2, and define $\mathbb{G}^\Gamma$ accordingly. Then any involution $\theta \in \mathrm{Aut}(\mathbb{G})$ in this inner class is of the form

(2.1)(a) $$\theta(g) = h\sigma(g)h^{-1}$$

for some $h \in \mathbb{G}$ such that $h\sigma(h) \in Z(\mathbb{G})$. In other words

(2.1)(b) $$\theta(g) = xgx^{-1} = \mathrm{int}(x)(g)$$

for some $x \in \mathbb{G}\sigma \subset \mathbb{G}^\Gamma$ with $x^2 \in Z(\mathbb{G})$.

Definition 2.2 *A strong real form of $\mathbb{G}$ is an element $x \in \mathbb{G}\sigma \subset \mathbb{G}^\Gamma$ such that $x\sigma(x) \in Z(\mathbb{G})$. Two strong real forms x, x' are equivalent if they are conjugate by $\mathbb{G}$.*

Example 2.3 If $\tau = 1$ then a strong real form may be identified with an element $x \in \mathbb{G}$ such that $x^2 \in Z(\mathbb{G})$. You can compute these: you may assume x is in a fixed Cartan subgroup on which $\mathbb{T}$, and equivalence is via the Weyl group.

The map $x \to \mathrm{int}(x)$ is a surjective map from the equivalence classes of strong real forms, to the equivalence classes of real forms. If $\mathbb{G}$ is adjoint this is a bijection, so we may safely ignore the concept of strong real form.

Example 2.4 If $\mathbb{G} = SO(3)$ (isomorphic to $PSL(2)$) there are two equivalence classes of strong real forms, given by $x = I$ and $x = \mathrm{diag}(-1, -1, 1)$. These correspond to $SO(3,0)$ and $SO(2,1)$ respectively.

If $\mathbb{G} = SL(2)$ there are 3 equivalence classes of strong real forms: $x = I, -I$ and $\mathrm{diag}(i, -i)$. These correspond to the real forms $SU(2), SU(2)$ and

$SU(1,1) \simeq SL(2,\mathbb{R})$, respectively. We may think of these as $SU(2,0)$, $SU(0,2)$ and $SU(1,1)$.

Thus the notion of strong real form is a refinement of the notion of real form.