Signatures of Invariant Hermitian Forms on Highest Weight Modules

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Motivation

Unitary Dual Problem: Classify the unitary irreps of a group

- abelian group: Pontrjagin
- compact, connected Lie group: Weyl, 1920s
- locally compact group–eg. reductive Lie group: open except for some special cases
- study a broader family of representations: those which admit an invariant Hermitian form
- real reductive Lie group: equivalent to classifying the irreducible Harish-Chandra modules (admissible, finitely-generated $(\mathfrak{g}, K)$-modules) which admit a positive-definite invariant Hermitian form
- Zuckerman 1978: construct all admissible $(\mathfrak{g}, K)$-modules by cohomological induction

- G - real reductive Lie group
- K - a maximal compact subgroup of G
- $\mathfrak{g}_0, \mathfrak{k}_0$ corresponding Lie algebras and $\mathfrak{g}$, $\mathfrak{k}$ their complexifications
- $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ parabolic subalgebra
- θ - Cartan involution corresponding to K
- begin with an $(\mathfrak{l}, L \cap K)$-module V where L is a Levi subgroup of G and $\mathfrak{l}$ its complexified Lie algebra
- Step 1: extend to a rep of $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ by allowing $\mathfrak{u}$ to act trivially, then apply induction functor

$$\mathrm{ind}_{(\mathfrak{q}, L\cap K)}^{(\mathfrak{g}, L\cap K)}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} V$$

- Step 2: apply a Zuckerman functor $\Gamma^j = j^{\mathrm{th}}$ derived functor of the left exact covariant functor Γ which takes the K-finite part of a representation

- $\mathrm{ind}_{(\mathfrak{q},L\cap K)}^{(\mathfrak{g},L\cap K)}(V)$ is a **generalized Verma module**, hence our interest in highest weight modules
- Strategy: relate the signature of invariant Hermitian form on V to signature of cohomologically induced module $\Gamma^j \mathrm{ind}_{(\mathfrak{q},L\cap K)}^{(\mathfrak{g},L\cap K)}(V)$
- **1984, Vogan:** Suppose $\mathfrak{q}$ is θ-stable. For an irreducible, unitarizable $(\mathfrak{l}, L\cap K)$-module V with infinitesimal character $\lambda \in \mathfrak{h}^*$, if

  $$\mathrm{Re}(\alpha, \lambda - \rho(\mathfrak{u})) \geq 0 \qquad \forall \alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$$

  then $\Gamma^m(\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V \otimes \wedge^{top}\mathfrak{u}))$ is also unitarizable, where $m = \dim \mathfrak{u} \cap \mathfrak{k}$.
  (Fact: $\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(V^h) := \mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V^h) \simeq (\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}} V)^h$.)
- **1984, Wallach:** more elementary proof of same result by computing the signature of the Shapovalov form on generalized Verma modules (invariant Hermitian form on the module obtained in Step 1 of cohomological induction)

- Potentially useful for unitary dual problem: signature of Shapovalov form on a generalized Verma module, when it exists, with no restrictions on value of infinitesimal character
- Today: irreducible Verma modules, irreducible highest weight modules of regular infinitesimal character

Invariant Hermitian Forms

Definition: **Invariant Hermitian form** $\langle \cdot, \cdot \rangle$ on V:

For all $v, w \in V$

- rep of G: $\langle gv, w \rangle = \langle v, g^{-1}w \rangle$ for all $g \in G$
- $\mathfrak{g}$-module: $\langle Xv, w \rangle + \langle v, \bar{X}w \rangle = 0$ for every $X \in \mathfrak{g}$, where $\bar{X}$ denotes the complex conjugate of X with respect to the real form $\mathfrak{g}_0$
- sesquilinear

When does a Verma module admit an invariant Hermitian form?

Theorem: An irreducible representation (π, V) admits a non-degenerate invariant Hermitian form if and only if it is isomorphic to a subrepresentation of its Hermitian dual (π^h, V^h).

Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be a Borel subalgebra of $\mathfrak{g}$ and $\Delta^+(\mathfrak{g}, \mathfrak{h})$ the corresponding system of positive roots.

$M(\lambda) = \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ so $M(\lambda)^h = \mathrm{pro}_{\bar{\mathfrak{b}}}^{\mathfrak{g}}(\mathbb{C}_\lambda^h) = \mathrm{Hom}_{\bar{\mathfrak{b}}}(U(\mathfrak{g}), \mathbb{C}_{-\bar{\lambda}})$

We see that $M(\lambda)$ embeds into $M(\lambda)^h$ if $\bar{\lambda} = -\lambda$ and $\overline{\Delta^+(\mathfrak{g}, \mathfrak{h})} = -\Delta^+(\mathfrak{g}, \mathfrak{h})$. When does this happen?

For $\mu \in \mathfrak{h}^*$, define:	$(\theta\mu)(H) = \mu(\theta^{-1}H)$	$(\bar{\mu})(H) = \overline{\mu(\bar{H})}$
Then:	$\theta\mathfrak{g}_\alpha = \mathfrak{g}_{\theta\alpha}$	$\bar{\mathfrak{g}}_\alpha = \mathfrak{g}_{\bar{\alpha}}$

Theorem: If $\mathfrak{h}$ is θ-stable and maximally compact, λ is imaginary, and $\theta\Delta^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{g}, \mathfrak{h})$, then $M(\lambda)$ admits a non-degenerate invariant Hermitian form.

- by $\mathfrak{h}$-invariance, the $\lambda - \mu$ weight space is orthogonal to the $\lambda - \nu$ weight space if $\nu \neq -\bar{\mu}$
- each weight space is finite dimensional, so it makes sense to talk about signatures and the determininants

Constructing the form:

For $X \in \mathfrak{g}$, let $X^* = -\bar{X}$ and extend $X \mapsto X^*$ to an involutive anti-automorphism of $U(\mathfrak{g})$ by $1^* = 1$ and $(xy)^* = y^* x^*$.

We have the decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n} + \mathfrak{n}^{op} U(\mathfrak{g}))$.
Let p be the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ under this direct sum.

- For $x, y \in U(\mathfrak{g})$, by invariance, $\langle x v_\lambda, y v_\lambda \rangle_\lambda = \langle y^* x v_\lambda, v_\lambda \rangle_\lambda$.
- $((U(\mathfrak{g})\mathfrak{n} + \mathfrak{n}^{op} U(\mathfrak{g})) v_\lambda, v_\lambda) = \{0\}$.
- $\langle x v_\lambda, y v_\lambda \rangle_\lambda = \langle p(y^* x) v_\lambda, v_\lambda \rangle_\lambda = \lambda(p(y^* x)) \langle v_\lambda, v_\lambda \rangle_\lambda$
- See that an invariant Hermitian form on a Verma module is unique up to a real scalar. When $\langle v_\lambda, v_\lambda \rangle_\lambda = 1$: **Shapovalov form**

Theorem: (Shapovalov determinant formula) The determinant of the Shapovalov form on the $\lambda - \mu$ weight space is

$$\prod_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})} \prod_{n=1}^{\infty} \left((\lambda+\rho, \alpha^\vee) - n\right)^{P(\mu - n\alpha)}$$

up to multiplication by a scalar, where P denotes Kostant's partition function. (**Assumption: $\mathfrak{h}$ is compact.**)

- radical of Shapovalov form = unique maximal submodule of $M(\lambda)$
- form non-degenerate precisely for the irreducible Verma modules
- according to Shapovalov determinant formula, $M(\lambda)$ is reducible on the affine hyperplanes $H_{\alpha,n} := \{\lambda + \rho \,|\, (\lambda+\rho, \alpha^\vee) = n\}$ where α is a positive root and n is a positive integer
- in any connected set of purely imaginary λ avoiding these reducibility hyperplanes, as the Shapovalov form never becomes degenerate, the signature corresponding to fixed μ remains constant

Definition: The largest of such regions, which we name the **Wallach region**, is the intersection of the negative open half spaces

$$\left(\bigcap_{\alpha\in\Pi} H^-_{\alpha,1}\right)\bigcap H^-_{\widetilde{\alpha},1}$$

with $i\mathfrak{h}_0^*$, where $\widetilde{\alpha}^\vee =$ the highest coroot, $\Pi =$ simple roots corresponding to Δ^+, and $H^-_{\beta,n} = \{\lambda+\rho|(\lambda+\rho,\beta^\vee) < n\}$.

Definition: If the signature of the Shapovalov form on $M(\lambda)_{\lambda-\mu}$ is $(p(\mu),q(\mu))$, the **signature character** of $\langle\cdot,\cdot\rangle_\lambda$ is

$$ch_s M(\lambda) = \sum_{\mu\in\Lambda_r^+} (p(\mu)-q(\mu))\, e^{\lambda-\mu}$$

Pick λ,ξ so that $\lambda+t\xi$ stays in the Wallach region for $t\geq 0$. An asymptotic argument (degree of t on the diagonal $>$ degree off the diagonal) leads to:

Theorem: (Wallach) The signature character of $M(\lambda)$ for $\lambda+\rho$ in the Wallach region is

$$ch_s M(\lambda) = \frac{e^{\lambda}}{\prod_{\alpha\in\Delta^+(\mathfrak{p},\mathfrak{t})}\left(1-e^{-\alpha}\right)\prod_{\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})}\left(1+e^{-\alpha}\right)}.$$

Goal: be able to find the signature *everywhere*.

Idea: determine how the signature changes as you cross a reducibility hyperplane. Combine this with induction.

- take λ s.t. $\lambda+\rho$ lies in exactly one reducibility hyperplane $H_{\alpha,n}$
- for reg ξ and non-zero t in a nbd of 0, $\langle\cdot,\cdot\rangle_{\lambda+t\xi}$ is non-degenerate
- $\langle\cdot,\cdot\rangle_{\lambda}$ has radical isom to the irreducible Verma module $M(\lambda-n\alpha)$
- **therefore signature must change by plus or minus the signature of $\langle\cdot,\cdot\rangle_{\lambda-n\alpha}$ across $H_{\alpha,n}$**

This can be made rigorous by using the **Jantzen filtration.**

- the $H_{\alpha,n}$'s where α is a root, n an integer, partition $\mathfrak{h}^*$ into **alcoves**

Definition: For an alcove A, $\exists$ constants c_μ^A for $\mu \in \Lambda_r^+$ such that

$$R^A(\lambda) := \sum_{\mu \in \Lambda_r^+} c_\mu^A e^{\lambda - \mu}$$

is the signature character of $\langle \cdot, \cdot \rangle_\lambda$ when $\lambda + \rho$ lies in the alcove A.

Our description of how signatures change as you cross a reducibility hyerplane may be expressed:

Lemma 1: If A, A' are adjacent alcoves separated by $H_{\alpha,n}$,

$$\text{then} \qquad R^A(\lambda) = R^{A'}(\lambda) + 2\varepsilon(A, A') R^{A - n\alpha}(\lambda - n\alpha)$$

where $\varepsilon(A, A')$ is zero if $H_{\alpha,n}$ is not a reducibility hyperplane and plus or minus one otherwise.

- use $R(\lambda)$ to denote common signature character for alcoves in Wallach region

We use the affine Weyl group, whose action on $\mathfrak{h}^*$ partitions $\mathfrak{h}^*$ into precisely the alcoves with walls $H_{\alpha,n}$ as described above.

Definition The **fundamental alcove** is

$$A_0 = \{\lambda + \rho \,|\, (\lambda + \rho, \alpha^\vee) < 0 \quad \forall \alpha \in \Pi, \quad (\lambda + \rho, \widetilde{\alpha}^\vee) > -1\}\,.$$

- reflections through walls of A_0 generate the affine Weyl group, W_a: reflections $s_{\alpha,0}$ for each simple root α and $s_{\widetilde{\alpha},-1}$ generate W_a
- omit $s_{\widetilde{\alpha},-1} \rightarrow$, generate the Weyl group W as a subgroup of W_a
- these generators compatible with reflection through walls of the fundamental Weyl chamber $\mathfrak{C}_0$, which we choose to contain A_0:

$$\mathfrak{C}_0 = \bigcap_{\alpha \in \Pi} H^-_{\alpha,0}.$$

Definition We will define two maps $\bar{\cdot}$ and $\tilde{\cdot}$ from the affine Weyl group to the Weyl group as follows:

- $\bar{\cdot}$ comes from structure of W_a as semidirect product of translation by the root lattice and the Weyl group: $\overline{w} = s$ if $w = ts$ with $t =$ translation by an element of Λ_r, $s \in W$
- We let $\widetilde{w}$ be such that wA_0 lies in the Weyl chamber $\widetilde{w}\mathfrak{C}_0$.

- $\bar{\cdot}$ is a group homomorphism
- $\tilde{\cdot}$ is not a group homomorphism
- $\overline{s_{\alpha,n}} = s_\alpha$, and $s_{\alpha,0}s_{\alpha,n}\mu = \mu - n\alpha$

Observe that we can rewrite Lemma 1 as

$$\begin{aligned} R^{wA_0}(\lambda) &= R^{w'A_0}(\lambda) + 2\varepsilon(wA_0, w'A_0) R^{s_{\alpha,0}s_{\alpha,n}wA_0}(s_{\alpha,0}s_{\alpha,n}\lambda) \\ &= R^{w'A_0}(\lambda) + 2\varepsilon(wA_0, w'A_0) R^{\overline{s_{\alpha,n}}w'A_0}(\overline{s_{\alpha,n}}s_{\alpha,n}\lambda). \quad (1) \end{aligned}$$

For w in the affine Weyl group, let $wA_0 = C_0 \xrightarrow{r_1} C_1 \xrightarrow{r_2} \cdots \xrightarrow{r_\ell} C_\ell = \widetilde{w}A_0$ be a (not necessarily reduced) path from wA_0 to $\widetilde{w}A_0$. Applying (1), ℓ times, we obtain

$$\begin{aligned} R^{wA_0}(\lambda) &= R^{\widetilde{w}A_0}(\lambda) + \sum_{j=1}^{\ell} \varepsilon(C_{j-1}, C_j) 2R^{\overline{r_j}C_j}(\overline{r_j}r_j\lambda) \\ &= R(\lambda) + 2\sum_{j=1}^{\ell} \varepsilon(C_{j-1}, C_j) R^{\overline{r_j}C_j}(\overline{r_j}r_j\lambda). \end{aligned}$$

Observe that a path from $\overline{r_j}C_j$ to $\overline{r_j}C_\ell$ is

$$\overline{r_j}C_j \xrightarrow{\overline{r_j}r_{j+1}\overline{r_j}} \overline{r_j}C_{j+1} \xrightarrow{\overline{r_j}r_{j+2}\overline{r_j}} \cdots \xrightarrow{\overline{r_j}r_\ell\overline{r_j}} \overline{r_j}C_\ell.$$

Applying induction on path length, we arrive at the following:

Theorem 2: For w in the affine Weyl group, let $wA_0 = C_0 \xrightarrow{r_1} C_1 \xrightarrow{r_2} \cdots \xrightarrow{r_\ell} C_\ell = \widetilde{w}A_0$ be a (not necessarily reduced) path from wA_0 to $\widetilde{w}A_0$.

$R^{wA_0}(\lambda)$ equals

$$\sum_{S=\{i_1<\cdots<i_k\}\subset\{1,\ldots,\ell\}} \varepsilon(S) 2^{|S|} R^{\overline{r_{i_1}}\cdots\overline{r_{i_k}}\widetilde{w}A_0}\left(\overline{r_{i_1}r_{i_2}}\cdots\overline{r_{i_k}}r_{i_k}r_{i_{k-1}}\cdots r_{i_1}\lambda\right)$$
$$= \sum_{S=\{i_1<\cdots<i_k\}\subset\{1,\ldots,\ell\}} \varepsilon(S) 2^{|S|} R\left(\overline{r_{i_1}r_{i_2}}\cdots\overline{r_{i_k}}r_{i_k}r_{i_{k-1}}\cdots r_{i_1}\lambda\right)$$

where $\varepsilon(\emptyset) = 1$ and

$$\varepsilon(S) = \varepsilon(C_{i_1-1}, C_{i_1})\varepsilon(\overline{r_{i_1}}C_{i_2-1}, \overline{r_{i_1}}C_{i_2})\cdots\varepsilon(\overline{r_{i_1}}\cdots\overline{r_{i_{k-1}}}C_{i_k-1}, \overline{r_{i_1}}\cdots\overline{r_{i_{k-1}}}C_{i_k}).$$

Calculating ε: difficult.

Calculating ε

The strategy for computing ε is as follows:

- We show that for a fixed hyperplane $H_{\alpha,n}$, the value of ε for crossing from $H^+_{\alpha,n}$ to $H^-_{\alpha,n}$ depends only on the Weyl chamber to which the point of crossing belongs.
- We consider rank 2 root systems of types A_2 and B_2, generated by simple roots α_1 and α_2, and calculate the values for ε by calculating changes that occur at the Weyl chamber walls. Our proofs do not depend on simplicity of the α_i.
- For an arbitrary positive root γ in a generic irreducible root system which is not type G_2, we develop a formula for ε inductively by replacing the α_i from the previous step with appropriate roots. Key in the induction is the independence of our rank 2 arguments from the simplicity of the α_i.

Let's begin with something simple: calculate ε for α simple.

Lemma 2: Let δ_α be -1 if α is noncompact, and 1 if it is compact. If α is simple and n is positive and if $H_{\alpha,n}$ separates wA_0 and $w'A_0$ with $wA_0 \subset H^+_{\alpha,n}$ and $w'A_0 \subset H^-_{\alpha,n}$, then $\varepsilon(wA_0, w'A_0) = \delta^n_\alpha$.

Proof: Choose $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, and $H_\alpha = [X_\alpha, Y_\alpha]$, a standard triple so that $\mu(H_\alpha) = (\mu, \alpha^\vee) \forall \mu \in \mathfrak{h}^*$. We may arrange so that

$$-\bar{Y}_\alpha = \delta_\alpha X_\alpha.$$

The $\lambda - n\alpha$ weight space of $M(\lambda)$ is one-dimensional and spanned by the vector $Y^n_\alpha v_\lambda$. We know that

$$\begin{aligned} \langle Y^n_\alpha v_\lambda, Y^n_\alpha v_\lambda \rangle_\lambda &= \delta^n_\alpha \langle v_\lambda, X^n_\alpha Y^n_\alpha v_\lambda \rangle_\lambda \\ &= \delta^n_\alpha n! \langle v_\lambda, H_\alpha (H_\alpha - 1) \cdots (H_\alpha - (n-1)) v_\lambda \rangle_\lambda \end{aligned}$$

from $\mathfrak{sl}_2$ theory. We conclude that

$$\varepsilon(wA_0, w'A_0) = \delta^n_\alpha.$$

Dependence on Weyl Chambers

Proposition 1: Suppose α is a positive root and $n \in \mathbb{Z}^+$ and suppose $H_{\alpha,n}$ separates adjacent alcoves wA_0 and $w'A_0$, with $wA_0 \subset H^+_{\alpha,n}$ and $w'A_0 \subset H^-_{\alpha,n}$. The value of $\varepsilon(w,w')$ depends only on $H_{\alpha,n}$ and on $\widetilde{w}(=\widetilde{w}')$.

We begin by refining Theorem 2: if we take an arbitrary C_ℓ, the formula becomes

$$R^{wA_0}(\lambda) = \sum_{I=\{i_1<\cdots<i_k\}\subset\{1,\ldots,\ell\}} \varepsilon(I) 2^{|I|} R^{\overline{r_{i_1}}\cdots\overline{r_{i_k}}C_\ell}\left(\overline{r_{i_1}}\cdots\overline{r_{i_k}} r_{i_k}\cdots r_{i_1}\lambda\right).$$

If we choose in particular $C_\ell = C_0$, we have

$$R^{C_0}(\lambda) = \sum_{\substack{I=\{i_1<\cdots<i_k\}\\ \subset\{1,\ldots,\ell\}}} \varepsilon(I) 2^{|I|} R^{\overline{r_{i_1}}\cdots\overline{r_{i_k}}C_0}\left(\overline{r_{i_1}}\cdots\overline{r_{i_k}} r_{i_k}\cdots r_{i_1}\lambda\right). \qquad (2)$$

We begin by proving the proposition in the special case where $wA_0 = C_i$ and $w'A_0 = C_{i+1}$ as described in the following figure:

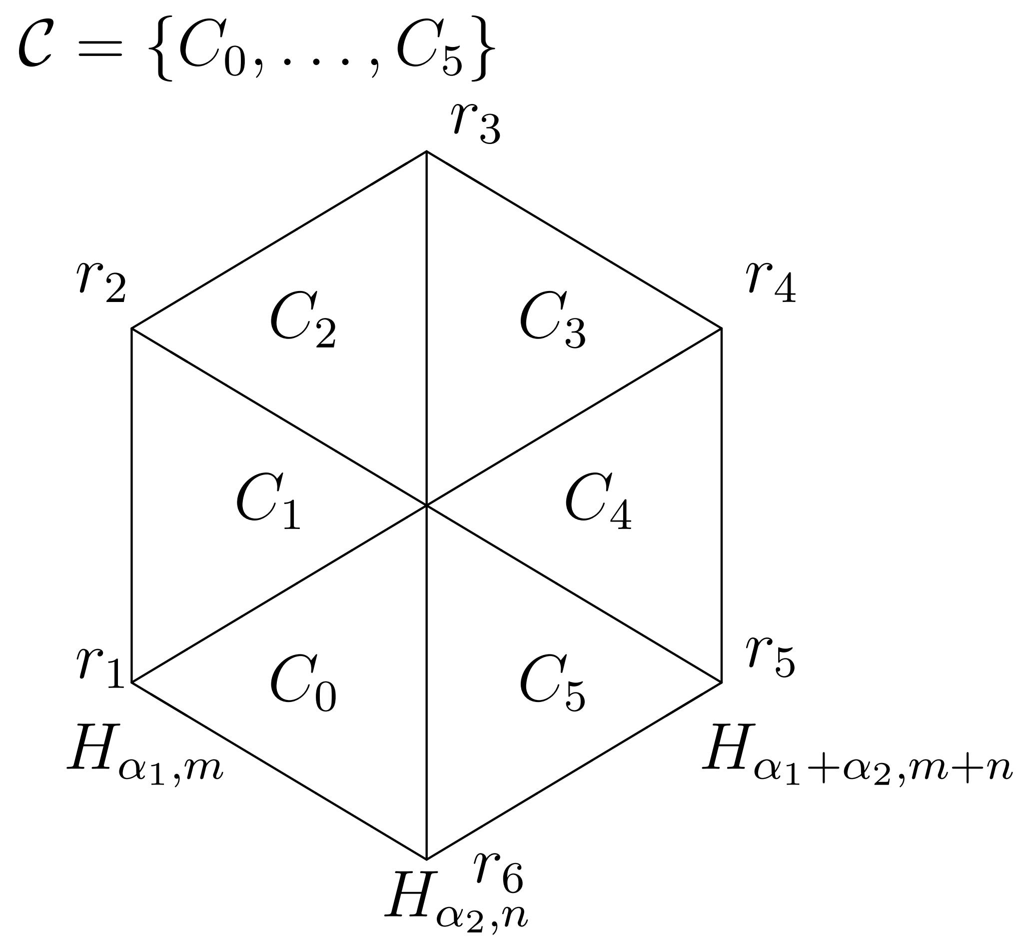


Figure 1: Type A_2

Lemma 3: Let $\mathcal{C} = \{C_i\}_{i=0,\ldots,\ell-1}$ be a set of alcoves that lie in the interior of some Weyl chamber and suppose the reflections $\{r_j\}_{j=1,\cdots,k}$ preserve $\mathcal{C}$. If $w, v \in W_a$ are generated by the r_j then

$$\overline{w}^{-1}w = \overline{v}^{-1}v \iff w = v.$$

Proof: $\Rightarrow$: By simple transitivity of the action of W_a on the alcoves, $\overline{w}^{-1}w = \overline{v}^{-1}v \iff \overline{w}^{-1}wC = \overline{v}^{-1}vC$ for any alcove C. Choose in particular $C = C_i$. The alcoves $\overline{w}^{-1}wC_i$ and $\overline{v}^{-1}vC_i$ belong to the same Weyl chamber as they are the same alcove. As the r_j's preserve $\mathcal{C}$ which lies in the interior of some Weyl chamber, wC_i and vC_i belong to the same Weyl chamber. Thus $\overline{w}^{-1} = \overline{v}^{-1}$, whence $w = v$. The other direction is trivial.

Note: $\mathcal{C}$ in the figure satisfies the conditions of Lemma 3.

To prove the proposition for the figure, we need to show that $\varepsilon(C_i, C_{i+1}) + \varepsilon(C_{i+3}, C_{i+4}) = 0$, $(C_6 = C_0)$.

For $I = \{i_1 < \cdots < i_k\}$, we define $w_I = r_{i_k} r_{i_{k-1}} \cdots r_1$. We rewrite (2) as

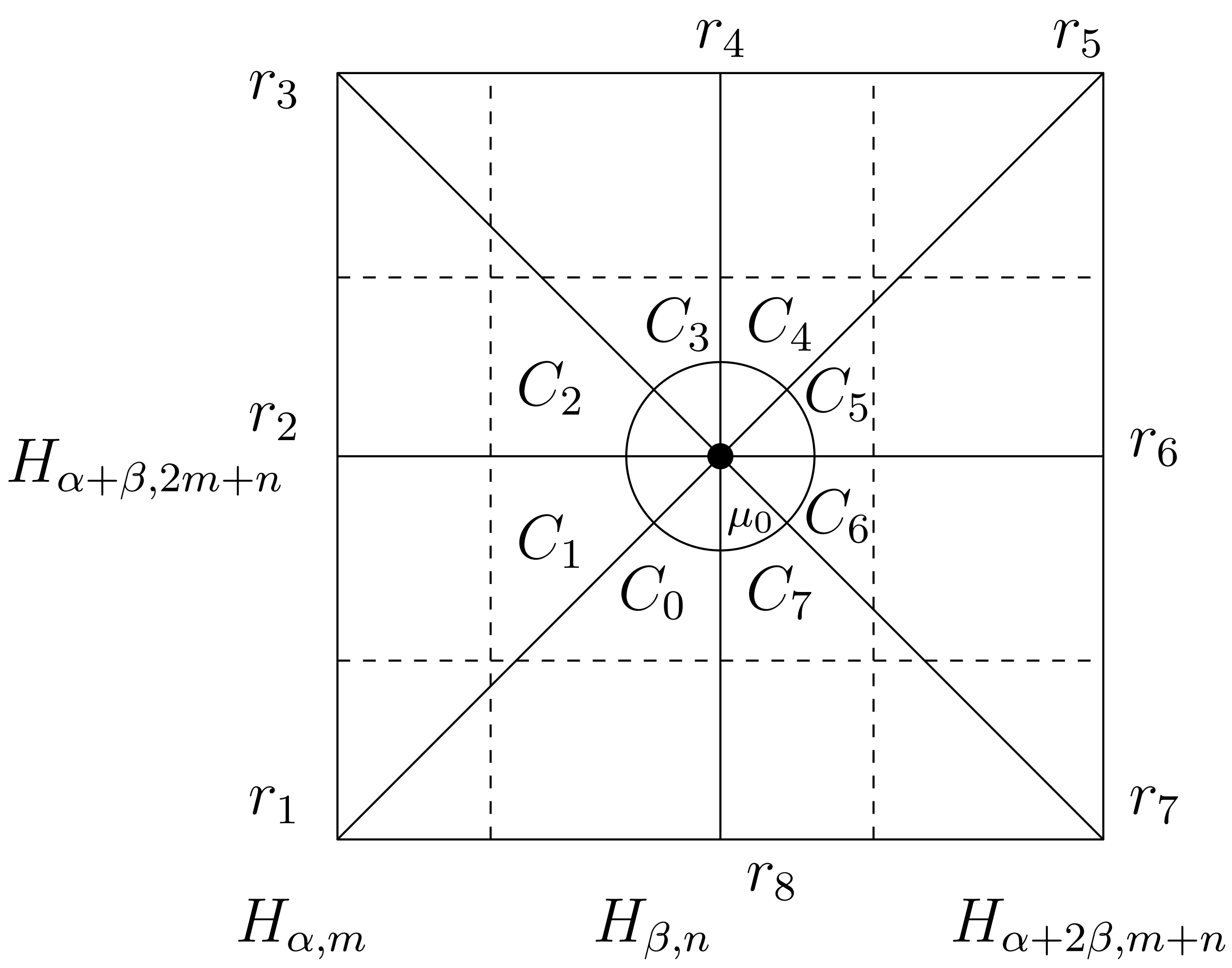
$$\sum_{\emptyset \neq I \subset \{1,\ldots,\ell\}} 2^{|I|} \varepsilon(I) R^{\overline{w_I}^{-1} C_0} \left(\overline{w_I}^{-1} w_I \lambda\right) = 0 \tag{3}$$

Using Lemma 3 and the partial ordering on Λ, we obtain

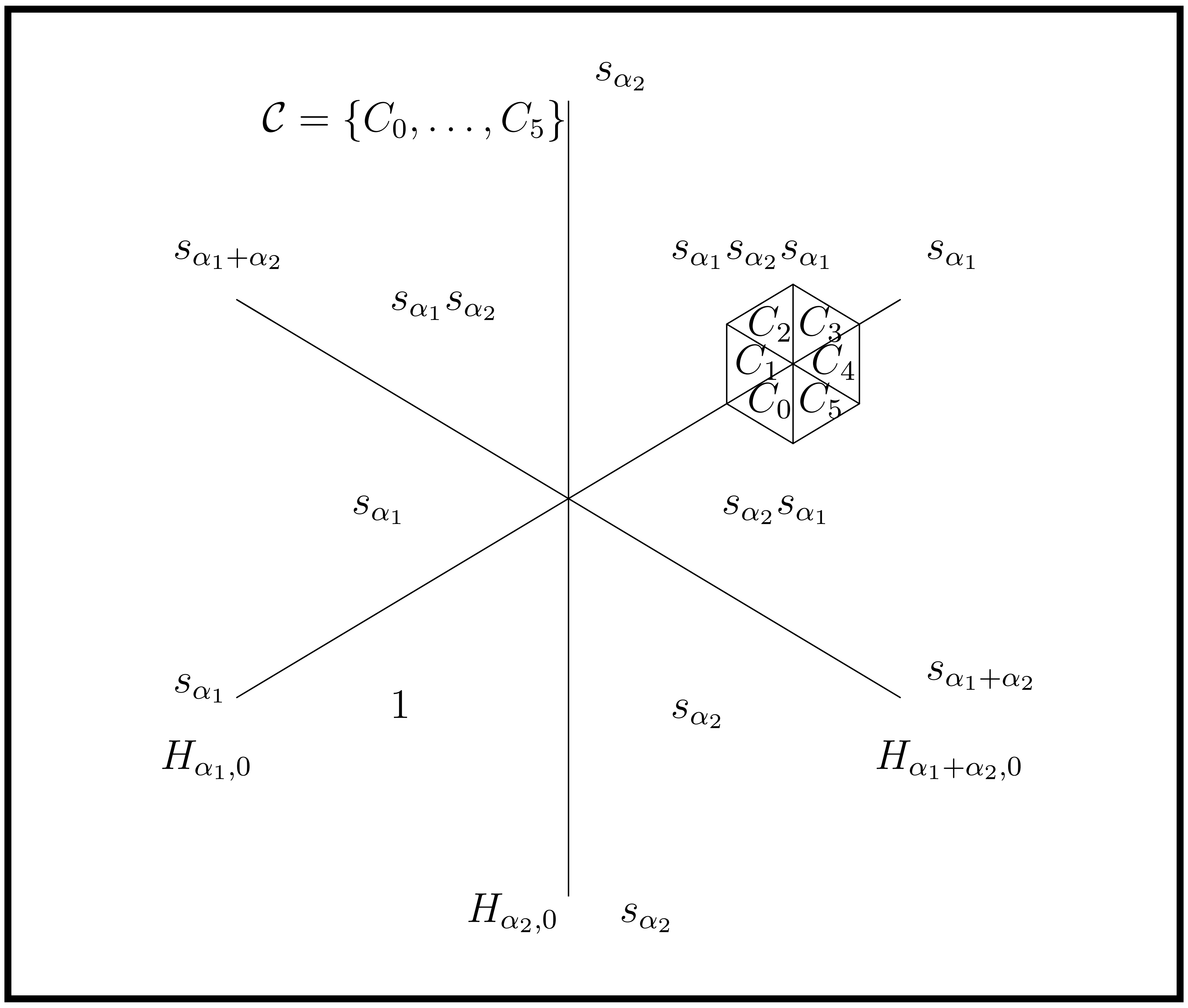
$$\sum_{\substack{\emptyset \neq I \subset \{1,\ldots,\ell\} \\ \overline{w_I}^{-1} w_I = \mu}} 2^{|I|} \varepsilon(I) = 0 \tag{4}$$

for every $\mu \in \Lambda$.

Suppose $\mu = m\alpha_1$. The subsets I of length less than 3 for which $\overline{w_I}^{-1} w_I = \mu$ are $I = \{1\}, \{4\}$. By considering equation (4) modulo 8, we obtain $\varepsilon(C_0, C_1) + \varepsilon(C_3, C_4) = 0$ which gives the desired result for $H_{\alpha_1,m}$. The same proof can be used for the other hyperplanes and also for type B_2.



Generalization: $\mathcal{C} = \{$alcoves containing μ_0 in their closures$\}$.
Conditions of Lemma 3 satisfied, argue as before.


s_{α_2}
$\mathcal{C} = \{C_0, \ldots, C_5\}$
$s_{\alpha_1+\alpha_2}$
$s_{\alpha_1} s_{\alpha_2}$
$s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$
s_{α_1}
C_2
C_3
C_1
C_4
C_0
C_5
s_{α_1}
$s_{\alpha_2} s_{\alpha_1}$
s_{α_1}
1
s_{α_2}
$s_{\alpha_1+\alpha_2}$
$H_{\alpha_1,0}$
$H_{\alpha_1+\alpha_2,0}$
$H_{\alpha_2,0}$
s_{α_2}

Calculating ε for Type A_2

- know how to calculate ε for hyperplanes corresponding to simple roots, so we know how to calculate ε in the Weyl chambers adjacent to the fundamental Weyl chamber
- again, changes along a closed path should sum to zero
- so previous diagram, where $\mathcal{C}$ overlaps with two Weyl chambers, allows you to relate values of ε in one chamber to values in an adjacent chamber

Weyl chamber walls in $\mathcal{C}$	**Equations**
$H_{\alpha_1,0}$	$\varepsilon(C_2, C_3) + \varepsilon(C_5, C_6) = 0$ $\varepsilon(C_1, C_2) + \varepsilon(C_4, C_5) + 2\varepsilon(C_2, C_3)\varepsilon(\overline{r_3}C_4, \overline{r_3}C_5) = 0$
$H_{\alpha_2,0}$	$\varepsilon(C_0, C_1) + \varepsilon(C_3, C_4) = 0$ $\varepsilon(C_1, C_2) + \varepsilon(C_4, C_5) + 2\varepsilon(C_0, C_1)\varepsilon(\overline{r_1}C_1, \overline{r_1}C_2) = 0$
$H_{\alpha_1+\alpha_2,0}$	$\varepsilon(C_0, C_1) + \varepsilon(C_3, C_4) = 0$ $\varepsilon(C_2, C_3) + \varepsilon(C_5, C_0) = 0$

Final Formula for ε

Notation: $\varepsilon(H_{\gamma,N}, s) = \varepsilon(A, A')$ where $A \subset H^+_{\gamma,N}$, $A' \subset H^-_{\gamma,N}$, A and A' are adjacent, and $A \subset s\mathfrak{C}_0$.

Using induction on height:

Theorem 3: Let γ be a positive root, and let $\gamma = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$ be such that $\mathrm{ht}(s_{i_j} \cdots s_{i_{k-1}} \alpha_{i_k})$ decreasesas j increases. Let $w_\gamma = s_{i_1} \cdots s_{i_k}$. If γ hyperplanes are positive on $s\mathfrak{C}_\mathfrak{o}$, then

$$\begin{aligned} \varepsilon(H_{\gamma,N}, s) = \quad & (-1)^{N\#\{\text{noncompact } \alpha_{i_j} : |\alpha_{i_j}| \geq |\gamma|\}} \\ \times \quad & (-1)^{\#\{\beta \in \Delta(w_\gamma^{-1}) : |\beta| = |\gamma|, \beta \neq \gamma, \text{ and } \beta, s_\beta\gamma \in \Delta(s^{-1})\}} \\ \times \quad & (-1)^{\#\{\beta \in \Delta(w_\gamma^{-1}) : |\beta| \neq |\gamma| \text{ and } \beta, -s_\beta s_\gamma \beta \in \Delta(s^{-1})\}}. \end{aligned}$$

Extending results so that we know how to compute signature characters for non-compact Cartan subalgebras: use formulas for singular vectors.

Irreducible Highest Weight Modules

- the Shapovalov form on $M(\lambda)$ descends to an invariant Hermitian form on the irreducible highest weight module $L(\lambda)$

Let λ be antidominant, regular, and $x \in W_\lambda$. The Jantzen filtration of $M(x \cdot \lambda)$ $(x \cdot \lambda = x(\lambda + \rho) - \rho)$ is

$$M(x \cdot \lambda) = M(x \cdot \lambda)^0 \supset M(x \cdot \lambda)^1 \supset \cdots \supset M(x \cdot \lambda)^N = \{0\}$$

where, for fixed δ regular,

$$M(x{\cdot}\lambda)^j = \left\{ \text{vectors } av_{x\cdot\lambda} \in M(x \cdot \lambda) \;\middle|\; \begin{array}{c} \langle av_{x\cdot\lambda+\delta t}, bv_{x\cdot\lambda+\delta t} \rangle_{x\cdot\lambda+\delta t} \\ \text{vanishes at least to order} \\ j \text{ at } t = 0 \, \forall \, b \in U(n^{op}) \end{array} \right\}.$$

• $M(x \cdot \lambda)_j = M(x \cdot \lambda)^j / M(x \cdot \lambda)^{j+1}$ is semisimple

• get a non-degenerate invariant Hermitian form $\langle \cdot, \cdot \rangle_j$ on $M(x \cdot \lambda)_j$

• Kazhdan-Lusztig polynomials tell you: $[M(x \cdot \lambda)_j : L(y \cdot \lambda)] =$ coefficient of $q^{(\ell(x)-\ell(y)-j)/2}$ in $P_{w_\lambda x, w_\lambda y}(q)$

• define analogous polynomials keeping track of signatures: form on each copy of $L(y \cdot \lambda)$ in j^{th} level of filtration has signature $\pm$ signature of the Shapovalov form on $L(y \cdot \lambda)$

• Jantzen filtration does not depend on choice of δ

• form on j^{th} level, however, does; $ch_s M(x \cdot \lambda + \delta t)$ equals:

$\sum_j ch_s \langle \cdot, \cdot \rangle_j$ for small $t > 0$

$\sum_{j \text{ even}} ch_s \langle \cdot, \cdot \rangle_j - \sum_{j \text{ odd}} ch_s \langle \cdot, \cdot \rangle_j$ for small $t < 0$

More precisely, the signature of the form depends on the (integral) Weyl chamber containing δ: if $\delta \in w\mathfrak{C}_0$, there are integers $a^{x\cdot\lambda,w}_{y,j}$ such that

$$ch_s \left\langle \cdot, \cdot \right\rangle_j = \sum_{y \leq w} a^{x\lambda,w}_{y,j} ch_s L(y \cdot j)$$

$$R^{wA_0+x\lambda}(x\lambda) = \sum_j \sum_{y \leq w} a^{x\lambda,w}_{y,j} ch_s L(y \cdot j)$$

Proposition: Letting $a^{x\lambda,w}_{y} = \sum_j a^{x\lambda,w}_{y,j}$,

$$ch_s L(x\lambda) = \sum_{y_1 < \cdots < y_j = x} (-1)^{j-1} \left(\prod_{i=2}^{i=j} a^{y_i\lambda,w}_{y_{i-1}} \right) R^{y_1\lambda + wA_0}(y_1\lambda).$$

The usual Kazhdan-Lusztig polynomials may be computed via the inductive formulas:

a) $P_{w_\lambda x, w_\lambda y} = P_{w_\lambda xs, w_\lambda y}$ if $ys > y$ and $x, xs \geq y$, s simple

a') $P_{w_\lambda x, w_\lambda y} = P_{w_\lambda sx, w_\lambda y}$ if $sy > y$ and $x, sx \geq y$, s simple

b) If $y > ys$ then

$$q^c P_{w_\lambda xs, w_\lambda y} + q^{1-c} P_{w_\lambda x, w_\lambda y} = \sum_{z \in W_\lambda | zs > z} \mu(w_\lambda z, w_\lambda y) q^{\frac{\ell(z)-\ell(y)+1}{2}} P_{w_\lambda x, w_\lambda z} + P_{w_\lambda x, w_\lambda ys}$$

Signed versions: inductive formulas simlar. Have to include some signs which depend on x, λ, w, $s = s_\alpha$.

We would like to extend this work to generalized Verma modules for the purpose of studying invariant Hermitian forms on Harish-Chandra modules. Open problems which need to be solved for this purpose:

- reducibility of generalized Verma modules
  - computing the determinant of the Shapovalov form in some special cases: Khomenko-Mazorchuk
  - sufficient conditions for certain principal series representations: Speh-Vogan
- determining the composition series of a generalized Verma module (i.e. what are the irreducible factors, and what are their multiplicities)
  - composition series for generalized principal series representations $\Rightarrow$ determine reducibility of representation induced from a parabolic subgroup