Petite K-types for Exceptional Groups
(joint work with Dan Barbasch)

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Contents

Chapter 1

Petite $\tilde{K}$-types for E_6

- $G = E_6$; $\tilde{K} = SP_4$; $K = \frac{SP_4}{\{\pm I\}}$
- **Fundamental weights of** $\tilde{K}$: $w_j = \sum_{i=1}^{j} \epsilon_i \quad j = 1 \dots 4$
- **Representations of** $\tilde{K}$: $\mu = \sum_{j=1}^{4} b_j w_j \quad b_j \geq 0$
- **Genuine** $\tilde{K}$**-types**: "$-I$ does not act trivially"

A petite $\tilde{K}$-type is either genuine of level $\leq 3/2$ or non-genuine of level ≤ 3

◇ μ is petite if and only if it has level 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, or 3

◇ The level of $\mu = (\sum_{j=1}^{4} b_j w_j)$ is $\frac{1}{2}b_1 + b_2 + \frac{3}{2}b_3 + 2b_4$.

Representations of $\tilde{M}$:

repr. of $\tilde{M}$	**dimension**	W_δ^0	**fine $\tilde{K}$-type**
δ_1	1	$W(E6)$	0
$\boxed{\delta_8}$	8	$W(E6)$	w_1
δ_{27}	$27 \cdot 1$	$W(D_5)$	w_2
δ_{36}	$36 \cdot 1$	$W(A_5A_1)$	$2w_1$

Tensor products of representations of $\tilde{M}$:

$\otimes$	δ_8	δ_{27}	δ_{36}
δ_8	$\delta_1 + \delta_{27} + \delta_{36}$	$27\delta_8$	$36\delta_8$
δ_{27}	$27\delta_8$	$27\delta_1 + 10\delta_{27} + 12\delta_{36}$	$16\delta_{27} + 15\delta_{36}$
δ_{36}	$36\,\delta_8$	$16\delta_{27} + 15\delta_{36}$	$36\delta_1 + 20\delta_{27} + 20\delta_{36}$

A complete list of petite $\tilde{K}$-types:

petite $\tilde{K}$-type	**level**	**$\tilde{M}$-decomposition**
0	0	δ_1
w_1	$1/2$	δ_8
$2w_1$	1	δ_{36}
w_2	1	δ_{27}
$3w_1$	$3/2$	$15\delta_8$
$w_1 + w_2$	$3/2$	$20\delta_8$
w_3	$3/2$	$6\delta_8$
$4w_1$	2	$15\delta_1 + 5\delta_{27} + 5\delta_{36}$
$2w_1 + w_2$	2	$10\delta_{27} + 9\delta_{36}$
$w_1 + w_3$	2	$5\delta_{27} + 5\delta_{36}$
$2w_2$	2	$20\delta_1 + 4\delta_{27} + 5\delta_{36}$
w_4	2	$6\delta_1 + \delta_{36}$
$6w_1$	3	$24\delta_1 + 24\delta_{27} + 29\delta_{36}$

A complete list of petite $\tilde{K}$-types (continued)

petite $\tilde{K}$-type	level	$\tilde{M}$-decomposition
$4w_1 + w_2$	3	$60\delta_1 + 70\delta_{27} + 65\delta_{36}$
$3w_1 + w_3$	3	$60\delta_1 + 60\delta_{27} + 56\delta_{36}$
$2w_1 + 2w_2$	3	$81\delta_1 + 71\delta_{27} + 81\delta_{36}$
$2w_1 + w_4$	3	$30\delta_1 + 15\delta_{27} + 20\delta_{36}$
$w_1 + w_2 + w_3$	3	$64\delta_1 + 64\delta_{27} + 64\delta_{36}$
$3w_2$	3	$24\delta_1 + 40\delta_{27} + 30\delta_{36}$
$w_2 + w_4$	3	$16\delta_{27} + 10\delta_{36}$
$2w_3$	3	$15\delta_1 + 10\delta_{27} + 15\delta_{36}$

The petite $\tilde{K}$-types containing δ_1:

petite $\tilde{K}$-type	**level**	**mult. of δ_1**	**representation of $W(E_6)$**
0	0	1	$\boxed{1_p}$
$4w_1$	2	15	$\boxed{15_q}$
$2w_2$	2	20	$\boxed{20_p}$
w_4	2	6	$\boxed{6_p}$
$6w_1$	3	24	$24_{p'}$
$4w_1 + w_2$	3	60	60_s
$3w_1 + w_3$	3	60	60_p
$2w_1 + 2w_2$	3	81	81_p
$2w_1 + w_4$	3	30	$\boxed{30_p}$
$w_1 + w_2 + w_3$	3	64	64_p
$3w_2$	3	24	24_p
$2w_3$	3	15	15_p

The relevant $W(E_6)$-types are: 1_p, 6_p, 15_q, 20_p, 30_p. *We can match all of them!*

The petite (genuine) $\tilde{K}$-types containing δ_8:

petite $\tilde{K}$-type	mult. of δ_8	repr. of $W(E_6)$
w_1	1	$\boxed{1_p}$
w_3	6	$\boxed{6_p}$
$3w_1$	15	$\boxed{15_q}$
$w_1 + w_2$	20	$\boxed{20_p}$

The relevant types for $W(E_6)$ are: 1_p, 6_p, 15_q, 20_p, 30_p .

We cannot match 30_p . Indeed this W-type makes its first appearance in (not petite) genuine $\tilde{K}$-types of level 5/2.

The petite $\tilde{K}$-types containing δ_{27}:

petite $\tilde{K}$-type	representation of $W(D_5)$
w_2	$\boxed{(5\|0)}$
$4w_1$	$\boxed{(32\|0)}$
$2w_1 + w_2$	$\boxed{(3\|2)}$
$w_1 + w_3$	$\boxed{(4\|1)}$
$2w_2$	$\boxed{(41\|0)}$
$2w_1 + w_4$	$(31\|1)$
$6w_1$	$(2111\|0) + (21\|11)$
$4w_1 + w_2$	$(32\|0) + (3\|2) + (221\|0) + (22\|1) + (21\|2) + (21\|11)$
$3w_1 + w_3$	$(32\|0) + (3\|2) + (31\|1) + (21\|2) + (22\|1)$
$2w_1 + 2w_2$	$(311\|0) + (31\|1) + (3\|11) + (21\|2) + (21\|11)$
$w_1 + w_2 + w_3$	$(4\|1) + (41\|0) + (3\|2) + (31\|1) + (3\|11) + (21\|2)$
$3w_2$	$(5\|0) + (41\|0) + (4\|1) + (3\|2) + (21\|2)$

The petite $\tilde{K}$-types containing δ_{27} (continued)

petite $\tilde{K}$-type	**representation of $W(D_5)$**
$w_2 + w_4$	$(5\|0) + (4\|1) + (3\|2)$
$2w_3$	$(3\|11)$

The relevant types for $W(D_5)$ are:

$$(5|0) \quad (41|0) \quad (32|0) \quad (4|1) \quad (3|2).$$

We can match all of them!

The petite $\tilde{K}$-types containing δ_{36}:

petite $\tilde{K}$-type	representation of $W(A_5 \times A_1)$
$2w_1$	$\boxed{6 \times 2}$
$4w_1$	$\boxed{33 \times 11}$
w_4	$\boxed{6 \times 11}$
$2w_1 + w_2$	$\boxed{42 \times 2}$
$w_1 + w_3$	$\boxed{51 \times 2}$
$2w_2$	$\boxed{51 \times 11}$
$2w_1 + w_4$	$6 \times 11 + 51 \times 2 + \boxed{42 \times 11} + \boxed{33 \times 2}$
$6w_1$	$6 \times 2 + 42 \times 2 + 222 \times 2 + 33 \times 11 + 2211 \times 11$
$4w_1 + w_2$	$42 \times 2 + 321 \times 2 + 321 \times 11 + 33 \times 11 + 3111 \times 2 + 2211 \times 11$
$3w_1 + w_3$	$42 \times 11 + 411 \times 2 + 33 \times 2 + 321 \times 2 + 321 \times 11$
$2w_1 + 2w_2$	$6 \times 2 + 51 \times 2 + 51 \times 11 + 2 \cdot (42 \times 2) + 411 \times 11 +$ $+33 \times 11 + 321 \times 2 + 321 \times 11 + 222 \times 2$
$w_1 + w_2 + w_3$	$51 \times 2 + 51 \times 11 + 42 \times 2 + 42 \times 11 + 411 \times 2 + 411 \times 11 + 321 \times 2$

The petite $\tilde{K}$-types containing δ_{36}:

petite $\tilde{K}$-type	**representation of $W(A_5 \times A_1)$**
$3w_2$	$411 \times 2 + 411 \times 11 + 311 \times 2$
$w_2 + w_4$	411×2
$2w_3$	$6 \times 2 + 51 \times 11 + 42 \times 2$

The relevant types for $W(A_5 \times A_1)$ are:

$$6 \times 2,\ 6 \times 11,\ 51 \times 2,\ 51 \times 11,\ 42 \times 2,\ 42 \times 11,\ 33 \times 2,\ 33 \times 11.$$

We can match all of them!

Conclusions

- If $\delta = \delta_1$ is the trivial representation of $\tilde{M}$, then $W^0_\delta = W(E_6)$. The relevant types for $W(E_6)$ are: 1_p, 6_p, 15_q, 20_p, 30_p. Our computations show that

  every relevant type for $W(E_6)$ is contained in the representation of $W(E_6)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_1})$, for some μ petite.

  Remark: 1_p, 6_p, 15_q and 20_p appear in (spherical) petite $\tilde{K}$-types of level less than or equal to 2, while 30_p appears only in petite $\tilde{K}$-types of level 3.

- If $\delta = \delta_{27}$, then $W^0_\delta = W(D_5)$. Our computations show that

  every relevant type for $W(D_5)$ is contained in the representation of $W(D_5)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_{27}})$, for some μ petite.

- If $\delta = \delta_{36}$, then $W^0_\delta = W(A_5 \times A_1)$. Our computations show that

  every relevant type for $W(A_5 \times A_1)$ is contained in the representation of $W(A_5 \times A_1)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_{36}})$, for some μ petite.

- If $\delta = \delta_8$ is the genuine representation of $\tilde{M}$, then $W^0_\delta = W(E_6)$. Our computations show that

  not every relevant type for $W(E_6)$ is contained in the representation of $W(E_6)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_8})$, for some μ petite.

  Indeed, only 1_p, 6_p, 15_q and 20_p have this property. A comparison with the spherical case shows that 1_p, 6_p, 15_q and 20_p are exactly the relevant $W(E_6)$-types that appear in spherical petite $\tilde{K}$-types of level less than or equal to 2.
  The "missing" relevant $W(E_6)$-type, 30_p, can only be found in non-petite genuine $\tilde{K}$-types. For completeness, we include the table of $W(E_6)$-representations on the δ_8-isotypic of genuine $\tilde{K}$-types of level 5/2.

The (non-petite) genuine $\tilde{K}$-types of level $5/2$:

$\tilde{K}$**-type**	**mult. of** δ_8	**repr. of** $W(E_6)$
$5w_1$	99	$15_q + 24_{p'} + 60_s$
$3w_1 + w_2$	216	$15_q + 60_s + 60_p + 81_p$
$2w_1 + w_3$	154	$\boxed{30_p} + 60_p + 64_p$
$w_1 + 2w_2$	189	$20_p + 24_p + 64_p + 81_p$
$w_1 + w_4$	36	$6_p + \boxed{30_p}$
$w_2 + w_3$	99	$15_p + 20_p + 64_p$

Chapter 2

Petite $\tilde{K}$-types for E_7

- $G = E_7$; $\tilde{K} = SU_8$; $K = \frac{SU(8)}{\{\pm I\}}$
- **Fundamental weights of** $\tilde{K}$: $\{w_j\}_{j=1\ldots7}$
- **Representations of** $\tilde{K}$: $\mu = \sum_{j=1}^{7} a_j w_j \quad a_j \geq 0$
- **Genuine** $\tilde{K}$**-types**: "$-I$ does not act trivially"

> ***A petite*** $\tilde{K}$***-type is either genuine of level*** $\leq 3/2$, ***or not genuine of level*** ≤ 3

◇ μ is petite if and only if it has level 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, or 3

◇ The level of $\mu = (\sum_{j=1}^{7} a_j w_j)$ is: $\frac{1}{2}(a_1+a_7)+(a_2+a_6)+\frac{3}{2}(a_3+a_5)+2a_4$.

Representations of $\tilde{M}$:

repr. of $\tilde{M}$	**dimension**	W^0_δ	**fine $\tilde{K}$-types**
δ_1	1	$W(E_7)$	(0)
$\boxed{\delta_8}$	8	$W(E_7)$	w_1
$\boxed{\delta_8^\star}$	8	$W(E_7)$	w_7
δ_{28}	$28 \cdot 1$	$W(E_6)$	$w_2,\ w_6$
δ_{36}	$36 \cdot 1$	$W(A_7)$	$2w_1,\ 2w_7$
δ_{63}	$63 \cdot 1$	$W(D_6 A_1)$	$w_1 + w_7$

Tensor products of representations of $\tilde{M}$:

$\otimes$	δ_8	$\delta_8^\star$	δ_{28}	δ_{36}	δ_{63}
δ_8	$\delta_{28} + \delta_{36}$	$\delta_1 + \delta_{63}$	$28\delta^\star$	$36\delta^\star$	$63\delta_8$
$\delta_8^\star$	$\delta_1 + \delta_{63}$	$\delta_{28} + \delta_{36}$	$28\delta_8$	$36\delta_8$	$63\delta^\star$
δ_{28}	$28\delta^\star$	$28\delta_8$	$28\delta_1 + 12\delta_{63}$	$16\delta_{63}$	$27\delta_{28} + 28\delta_{36}$
δ_{36}	$36\delta^\star$	$36\delta_8$	$16\delta_{63}$	$36\delta_1 + 20\delta_{63}$	$36\delta_{28} + 35\delta_{36}$
δ_{63}	$63\delta_8$	$63\delta^\star$	$27\delta_{28} + 28\delta_{36}$	$36\delta_{28} + 35\delta_{36}$	$63\delta_1 + 62\delta_{63}$

A complete list of petite $\tilde{K}$-types:

level	**petite $\tilde{K}$-type**	**mult. of δ_1**	**mult. of δ_{28}**	**mult. of δ_{36}**	**mult. of δ_{63}**	**mult. of δ_8**	**mult. of $\delta^\star$**
0	(0)	1	0	0	0	0	0
1/2	w_1	0	0	0	0	1	0
1/2	w_7	0	0	0	0	0	1
1	w_2	0	1	0	0	0	0
1	w_6	0	1	0	0	0	0
1	$2w_1$	0	0	1	0	0	0
1	$2w_7$	0	0	1	0	0	0
1	$w_1 + w_7$	0	0	0	1	0	0
3/2	$3w_1$	0	0	0	0	0	15
3/2	$3w_7$	0	0	0	0	15	0
3/2	$2w_1 + w_7$	0	0	0	0	35	0
3/2	$w_1 + 2w_7$	0	0	0	0	0	35
3/2	w_3	0	0	0	0	0	7
3/2	w_5	0	0	0	0	7	0
3/2	$w_2 + w_7$	0	0	0	0	27	0
3/2	$w_1 + w_6$	0	0	0	0	0	27
3/2	$w_1 + w_2$	0	0	0	0	0	21
3/2	$w_6 + w_7$	0	0	0	0	21	0

A complete list of petite $\tilde{K}$-types (continued)

level	petite $\tilde{K}$-type	mult. of δ_1	mult. of δ_{28}	mult. of δ_{36}	mult. of δ_{63}	mult. of δ_8	mult. of $\delta^\star$
2	w_4	7	0	0	1	0	0
2	$w_1 + w_5$	0	6	7	0	0	0
2	$w_3 + w_7$	0	6	7	0	0	0
2	$w_1 + w_3$	0	0	0	6	0	0
2	$w_5 + w_7$	0	0	0	6	0	0
2	$w_2 + w_6$	27	0	0	11	0	0
2	$4w_1$	15	0	0	5	0	0
2	$4w_7$	15	0	0	5	0	0
2	$2w_2$	21	0	0	5	0	0
2	$2w_6$	21	0	0	5	0	0
2	$2w_1 + 2w_7$	35	0	0	19	0	0
2	$w_1 + w_2 + w_7$	0	20	20	0	0	0
2	$w_1 + w_6 + w_7$	0	20	20	0	0	0
2	$2w_1 + w_6$	0	0	0	15	0	0
2	$w_2 + 2w_7$	0	0	0	15	0	0
2	$2w_1 + w_2$	0	0	0	10	0	0
2	$w_6 + 2w_7$	0	0	0	10	0	0
2	$w_1 + 3w_7$	0	15	14	0	0	0
2	$3w_1 + w_7$	0	15	14	0	0	0

A complete list of petite $\tilde{K}$-types (continued)

level	**petite $\tilde{K}$-type**	**mult. of δ_1**	**mult. of δ_{28}**	**mult. of δ_{36}**	**mult. of δ_{63}**	**mult. of δ_8**	**mult. of $\delta^\star$**
3	$2w_3$	0	15	21	0	0	0
3	$2w_5$	0	15	21	0	0	0
3	$w_3 + w_5$	21	0	0	37	0	0
3	$3w_1 + w_3$	0	75	70	0	0	0
3	$w_5 + 3w_7$	0	75	70	0	0	0
3	$3w_1 + w_5$	105	0	0	90	0	0
3	$w_3 + 3w_7$	105	0	0	90	0	0
3	$2w_1 + w_3 + w_7$	189	0	0	173	0	0
3	$w_1 + w_5 + 2w_7$	189	0	0	173	0	0
3	$w_1 + w_3 + 2w_7$	0	189	183	0	0	0
3	$2w_1 + w_5 + w_7$	0	189	183	0	0	0
3	$w_1 + w_5 + w_6$	105	0	0	115	0	0
3	$w_2 + w_3 + w_7$	105	0	0	115	0	0
3	$2w_1 + w_4$	0	30	35	0	0	0
3	$w_4 + 2w_7$	0	30	35	0	0	0
3	$w_1 + w_2 + w_5$	120	0	0	120	0	0
3	$w_3 + w_6 + w_7$	120	0	0	120	0	0
3	$w_1 + w_2 + w_3$	0	84	84	0	0	0
3	$w_5 + w_6 + w_7$	0	84	84	0	0	0

A complete list of petite $\tilde{K}$-types (continued)

level	**petite $\tilde{K}$-type**	**mult. of δ_1**	**mult. of δ_{28}**	**mult. of δ_{36}**	**mult. of δ_{63}**	**mult. of δ_8**	**mult. of $\delta^\star$**
3	$4w_1 + 2w_7$	0	165	175	0	0	0
3	$2w_1 + 4w_7$	0	165	175	0	0	0
3	$w_1 + w_3 + w_6$	0	135	140	0	0	0
3	$w_2 + w_5 + w_7$	0	135	140	0	0	0
3	$3w_2$	0	45	35	0	0	0
3	$3w_6$	0	45	35	0	0	0
3	$w_1 + w_4 + w_7$	56	0	0	56	0	0
3	$2w_2 + w_6$	0	135	120	0	0	0
3	$w_2 + 2w_6$	0	135	120	0	0	0
3	$w_2 + w_4$	0	27	21	0	0	0
3	$w_4 + w_6$	0	27	21	0	0	0
3	$6w_1$	0	24	29	0	0	0
3	$6w_7$	0	24	29	0	0	0
3	$2w_1 + 2w_6$	0	160	175	0	0	0
3	$2w_2 + 2w_7$	0	160	175	0	0	0
3	$2w_1 + 2w_2$	0	81	91	0	0	0
3	$2w_6 + 2w_7$	0	81	91	0	0	0
3	$w_1 + 2w_2 + w_7$	189	0	0	195	0	0
3	$w_1 + 2w_6 + w_7$	189	0	0	195	0	0

A complete list of petite $\tilde{K}$-types (continued)

level	petite $\tilde{K}$-type	mult. of δ_1	mult. of δ_{28}	mult. of δ_{36}	mult. of δ_{63}	mult. of δ_8	mult. of $\delta^\star$
3	$5w_1 + w_7$	84	0	0	94	0	0
3	$w_1 + 5w_7$	84	0	0	94	0	0
3	$2w_1 + w_2 + w_6$	0	235	245	0	0	0
3	$w_2 + w_6 + 2w_7$	0	235	245	0	0	0
3	$w_1 + w_2 + w_6 + w_7$	378	0	0	379	0	0
3	$3w_1 + 3w_7$	189	0	0	205	0	0
3	$4w_1 + w_2$	0	75	70	0	0	0
3	$w_6 + 4w_7$	0	75	70	0	0	0
3	$4w_1 + w_6$	0	135	126	0	0	0
3	$w_2 + 4w_7$	0	135	126	0	0	0
3	$3w_1 + w_2 + w_7$	216	0	0	216	0	0
3	$w_1 + w_6 + 3w_7$	216	0	0	216	0	0
3	$w_1 + w_2 + 3w_7$	280	0	0	280	0	0
3	$3w_1 + w_6 + w_7$	280	0	0	280	0	0
3	$2w_1 + w_2 + 2w_7$	0	324	315	0	0	0
3	$2w_1 + w_6 + 2w_7$	0	324	315	0	0	0

Spherical petite $\tilde{K}$-types:

petite $\tilde{K}$-type	mult. of δ_1	representation of $W(E_7)$
(0)	1	$\boxed{1_a}$
w_4	7	$\boxed{7_{a'}}$
$w_2 + w_6$	27	$\boxed{27_a}$
$4w_1$	15	$15_{a'}$
$4w_7$	15	$15_{a'}$
$2w_2$	21	$\boxed{21_{b'}}$
$2w_6$	21	$\boxed{21_{b'}}$
$2w_1 + 2w_7$	35	$\boxed{35_b}$
$w_3 + w_5$	21	21_a
$3w_1 + w_5$	105	$\boxed{105_b}$
$w_3 + 3w_7$	105	$\boxed{105_b}$
$2w_1 + w_3 + w_7$	189	$189_{b'}$
$w_1 + w_5 + 2w_7$	189	$189_{b'}$
$w_1 + w_5 + w_6$	105	$105_{a'}$
$w_2 + w_3 + w_7$	105	$105_{a'}$
$w_1 + w_2 + w_5$	120	120_a
$w_3 + w_6 + w_7$	120	120_a
$w_1 + w_4 + w_7$	56	$\boxed{56_{a'}}$

Spherical petite $\tilde{K}$-types (continued)

petite $\tilde{K}$-type	mult. of δ_1	representation of $W(E_7)$
$w_1 + 2w_2 + w_7$	189	$189_{c'}$
$w_1 + 2w_6 + w_7$	189	$189_{c'}$
$5w_1 + w_7$	84	$84_{a'}$
$w_1 + 5w_7$	84	$84_{a'}$
$w_1 + w_2 + w_6 + w_7$	378	$210_a + 168_a$
$3w_1 + 3w_7$	189	$84_a + 105_c$
$3w_1 + w_2 + w_7$	216	$216_{a'}$
$w_1 + w_6 + 3w_7$	216	$216_{a'}$
$w_1 + w_2 + 3w_7$	280	280_b
$3w_1 + w_6 + w_7$	280	280_b

The relevant types for $W(E_7)$ are: 1_a, $7_{a'}$, 27_a, $56_{a'}$, $21_{b'}$, 35_b, 105_b .

Petite $\tilde{K}$-types containing δ_8:

petite $\tilde{K}$-type	**mult. of δ_8**	**representation of $W(E_7)$**
w_1	1	$\boxed{1_a}$
$3w_7$	15	$15_{a'}$
$2w_1 + w_7$	35	$\boxed{35_b}$
w_5	7	$\boxed{7_{a'}}$
$w_2 + w_7$	27	$\boxed{27_a}$
$w_6 + w_7$	21	$\boxed{21_{b'}}$

The relevant types for $W(E_7)$ are: 1_a, $7_{a'}$, 27_a, $56_{a'}$, $21_{b'}$, 35_b, 105_b .

We cannot match $56_{a'}$ *and* 105_b. Indeed, these representations of $W(E_7)$ make their first appearance in (non-genuine) $\tilde{K}$-types of level $5/2$.[1]

The situation for $\delta_8^\star$ is an analogous, just take the dual $\tilde{K}$-types.

[1]We have: $w_1 + w_4 \to 7_{a'} + 56_{a'}$, and $w_3 + 2w_7 \to 120_a + 105_b$.

Petite $\tilde{K}$-types containing δ_{28}:

petite $\tilde{K}$-type	mult. of δ_{28}	representation of $W(E_6)$
w_2	1	$\boxed{1_p}$
$w_1 + w_2 + w_7$	20	$\boxed{20_p}$
$w_3 + w_7$	6	$\boxed{6_p}$
$3w_1 + w_7$	15	$\boxed{15_q}$
$2w_1 + w_4$	30	$\boxed{30_p}$
$\dots$	$\dots$	$\dots$

The relevant types for $W(E_6)$ are: 1_p, 6_p, 20_p, 30_p, 15_q .

We can match all of them!

Petite $\tilde{K}$-types containing δ_{36}:

petite $\tilde{K}$-type	mult. of δ_{36}	representation of $W(A_7)$
$2w_1$	1	$\boxed{(8)}$
$w_1 + w_2 + w_7$	20	$\boxed{(62)}$
$w_3 + w_7$	7	$\boxed{(71)}$
$3w_1 + w_7$	14	$\boxed{(44)}$
$2w_1 + w_4$	35	$\boxed{(53)} + (71)$
...	...	...

The relevant types for $W(A_7)$ are: (8), (71), (62), (53), (44).

We can match all of them!

Petite $\tilde{K}$-types containing δ_{63}:

petite $\tilde{K}$-type	mult. of δ_{63}	representation of $W(D_6 \times A_1)$
$w_1 + w_7$	1	$\boxed{(6\|0) \times 2}$
$w_1 + w_3$	6	$\boxed{(5\|1) \times 2}$
$2w_1 + w_2$	10	$\boxed{(3\|3)^- \times 2}$
$w_2 + 2w_7$	15	$\boxed{(4\|2) \times 2}$
w_4	1	$\boxed{(6\|0) \times 11}$
$2w_2$	5	$\boxed{(51\|0) \times 11}$
$w_2 + w_6$	11	$\boxed{(5\|1) \times 11} + \boxed{(51\|0) \times 2}$
$2w_1 + 2w_7$	19	$\boxed{(3\|3)^+ \times 11} + \boxed{(42\|0) \times 2}$
$4w_1$	5	$\boxed{(33\|0) \times 11}$
$3w_1 + w_5$	90	$\boxed{(33\|0) \times 2} + (32\|1) \times 11 + (31\|2) \times 2 + \boxed{(3\|3)^- \times 11}$
$2w_1 + w_3 + w_7$	173	$\boxed{(42\|0) \times 11} + \boxed{(3\|3)^+ \times 2} + \boxed{(4\|2) \times 11} + \ldots\ldots$
$\cdots$	$\cdots$	$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$

The relevant types for $W(D_6 \times A_1)$ are:

$$\begin{array}{llll} (6|0) \times 2 & (51|0) \times 2 & (42|0) \times 2 & (33|0) \times 2 \\ (5|1) \times 2 & (4|2) \times 2 & (3|3)^{+} \times 2 & (3|3)^{-} \times 2 \\ (6|0) \times 11 & (51|0) \times 11 & (42|0) \times 11 & (33|0) \times 11 \\ (5|1) \times 11 & (4|2) \times 11 & (3|3)^{+} \times 11 & (3|3)^{-} \times 11. \end{array}$$

We can match all of them!

Conclusions

- If $\delta = \delta_1$ is the trivial representation of $\tilde{M}$, then $W^0_\delta = W(E_7)$. The relevant types for $W(E_7)$ are: 1_a, $7_{a'}$, 27_a, $56_{a'}$, $21_{b'}$, 35_b, and 105_b. Our computations show that

  every relevant type for $W(E_7)$ is contained in the representation of $W(E_7)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_1})$, for some μ petite.

  Remark: 1_a, $7_{a'}$, 27_a, $21_{b'}$ and 35_b appear in (spherical) petite $\tilde{K}$-types of level less than or equal to 2, while $56_{b'}$ and 105_b appear only in petite $\tilde{K}$-types of level 3.

- If $\delta = \delta_{28}$, then $W^0_\delta = W(E_6)$. Our computations show that

  every relevant type for $W(E_6)$ is contained in the representation of $W(E_6)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_{28}})$, for some μ petite.

- If $\delta = \delta_{36}$, then $W^0_\delta = W(A_7)$. Our computations show that

  every relevant type for $W(A_7)$ is contained in the representation of $W(A_7)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_{36}})$, for some μ petite.

- If $\delta = \delta_{63}$, then $W^0_\delta = W(D_6 \times A_1)$. Our computations show that

  every relevant type for $W(D_6 \times A_1)$ is contained in the representation of $W(D_6 \times A_1)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_{63}})$, for some μ petite.

- If $\delta = \delta_8$ (or $\delta = \delta_8^\star$) is the genuine representation of $\tilde{M}$, then $W^0_\delta = W(E_7)$. Our computations show that

  not every relevant type for $W(E_7)$ is contained in the representation of $W(E_7)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_8})$ (or $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_8^\star})$), for some μ petite.

  Indeed, only 1_a, $7_{a'}$, 27_a, $21_{b'}$ and 35_b have this property. A comparison with the spherical case shows that 1_a, $7_{a'}$, 27_a, $21_{b'}$ and 35_b are exactly the relevant $W(E_7)$-types that appear in spherical petite $\tilde{K}$-types of level less than or equal to 2.
  The "missing" relevant $W(E_7)$-types, $56_{a'}$ and 105_b, can only be found in non-petite genuine $\tilde{K}$-types.

  For completeness, we include the table of the representations of $W(E_7)$ on the δ_8-isotypic of genuine $\tilde{K}$-types of level 5/2.

$\tilde{K}$-type of level $5/2$	mult. of δ_8	representation of $W(E_7)$
$w_1 + w_4$	63	$\boxed{56_{a'}} + 7_{a'}$
$w_1 + 2w_2$	210	$21_{b'} + 189_{c'}$
$2w_1 + w_3$	189	$189_{b'}$
$w_2 + w_3$	126	$21_{b'} + 105_{a'}$
$w_1 + w_2 + w_6$	525	$27_a + 120_a + 168_a + 210_a$
$3w_1 + w_2$	231	$15_{a'} + 216_{a'}$
$w_1 + w_2 + 2w_7$	693	$35_b + 168_a + 210_a + 280_b$
$w_1 + w_6 + 2w_7$	594	$189_{b'} + 189_{c'} + 216_{a'}$
$w_1 + 2w_6$	315	$21_{b'} + 105_{a'} + 189_{c'}$
$w_1 + w_5 + w_7$	350	$\boxed{56_{a'}} + 105_{a'} + 189_{b'}$
$w_3 + w_6$	168	$21_a + 27_a + 120_a$
$w_3 + 2w_7$	225	$\boxed{105_b} + 120_a$
$3w_1 + w_6$	385	$\boxed{105_b} + 280_b$
$5w_1$	99	$15_{a'} + 84_{a'}$
$3w_1 + 2w_7$	504	$35_b + 84_a + 105_c + 280_b$
$w_1 + 4w_7$	315	$15_{a'} + 84_{a'} + 216_{a'}$

Chapter 3

Petite $\tilde{K}$-types for E_8

- $G = E_8$; $\tilde{K} = Spin(16)$; $K = \frac{Spin(16)}{\{I,\omega\}}$
- **Fundamental weights of** $\tilde{K}$: $\{w_j\}_{j=1\dots8}$
- **Representations of** $\tilde{K}$: $\mu = \sum_{j=1}^{8} a_j w_j \quad a_j \geq 0$
- **Genuine** $\tilde{K}$**-types**: "ω does not act trivially"

A petite $\tilde{K}$-type is either genuine of level $\leq 3/2$ or not genuine of level ≤ 3

◇ The level of $\mu = (\sum_{j=1}^{8} a_j w_j)$ is:

$$\frac{1}{2}(a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_6 + 6a_6 + 3a_7 + 4a_8)$$

◇ μ is petite if and only if it has level 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, or 3

Representations of $\tilde{M}$:

repr. of $\tilde{M}$	**dimension**	W_δ^0	**fine K-type**
δ_1	1	$W(E_8)$	0
$\boxed{\delta_{16}}$	16	$W(E_8)$	w_1
δ_{120}	$120 \cdot 1$	$W(E_7 \times A_1)$	w_2
δ_{135}	$135 \cdot 1$	$W(D_8)$	$2w_1$

Tensor products of representations of $\tilde{M}$:

$\otimes$	δ_{16}	δ_{120}	δ_{135}
δ_{16}	$\delta_1 + \delta_{120} + \delta_{135}$	$120\delta_{16}$	$135\delta_{16}$
δ_{120}	$120\delta_{16}$	$120\delta_1 + 56\delta_{120} + 56\delta_{135}$	$63\delta_{120} + 64\delta_{135}$
δ_{135}	$135\,\delta_{16}$	$63\delta_{120} + 64\delta_{135}$	$135\delta_1 + 72\delta_{120} + 70\delta_{135}$

non-genuine petite K-type	**level**	**$\tilde{M}$-decomposition**
0	0	δ_1
w_2	1	δ_{120}
$2w_1$	1	δ_{135}
$2w_2$	2	$84\delta_1 + 21\delta_{120} + 20\delta_{135}$
$w_1 + w_3$	2	$27\delta_{120} + 28\delta_{135}$
w_4	2	$35\delta_1 + 7\delta_{120} + 7\delta_{135}$
$2w_1 + w_2$	2	$35\delta_{120} + 35\delta_{135}$
$4w_1$	2	$50\delta_1 + 15\delta_{120} + 14\delta_{135}$
$3w_2$	3	$525\delta_1 + 525\delta_{120} + 490\delta_{135}$
$w_1 + w_2 + w_3$	3	$1344\delta_1 + 1344\delta_{120} + 1344\delta_{135}$
$w_2 + w_4$	3	$567\delta_1 + 567\delta_{120} + 539\delta_{135}$
$2w_1 + w_4$	3	$700\delta_1 + 680\delta_{120} + 700\delta_{135}$
$2w_3$	3	$300\delta_1 + 336\delta_{120} + 364\delta_{135}$
$w_1 + w_5$	3	$210\delta_1 + 231\delta_{120} + 238\delta_{135}$

non-genuine petite K-type	**level**	**$\tilde{M}$-decomposition**
$3w_1 + w_3$	3	$1050\delta_1 + 1008\delta_{120} + 987\delta_{135}$
w_6	3	$28\delta_1 + 35\delta_{120} + 28\delta_{135}$
$2w_1 + 2w_2$	3	$972\delta_1 + 1000\delta_{120} + 1036\delta_{135}$
$4w_1 + w_2$	3	$700\delta_1 + 714\delta_{120} + 700\delta_{135}$
$6w_1$	3	$168\delta_1 + 189\delta_{120} + 204\delta_{135}$
w_8	2	$8\delta_1 + \delta_{120}$
$2w_1 + w_8$	3	$56\delta_{120} + 64\delta_{135}$
$w_1 + w_7$	2	$7\delta_{120} + 8\delta_{135}$
$w_2 + w_8$	3	$112\delta_1 + 56\delta_{120} + 48\delta_{135}$
$w_1 + w_2 + w_7$	3	$560\delta_1 + 553\delta_{120} + 552\delta_{135}$
$w_3 + w_7$	3	$160\delta_1 + 216\delta_{120} + 224\delta_{135}$
$3w_1 + w_7$	3	$400\delta_1 + 344\delta_{120} + 336\delta_{135}$
$2w_7$	3	$21\delta_{120} + 29\delta_{135}$

genuine petite K-type	**level**	**$\tilde{M}$-decomposition**
w_1	$1/2$	δ_{16}
w_7	$3/2$	$8\delta_{16}$
w_3	$3/2$	$35\delta_{16}$
$3w_1$	$3/2$	$50\delta_{16}$
$w_1 + w_2$	$3/2$	$84\delta_{16}$

We have listed *all* the petite $\tilde{K}$-types.

The petite $\tilde{K}$-types containing δ_1:

spherical petite K-type	mult. of δ_1	representation of $W(E_8)$
0	1	$\boxed{1_x}$
$2w_2$	84	$\boxed{84_x}$
w_4	35	$\boxed{35_x}$
$4w_1$	50	$\boxed{50_x}$
$3w_2$	525	525_x
$w_1 + w_2 + w_3$	1344	1344_x
$w_2 + w_4$	567	567_x
$2w_1 + w_4$	700	700_x
$2w_3$	300	$\boxed{300_x}$
$w_1 + w_5$	210	$\boxed{210_x}$
$3w_1 + w_3$	1050	1050_x
w_6	28	28_x

The petite $\tilde{K}$-types containing δ_1 (continued)

spherical petite K-type	**mult. of δ_1**	**representation of $W(E_8)$**
$2w_1 + 2w_2$	972	972_x
$4w_1 + w_2$	700	700_{xx}
$6w_1$	168	168_y
w_8	8	$\boxed{8_z}$
$w_2 + w_8$	112	$\boxed{112_z}$
$w_1 + w_2 + w_7$	560	560_z
$w_3 + w_7$	160	160_z
$3w_1 + w_7$	400	$\boxed{400_z}$

The relevant types for $W(E_8)$ are:

$$1_x,\, 8_z,\, 35_x,\, 50_x,\, 84_x,\, 112_z,\, 400_z,\, 300_x,\, 210_x\,.$$

We can match all of them!

The (genuine) petite$\tilde{K}$-types containing δ_{16}:

petite K-type	mult. of δ_{16}	repr. of $W(E_8)$
w_1	1	$\boxed{1_x}$
w_7	8	$\boxed{8_z}$
w_3	35	$\boxed{35_x}$
$3w_1$	50	$\boxed{50_x}$
$w_1 + w_2$	84	$\boxed{84_x}$

The relevant types for $W(E_8)$ are:

$$1_x,\ 8_z,\ 35_x,\ 50_x,\ 84_x,\ 112_z,\ 400_z,\ 300_x,\ 210_x$$

We cannot match 112_z, 400_z, 300_x, 210_x . Indeed, these representations make their first appearance in (non-petite) $\tilde{K}$-types of level 5/2:

$$\begin{array}{lcl} 2w_1 + w_7 & \rightarrow & 400_z + 560_z \\ w_2 + w_3 & \rightarrow & 300_x + 84_x + 567_x + 1344_x \\ w_1 + w_8 & \rightarrow & 8_z + 112_z \\ w_5 & \rightarrow & 210_x + 28_x + 35_x \,. \end{array}$$

Remark In the p-adic case, if a spherical principal series $X_P(triv. \otimes \nu)$ is not unitary, then there is at least one relevant W-type τ such that the intertwining operator on τ is not unitary.
When τ is equal to 1_x, 8_z, 35_x, 50_x or 84_x, we can conclude that the *real* genuine principal series $X_P(\delta_{16} \otimes \nu)$ is not unitary.
We do not get the same conclusion if the unitarity of $X_P(triv. \otimes \nu)$ is ruled out by 400_z, 300_x or 210_x, indeed the *real* series $X_P(\delta_{16} \otimes \nu)$ could still be unitary.[1]

[1] This case can never happen if $X_P(triv. \otimes \nu)$ is irreducible. Indeed the unitarity of a generic principal series can be detected using only 1_x, 8_z and 35_x.

The petite $\tilde{K}$-types containing δ_{120}:

petite K-type	mult. of δ_{120}	representation of $W(E_7 \times A_1)$
w_2	1	$\boxed{1_a \times 2}$
$2w_2$	21	$\boxed{21_{b'} \times 11}$
$w_1 + w_3$	27	$\boxed{27_a \times 2}$
w_4	7	$\boxed{7_{a'} \times 11}$
$2w_1 + w_2$	35	$\boxed{35_b \times 2}$
$4w_1$	15	$15_{a'} \times 11$
$3w_2$	525	$1_a \times 2 + 21_{b'} \times 11 + 27_a \times 2 + 35_b \times 2+$ $+189_{c'} \times 11 + 84_a \times 2 + 168_a \times 2$
$w_1 + w_2 + w_3$	1344	$35_b \times 2 + 27_a \times 2 + 105_{a'} \times 11 + +168_a \times 2+$ $+280_b \times 2 + 21_{b'} \times 11 + 189_{b'} \times 11+$ $189_{c'} \times 11 + 210_a \times 2 + 120_a \times 2$
$w_2 + w_4$	567	$2 \cdot (27_a \times 2) + 35_b \times 2 + 1_a \times 2 + 56_{a'} \times 11+$ $+120_a \times 2 + 168_a \times 2 + 7_{a'} \times 11+$ $+21_{b'} \times 11 + 105_{a'} \times 11$
$2w_1 + w_4$	680	$56_{a'} \times 11 + \boxed{105_b \times 2} + 120_a \times 2+$ $+189_{b'} \times 11 + 210_a \times 2$

The petite $\tilde{K}$-types containing δ_{120} (continued)

petite K-type	mult. of δ_{120}	representation of $W(E_7 \times A_1)$
$w_1 + w_5$	231	$27_a \times 2 + 7_{a'} \times 11 + \boxed{56_{a'} \times 11} + +120_a \times 2 + 21_a \times 2$
$3w_1 + w_3$	1008	$15_{a'} \times 11 + 35_b \times 2 + 105_b \times 2 + 280_b \times 2+ +189_{b'} \times 11 + 216_{a'} \times 11 + 168_a \times 2$
w_6	35	$7_{a'} \times 11 + 1_a \times 2 + 27_a \times 2$
$2w_1 + 2w_2$	1000	$105_c \times 2 + 189_{c'} \times 11 + 210_a \times 2+ +216_{a'} \times 11 + 280_b \times 2$
$4w_1 + w_2$	714	$35_b \times 2 + 15_{a'} \times 11 + 84_{a'} \times 11+ +216_{a'} \times 11 + 280_b \times 2 + 84_a \times 2$
$6w_1$	189	$84_{a'} \times 11 + 105_c \times 2$
w_8	1	$\boxed{1_a \times 11}$
$2w_1 + w_8$	56	$\boxed{56_{a'} \times 2}$
$w_1 + w_7$	7	$7_{a'} \times 2$
$w_2 + w_8$	56	$\boxed{21_{b'} \times 2} + 27_a \times 11 + \boxed{7_{a'} \times 2} + 1_a \times 11$
$w_1 + w_2 + w_7$	553	$\boxed{35_b \times 11} + 56_{a'} \times 2 + 189_{b'} \times 2+ +21_{b'} \times 2 + \boxed{27_a \times 11} + 105_{a'} \times 2$

The petite $\tilde{K}$-types containing δ_{120} (continued)

petite K-type	mult. of δ_{120}	representation of $W(E_7 \times A_1)$
$w_3 + w_7$	216	$27_a \times 11 + 56_{a'} \times 2 + 105_{a'} \times 2 +$ $+ 7_{a'} \times 2 + 21_a \times 11$
$3w_1 + w_7$	344	$15_{a'} \times 2 + 35_b \times 11 + \boxed{105_b \times 11} + 189_{b'} \times 2$
$2w_7$	21	$21_a \times 2$
$2w_3$	336	$105_{a'} \times 11 + 21_a \times 2 + 210_a \times 2$

The relevant types for $W(E_7 \times A_1)$ are:

$$\begin{array}{ccccccc} 1_a \times 2 & 7_{a'} \times 2 & 27_a \times 2 & 56_{a'} \times 2 & 21_{b'} \times 2 & 35_b \times 2 & 105_b \times 2 \\ 1_a \times 11 & 7_{a'} \times 11 & 27_a \times 11 & 56_{a'} \times 11 & 21_{b'} \times 11 & 35_b \times 11 & 105_b \times 11. \end{array}$$

We can match all of them!

The petite $\tilde{K}$-types containing δ_{135}:

petite K-type	**mult. of δ_{135}**	**representation of $W(D_8)$**
$2w_1$	1	$\boxed{(0\|8)}$
$2w_2$	20	$\boxed{(0\|62)}$
$w_1 + w_3$	28	$\boxed{(2\|6)}$
w_4	7	$\boxed{(0\|71)}$
$2w_1 + w_2$	35	$\boxed{(4\|4)^-}$
$4w_1$	14	$\boxed{(0\|44)}$
$3w_2$	490	$(31\|31)^- + (11\|51) + (0\|511)$
$w_1 + w_2 + w_3$	1344	$(2\|6) + (0\|62) + (4\|4)^- + (22\|4) + (31\|4) + (0\|521)+$ $+(2\|42) + (31\|31)^- + (11\|51) + (2\|51)$
$w_2 + w_4$	539	$(31\|4) + (2\|51) + (0\|611) + (11\|51) + (11\|6)$
$2w_1 + w_4$	700	$\boxed{(0\|53)} + \boxed{(4\|4)^+} + (0\|71) + (2\|6) + (2\|51) + (31\|4) + (2\|42)$
$2w_3$	364	$(2\|6) + (0\|62) + (0\|8) + (4\|4)^- + (22\|4) + (2\|51)$
$w_1 + w_5$	238	$(2\|51) + (4\|4)^+ + (11\|6) + (0\|71) + (2\|6)$

The petite $\tilde{K}$-types containing δ_{135} (continued)

petite K-type	mult. of δ_{135}	representation of $W(D_8)$
$3w_1 + w_3$	987	$(0\|431) + (31\|4) + (2\|42) + (31\|31)^- + (11\|33)$
w_6	28	$(11\|6)$
$2w_1 + 2w_2$	1036	$(2\|6) + (0\|62) + (0\|8) + 2 \cdot (4\|4)^- + (31\|31)^- +$ $+(22\|22)^- + (22\|4) + (2\|42) + (0\|422) + (0\|44)$
$4w_1 + w_2$	700	$(0\|44) + (4\|4)^- + (22\|22)^- + (11\|33) + (31\|31)^- + (0\|3311)$
$6w_1$	204	$(22\|22)^- + (0\|2222) + (0\|44) + (4\|4)^- + (0\|8)$
$2w_1 + w_8$	64	$(1\|7) + \boxed{(3\|5)}$
$w_1 + w_7$	8	$\boxed{(1\|7)}$
$w_2 + w_8$	48	$(1\|61)$
$w_1 + w_2 + w_7$	552	$(3\|41) + (3\|5) + (21\|5) + (1\|52) + (1\|61)$
$w_3 + w_7$	224	$(3\|5) + (21\|5) + (1\|61) + (1\|7)$
$3w_1 + w_7$	336	$(1\|43) + (3\|41)$
$2w_7$	29	$(2\|6) + (0\|8)$

The relevant types for $W(D_8)$ are:

$$\begin{array}{lllll} (0|8) & (0|71) & (0|62) & (0|53) & (0|44) \\ (1|7) & (2|6) & (3|5) & (4|4)^+ & (4|4)^- \end{array}$$

We can match all of them!

Conclusions

- If $\delta = \delta_1$ is the trivial representation of $\tilde{M}$, then $W_\delta^0 = W(E_8)$. The relevant types for $W(E_8)$ are: 1_x, 8_z, 35_x, 50_x, 84_x, 112_z, 400_z, 300_x, 210_x. Our computations show that

  every relevant type for $W(E_8)$ is contained in the representation of $W(E_8)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_1})$, for some μ petite.

  Remark: 1_x, 8_z, 35_x, 50_x and 84_x appear in (spherical) petite $\tilde{K}$-types of level less than or equal to 2, while 112_z, 400_z, 300_x and 210_x appear only in petite $\tilde{K}$-types of level 3.

- If $\delta = \delta_{120}$, then $W_\delta^0 = W(E_7 \times A_1)$. Our computations show that

  every relevant type for $W(E_7 \times A_1)$ is contained in the representation of $W(E_7 \times A_1)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_{120}})$, for some μ petite.

- If $\delta = \delta_{135}$, then $W_\delta^0 = W(D_8)$. Our computations show that

  every relevant type for $W(D_8)$ is contained in the representation of $W(D_8)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_{135}})$, for some μ petite.

- If $\delta = \delta_{16}$ is the genuine representation of $\tilde{M}$, then $W_\delta^0 = W(E_8)$. Our computations show that

  not every relevant type for $W(E_8)$ is contained in the representation of $W(E_8)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_{16}})$, for some μ petite.

  Indeed, only 1_x, 8_z, 35_x, 50_x and 84_x have this property. A comparison with the spherical case shows that 1_x, 8_z, 35_x, 50_x and 84_x are exactly the relevant $W(E_8)$-types that appear in spherical petite $\tilde{K}$-types of level less than or equal to 2.
  The "missing" relevant $W(E_8)$-types, 112_z, 400_z, 300_x and 210_x, can only be found in non-petite genuine $\tilde{K}$-types.

  For completeness, we include the table of the representations of $W(E_8)$ on the δ_{16}-isotypic of genuine $\tilde{K}$-types of level 5/2.

The (non-petite) genuine $\tilde{K}$-types of level 5/2:

$\tilde{K}$-type	mult. of δ_{16}	repr. of $W(E_8)$
w_5	273	$\boxed{210_x} + 28_x + 35_x$
$w_1 + 2w_2$	2925	$84_x + 525_x + 972_x + 1344_x$
$2w_1 + w_3$	3094	$700_x + 1050_x + 1344_x$
$w_2 + w_3$	2295	$84_x + \boxed{300_x} + 567_x + 1344_x$
$w_1 + w_4$	1512	$35_x + \boxed{210_x} + 567_x + 700_x$
$3w_1 + w_2$	2772	$50_x + 700_{xx} + 972_x + 1050_x$
$5w_1$	918	$50_x + 168_y + 700_{xx}$
$2w_1 + w_7$	960	$\boxed{400_z} + 560_z$
$w_1 + w_8$	120	$8_z + \boxed{112_z}$
$w_2 + w_7$	832	$\boxed{112_z} + 160_z + 560_z$

Chapter 4

Petite $\tilde{K}$-types for F_4

- $G = F_4$; $\tilde{K} = SP_1 \times SP(3)$; $K = \frac{SP_1 \times SP(3)}{\{\pm I\}}$
- **Representations of** $\tilde{K}$:[1] $\mu = \sum_{j=1}^{4} a_j \epsilon_j \quad a_1 \geq 0;\ a_2 \geq a_3 \geq a_4 \geq 0$
- **Genuine** $\tilde{K}$**-types**: "$-I$ does not act trivially"

A petite $\tilde{K}$-type is either genuine of level $\leq 3/2$ or not genuine of level ≤ 3

◇ If $(\sum_{j=1}^{4} a_j)$ is odd, then μ is genuine, and you require:

$$a_1 + a_2 + a_3 + a_4 \leq 5 \quad \text{and} \quad a_2 + a_3 \leq 2$$

◇ If $(\sum_{j=1}^{4} a_j)$ is even, then μ is not genuine, and you require:

$$a_1 + a_2 + a_3 + a_4 \leq 6 \quad \text{and} \quad a_2 + a_3 \leq 3.$$

[1] Classification by highest weight.

Representations of $\tilde{M}$:

repr.of $\tilde{M}$	**dimension**	W^0_δ	**fine $\tilde{K}$-type**
δ_0	1	$W(F_4)$	$(0\|0,0,0)$
$\boxed{\delta_2}$	2	$W(F_4)$	$(1\|0,0,0)$
δ_3	$3 \cdot 1$	$W(C_4)$	$(2\|0,0,0)$
$\boxed{\delta_6}$	$3 \cdot 2$	$W(B_4)$	$(0\|1,0,0)$
δ_{12}	$12 \cdot 1$	$W(B_3A_1)$	$(1\|1,0,0)$

Tensor products of representationsof $\tilde{M}$:

$\otimes$	δ_2	δ_3	δ_6	δ_{12}
δ_2	$\delta_0 + \delta_3$	$3\delta_2$	δ_{12}	$4\delta_6$
δ_3	$3\delta_2$	$3\delta_0 + 2\delta_3$	$3\delta_6$	$3\delta_{12}$
δ_6	δ_{12}	$3\delta_6$	$3\delta_0 + 3\delta_3 + 2\delta_{12}$	$12\delta_2 + 8\delta_6$
δ_{12}	$4\,\delta_6$	$3\delta_{12}$	$12\delta_2 + 8\delta_6$	$12\delta_0 + 12\delta_3 + 8\delta_{12}$

A complete list of petite $\tilde{K}$-types:

petite $\tilde{K}$-type	$\tilde{M}$-decomposition
$(0\|0,0,0)$	δ_0
$(1\|0,0,0)$	δ_2
$(2\|0,0,0)$	δ_3
$(0\|1,0,0)$	δ_6
$(1\|1,0,0)$	δ_{12}
$(3\|0,0,0)$	$2\delta_2$
$(2\|1,0,0)$	$3\delta_6$
$(4\|0,0,0)$	$2\delta_0 + \delta_3$
$(3\|1,0,0)$	$2\delta_{12}$
$(0\|2,0,0)$	$3\delta_3 + \delta_{12}$
$(0\|1,1,0)$	$2\delta_0 + \delta_{12}$
$(1\|2,0,0)$	$9\delta_2 + 4\delta_6$
$(1\|1,1,0)$	$2\delta_2 + 4\delta_6$
$(2\|2,0,0)$	$9\delta_0 + 6\delta_3 + 3\delta_{12}$
$(2\|1,1,0)$	$2\delta_3 + 3\delta_{12}$
$(4\|1,0,0)$	$5\delta_6$
$(5\|1,0,0)$	$3\delta_{12}$
$(3\|2,0,0)$	$18\delta_2 + 8\delta_6$
$(3\|1,1,0)$	$4\delta_2 + 8\delta_6$

A complete list of petite $\tilde{K}$-types (continued)

petite $\tilde{K}$-type	**$\tilde{M}$-decomposition**
$(4\|2,0,0)$	$9\delta_0 + 12\delta_3 + 5\delta_{12}$
$(4\|1,1,0)$	$4\delta_0 + 2\delta_3 + 5\delta_{12}$
$(1\|3,0,0)$	$4\delta_0 + 4\delta_3 + 8\delta_{12}$
$(1\|2,1,0)$	$8\delta_0 + 8\delta_3 + 8\delta_{12}$
$(0\|1,1,1)$	$4\delta_2 + \delta_6$
$(5\|0,0,0)$	$3\delta_2$
$(6\|0,0,0)$	$\delta_0 + 2\delta_3$
$(1\|1,1,1)$	$4\delta_0 + 4\delta_3 + \delta_{12}$
$(2\|1,1,1)$	$12\delta_2 + 3\delta_6$
$(3\|1,1,1)$	$8\delta_0 + 8\delta_3 + 2\delta_{12}$
$(3\|2,1,0)$	$16\delta_0 + 16\delta_3 + 16\delta_{12}$
$(2\|2,1,1)$	$9\delta_0 + 7\delta_3 + 15\delta_{12}$
$(0\|2,1,1)$	$\delta_0 + 3\delta_3 + 5\delta_{12}$
$(3\|3,0,0)$	$8\delta_0 + 8\delta_3 + 16\delta_{12}$

The petite $\tilde{K}$-types containing δ_0:

petite $\tilde{K}$-type	**mult. of δ_0**	**representation of $W(F_4)$**
$(0\|0,0,0)$	1	$\boxed{1_1}$
$(4\|0,0,0)$	2	$\boxed{2_3}$
$(0\|1,1,0)$	2	2_1
$(2\|2,0,0)$	9	$\boxed{9_1}$
$(4\|1,1,0)$	4	4_1
$(1\|3,0,0)$	4	4_3
$(1\|2,1,0)$	8	$\boxed{8_1}$
$(6\|0,0,0)$	1	1_3
$(1\|1,1,1)$	4	$\boxed{4_2}$
$(3\|1,1,1)$	8	8_3
$(3\|2,1,0)$	16	16
$(2\|2,1,1)$	9	9_2
$(0\|2,1,1)$	1	1_2
$(3\|3,0,0)$	8	8_4
$(4\|2,0,0)$	9	9_3

The relevant types for $W(F_4)$ are: 1_1, 2_3, 4_2, 8_1, 9_1 .

We can match all of them!

The petite $\tilde{K}$-types containing δ_2:

petite $\tilde{K}$-type	mult. of δ_2	representation of $W(F_4)$
$(1\|0,0,0)$	1	$\boxed{1_1}$
$(3\|0,0,0)$	2	$\boxed{2_3}$
$(1\|2,0,0)$	9	$\boxed{9_1}$
$(1\|1,1,0)$	2	2_1
$(3\|2,0,0)$	18	$9_1 + 9_3$
$(3\|1,1,0)$	4	4_1
$(0\|1,1,1)$	4	$\boxed{4_2}$
$(5\|0,0,0)$	3	$1_3 + 2_3$
$(2\|1,1,1)$	12	$8_3 + 4_2$

The relevant types for $W(F_4)$ are: 1_1, 2_3, 4_2, 8_1, 9_1 .

We cannot match 8_1. Indeed, this W-representation makes its first appearance in the non-genuine $\tilde{K}$-type $(0|2,1,0)$, of level $5/2$.[2]

[2] $(0|2,1,0)$ contains exactly 8 copies of δ_2, and W acts on the δ_2 isotypic by 8_1.

The petite $\tilde{K}$-types containing δ_6:

petite $\tilde{K}$-type	mult. of δ_6	representation of $W(B_4)$
$(0\|1,0,0)$	1	$\boxed{(4\|0)}$
$(2\|1,0,0)$	3	$\boxed{(31\|0)}$
$(1\|2,0,0)$	4	$\boxed{(1\|3)}$
$(1\|1,1,0)$	4	$\boxed{(3\|1)}$
$(4\|1,0,0)$	5	$\boxed{(22\|0)} + (211\|0)$
$(3\|2,0,0)$	8	$(1\|21)$
$(3\|1,1,0)$	8	$(21\|1)$
$(0\|1,1,1)$	1	$\boxed{(0\|4)}$
$(2\|1,1,1)$	3	$(0\|31)$

The relevant types for $W(B_4)$ are: $(4|0)$, $(31|0)$, $(22|0)$, $(3|1)$, $(2|2)$, $(1|3)$, $(0|4)$.

We cannot match $(2|2)$. Indeed, this representation of $W(B_4)$ makes its first appearance in the non-genuine $\tilde{K}$-types $(0|3,0,0)$ and $(0|2,1,0)$, of level $5/2$.[3]

[3]We have: $(0|3,0,0) \to (22|0) + (2|2)$, and $(0|2,1,0) \to (4|0) + (0|4) + (2|2)$.

The petite $\tilde{K}$-types containing δ_3:

petite $\tilde{K}$-type	mult. of δ_3	representation of $W(C_4)$
$(2\|0,0,0)$	1	$\boxed{(4\|0)}$
$(4\|0,0,0)$	1	$\boxed{(0\|4)}$
$(0\|2,0,0)$	3	$\boxed{(31\|0)}$
$(2\|2,0,0)$	6	$\boxed{(2\|2)}$
$(2\|1,1,0)$	2	$\boxed{(22\|0)}$
$(1\|1,1,1)$	4	$\boxed{(3\|1)}$
$(3\|1,1,1)$	8	$(3\|1)+\boxed{(1\|3)}$
...	...	...

The relevant types for $W(C_4)$ are: $(4|0)$, $(31|0)$, $(22|0)$, $(3|1)$, $(2|2)$, $(1|3)$, $(0|4)$.

We can match all of them!

The petite $\tilde{K}$-types containing δ_{12}:

petite $\tilde{K}$-type	mult. of δ_{12}	representation of $W(B_3 \times A_1)$
$(1\|1,0,0)$	1	$\boxed{(3\|0) \times 2}$
$(3\|1,0,0)$	2	$\boxed{(21\|0) \times 2}$
$(0\|2,0,0)$	1	$\boxed{(0\|3) \times 2}$
$(0\|1,1,0)$	1	$\boxed{(3\|0) \times 11}$
$(2\|2,0,0)$	3	$\boxed{(1\|2) \times 11}$
$(2\|1,1,0)$	3	$\boxed{(2\|1) \times 2}$
$(4\|1,1,0)$	5	$\boxed{(21\|0) \times 11} + (11\|1) \times 2$
$(1\|2,1,0)$	8	$(3\|0) \times 2 + (0\|3) \times 11 + \boxed{(1\|2) \times 2} + \boxed{(2\|1) \times 11}$
$(1\|1,1,1)$	1	$\boxed{(0\|3) \times 11}$
...	...	....

The relevant types for $W(B_3 \times A_1)$ are:

$$(3|0) \times 2 \quad (21|0) \times 2 \quad (2|1) \times 2 \quad (1|2) \times 2 \quad (0|3) \times 2$$

$$(3|0) \times 11 \quad (21|0) \times 11 \quad (2|1) \times 11 \quad (1|2) \times 11 \quad (0|3) \times 11.$$

We can match all of them!

Conclusions

- If $\delta = \delta_0$ is the trivial representation of $\tilde{M}$, then $W^0_\delta = W(F_4)$.
  The relevant types for $W(F_4)$ are: 1_1, 2_3, 4_2, 8_1 and 9_1. Our computations show that

  every relevant type for $W(F_4)$ is contained in the representation of $W(F_4)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_0})$, for some μ petite.

- If $\delta = \delta_3$, then $W^0_\delta = W(C_4)$. Our computations show that

  every relevant type for $W(C_4)$ is contained in the representation of $W(C_4)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_3})$, for some μ petite.

- If $\delta = \delta_{12}$, then $W^0_\delta = W(B_3 \times A_1)$. Our computations show that

  every relevant type for $W(B_3 \times A_1)$ is contained in the representation of $W(B_3 \times A_1)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_{12}})$, for some μ petite.

- If δ is the genuine representation δ_2, then $W^0_\delta = W(F_4)$. Our computations show that

  not every relevant type for $W(F_4)$ is contained in the representation of $W(F_4)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_2})$, for some μ petite.

  Indeed, only 1_1, 2_3, 4_2 and 9_1 have this property. The "missing" relevant $W(F_4)$-type, 8_1, can only be found in non-petite genuine $\tilde{K}$-types.

  For completeness, we include the table of the representations of $W(F_4)$ on the δ_2-isotypic of genuine $\tilde{K}$-types of level $5/2$.

- If δ is the genuine representation δ_6, then $W^0_\delta = W(B_4)$. Our computations show that

  not every relevant type for $W(B_4)$ is contained in the representation of $W(B_4)$ on $\mathrm{Hom}_{\tilde{M}}(E_\mu, V^{\delta_6})$, for some μ petite.

  The "missing" relevant $W(B_4)$-type, $(2|2)$, can only be found in non-petite genuine $\tilde{K}$-types.

  For completeness, we include the table of the representations of $W(B_4)$ on the δ_6-isotypic of genuine $\tilde{K}$-types of level $5/2$.

The (non-petite) genuine $\tilde{K}$-types of level $5/2$:

$\tilde{K}$-type of level $5/2$	mult. of δ_2	representation of $W(F_4)$	mult. of δ_6	representation of $W(B_4)$
$(0\|3,0,0)$	4	4_3	8	$(22\|0) + \boxed{(2\|2)}$
$(2\|3,0,0)$	12	$4_3 + 8_4$	24	$(31\|0) + (211\|0) + \boxed{(2\|2)} + (2\|11) + (11\|2)$
$(0\|2,1,0)$	8	$\boxed{8_1}$	8	$(4\|0) + \boxed{(2\|2)} + (0\|4)$
$(2\|2,1,0)$	24	$\boxed{8_1} + 16$	24	$(31\|0) + \boxed{(2\|2)} + (2\|11) + (11\|2) + (0\|31)$
$(1\|2,1,1)$	10	$1_2 + 9_2$	20	$2 \cdot (3\|1) + (21\|1) + (1\|3)$