Notes on the unitary dual of real split groups

July 20, 2005

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Chapter 1

The unitary dual of a real split semi-simple Lie group

1.1 The Unitarity Problem

Let G be the set of real points of a linear connected reductive group. We denote by:

- $\mathfrak{g}$: the Lie algebra of G
- $\mathfrak{g}_{\mathbb{C}}$: the complexification of $\mathfrak{g}$
- θ: a Cartan involution
- $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: the corresponding Cartan decomposition of $\mathfrak{g}$
- K: the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$.

A representation $(\pi, \mathcal{H})$ of G on a Hilbert space is called *unitary* if $\mathcal{H}$ admits a G-invariant positive definite inner product.

PROBLEM: Classify all irreducible unitary representations of G, up to unitary equivalence.

By results of Harish-Chandra, this is equivalent to classifying all the unitary admissible representations of G, up to infinitesimal equivalence. We split this problem in three parts:

1. Describe all the *irreducible admissible* representations of G, up to infinitesimal equivalence
2. Understand which irreducible admissible representations of G are *Hermitian*, i.e. have a non-degenerate invariant Hermitian form

3. Understand which Hermitian irreducible admissible representations are *unitary*, i.e. decide whether the non-degenerate invariant Hermitian form is positive definite.

1.1.1 Irreducible admissible representations

We need to introduce more notations:

- $P = MAN$: a parabolic subgroup of G
- $\mathfrak{a}$: the Lie algebra of A
- $\mathfrak{a}_{\mathbb{C}}$: its complexification
- (δ, V_δ): an irreducible tempered unitary representation of M
- $\nu \in \mathfrak{a}_{\mathbb{C}}^*$: a linear functional with real part in the open positive Weyl chamber
- $X_P(\delta \otimes \nu)$: the principal series with parameters (P, δ, ν)
- $\overline{X}_P(\delta \otimes \nu)$: **the unique Langlands quotient** of $X_P(\delta \otimes \nu)$.

A brief recall: The principal series $X_P(\delta \otimes \nu)$ is obtained by inducing the representation $\delta \otimes \nu$ from P to G. It is defined as the representation of G by left translation on the space of functions:

$$\mathcal{H}^P_{\delta\otimes\nu} = \{F\colon G \to V^\delta\colon \mathrm{Res}_K(F) \in L^2(K, V^\delta) \text{ and}$$
$$F(gman) = e^{-(\nu+\rho)\log(a)}\delta(m)^{-1}F(g),\ \forall\, man \in P = MAN,\ \forall\, g \in G\}.$$

When $\Re(\nu)$ is strictly dominant, the principal series $X_P(\delta \otimes \nu)$ has a unique irreducible quotient, that we denote by $\overline{X}_P(\delta\otimes\nu)$. It is the quotient of $X_P(\delta\otimes\nu)$ by the Kernel of the intertwining operator

$$A(\overline{P}, P, \delta, \nu)\ :\ X_P(\delta, \nu) \to X_{\overline{P}}(\delta, \nu)$$

($\overline{P}$ is the opposite parabolic). More details are given in chapter D.

Classification

The classification of the irreducible admissible representations of G was given by Langlands in the early 1970s:

- Every irreducible admissible representation of G is infinitesimally equivalent to a **Langlands quotient** $\overline{X}_P(\delta \otimes \nu)$.
- Two Langlands quotients $\overline{X}_P(\delta \otimes \nu)$ and $\overline{X}_{P'}(\delta' \otimes \nu')$ are infinitesimally equivalent if and only if there exists an element ω of K such that

$$\omega P \omega^{-1} = P' \quad \omega \cdot \delta \cong \delta', \quad \omega \cdot \nu = \nu'.$$

1.1.2 Irreducible Hermitian admissible representations

Every irreducible Hermitian admissible representation of G is infinitesimally equivalent to a Hermitian Langlands quotient.
In 1976, Knapp and Zuckerman have proved that $\overline{X}_P(\delta \otimes \nu)$ admits **a non-degenerate invariant Hermitian form** if and only if there exists $\omega \in K$ satisfying

$$\omega P \omega^{-1} = \bar{P} \qquad \omega \cdot \delta \simeq \delta \qquad\qquad \omega \cdot \nu = -\bar{\nu}.$$

This condition follows from the facts that

- $\overline{X}_P(\delta \otimes \nu)$ is Hermitian if and only if it is infinitesimally equivalent to its Hermitian dual

- the Hermitian dual of $\overline{X}_P(\delta \otimes \nu)$ is $\overline{X}_{\bar{P}}(\delta \otimes -\bar{\nu})$.

1.1.3 Unitary irreducible admissible representations

Every unitary irreducible admissible representation of G is infinitesimally equivalent to a unitary Langlands quotient.
Knapp and Zuckerman have proved that every non-degenerate invariant Hermitian form on $\overline{X}_P(\delta \otimes \nu)$ is a real multiple of the form induced by the Hermitian operator

$$B = \delta(\omega) \circ R(\omega) \circ A(\bar{P} : P : \delta : \nu)$$

from $X_P(\delta \otimes \nu)$ to $X_P(\delta \otimes -\bar{\nu})$. So $\overline{X}_P(\delta \otimes \nu)$ is unitary if and only if the form induced by B is positive semidefinite.

Remark 1. *The unitarity problem is reduced to the analytic problem of computing the signature of the Hermitian operator B.*

1.2 The signature of the Hermitian operator B

We assume the existence of an element ω of K satisfying[1]

$$\omega P \omega^{-1} = \bar{P} \qquad \omega \cdot \delta \simeq \delta \qquad \omega \cdot \nu = -\bar{\nu}$$

and we discuss the signature of the Hermitian operator

$$B \colon X_P(\delta \otimes \nu) \to X_P(\delta \otimes -\bar{\nu}).$$

This is a very hard problem. The first reduction consists of computing the signature separately on each K-type appearing in the principal series.

[1] This is a necessary condition for the unitarity of the Langlands quotient $\overline{X}_P(\delta \otimes \nu)$.

Reduction to a K-type by K-type calculation...

For every K-type $(\mu,\, E_\mu)$, consider the Hermitian operator

$$R_\mu(\omega,\,\nu)\colon\ \mathrm{Hom}_K(E_\mu,\, X_P(\delta\otimes\nu)) \to \mathrm{Hom}_K(E_\mu,\, X_P(\delta\otimes -\bar{\nu}))$$

defined by applying B to the range. By Frobenius reciprocity:

$$R_\mu(\omega,\,\nu)\colon\ \mathrm{Hom}_{M\cap K}(E_\mu\mid_{M\cap K},\, V^\delta) \to \mathrm{Hom}_{M\cap K}(E_\mu\mid_{M\cap K},\, V^\delta).$$

If P is a minimal parabolic subgroup, then $M\cap K = M$, so we can write

$$R_\mu(\omega,\,\nu)\colon\ \mathrm{Hom}_M(E_\mu\mid_M,\, V^\delta) \to \mathrm{Hom}_M(E_\mu\mid_M,\, V^\delta).$$

Remark 2. *In order to solve the unitarity problem, we need to construct the operator $R_\mu(\omega,\,\nu)$ for every K-type μ appearing in the principal series.*
This is still a complicated issue. Therefore, we make some additional assumptions:

- *G is **split**[2], in particular $SL(n,\mathbb{R})$, $Sp(2n,\mathbb{R})$, $SO(n,n)$, F_4, E_6, E_7, E_8*
- *$P = MAN$ is a **minimal** parabolic subgroup of G*
- *ν is a **real** character of A.*

Then, a rank-one reduction is possible.

A rank-one reduction...

If P is a minimal parabolic, we can regard ω as an element of $W := N_K(\mathfrak{a})/M$. The operator $R_\mu(\omega,\nu)$ decomposes into a product of factors according to the decomposition of ω into a product of simple reflections (as in Gindikin-Karpelevic). These factors are induced from the corresponding intertwining operators on the root-$SL(2,\mathbb{R})$'s.

Root SL(2)'s

For each $\alpha\in\Delta(\mathfrak{g},\mathfrak{a})$, choose a map $\psi_\alpha : \mathfrak{sl}(2,\mathbb{R})\to\mathfrak{g}$ which commutes with θ, and satisfies

$$\psi_\alpha\Big(\begin{bmatrix}0&1\\0&0\end{bmatrix}\Big) = E_\alpha, \qquad \psi_\alpha\Big(\begin{bmatrix}0&0\\1&0\end{bmatrix}\Big) = E_{-\alpha},$$

where $E_{\pm\alpha}$ are the root vectors, and $\theta(E_\alpha) = -E_{-\alpha}$. Then ψ_α determines a map

$$\Psi_\alpha\ :\ SL(2,\mathbb{R})\to G$$

with image G^α, a connected group with Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Denote by

$$\sigma_\alpha := \Psi_\alpha\Big(\begin{bmatrix}0&1\\-1&0\end{bmatrix}\Big), \qquad m_\alpha := \sigma_\alpha^2,$$

[2] G is split if and only if the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ is trivial.

and let $Z_\alpha := E_\alpha - E_{-\alpha} \in \mathfrak{k}$.
The element Z_α generates a Lie algebra $\mathfrak{k}^\alpha$ isomorphic to $\mathfrak{so}(2)$. The group $K^\alpha = \exp(\mathfrak{k}^\alpha)$ is isomorphic to $SO(2)$. We will refer to K^α as the $SO(2)$-subgroup attached to α.

1.2.1 The Gindikin-Karpelevic decomposition of $R_\mu(\omega, \nu)$

We describe the main steps:

1. Take a *minimal* decomposition of ω as a product of simple reflections[3]
$$\omega = s_{\alpha_r} \cdots s_{\alpha_2} s_{\alpha_1}$$
2. Decompose the operator $A(\overline{P}, P, \delta, \nu)$ accordingly:
$$A(\overline{P}, P, \delta, \nu) = A(s_{\alpha_r})A(s_{\alpha_{r-1}}) \cdots A(s_{\alpha_1}).$$
For all $j = 1 \dots r$ we have set:
$$A(s_{\alpha_j}) = A(P^j, P^{j-1}, \delta^{j-1}, \nu^{j-1}) \colon X_{P^{j-1}}(\delta^{j-1}, \nu^{j-1}) \to X_{P^j}(\delta^j, \nu^j)$$
with[4]
$$\begin{aligned} P^{j-1} &= (s_{\alpha_{j-1}} \cdots s_{\alpha_2} s_{\alpha_1}) P (s_{\alpha_{j-1}} \cdots s_{\alpha_2} s_{\alpha_1})^{-1} \\ \delta^{j-1} &= (s_{\alpha_{j-1}} \cdots s_{\alpha_2} s_{\alpha_1}) \cdot \delta \\ \nu^{j-1} &= (s_{\alpha_{j-1}} \cdots s_{\alpha_2} s_{\alpha_1}) \cdot \nu. \end{aligned}$$
3. When s_α is a simple reflection, interpret the Hermitian operator $A_P(s_\alpha)$ as an intertwining operator for the rank-one subgroup MG^α, and compute the corresponding operator $R_\mu(s_\alpha)$. Then
$$R_\mu(\omega,\, \nu) = R(s_{\alpha_r})R(s_{\alpha_{r-1}}) \cdots R(s_{\alpha_1}).$$
4. Using the results already known for $SL(2,\, \mathbb{R})$, compute the various operators $R(s_\alpha)$.[5]

Remark 3. *$R_\mu(\omega,\, \nu)$ can be decomposed as a product of operators corresponding to simple reflections s_α, and for these operators an explicit formula exists.* ***This formula depends on the decomposition of E_μ in irreducible K^α-types.*** *Because the decomposition changes with α, it is very hard to keep track of the different decompositions when you multiply the various rank-one operators to obtain $R_\mu(\omega,\, \nu)$.*

To solve the unitarity problem we need a formula to compute $R(s_\alpha)$ which is independent of the decomposition of E_μ in K^α-types. When the K-type is **petite** such a formula exists.

Definition. *A K-type is called* ***petite,*** *if $\mu(iZ_\alpha) = 0, \pm 1, \pm 2, \pm 3$.*

[3]Such a decomposition is called minimal if ω has length r.
[4]For $j = 1$, set $P^0 = P$, $\delta^0 = \delta$, $\nu^0 = \nu$.
[5]Details are given in the next chapters.

Chapter 2

The spherical unitary dual

2.1 Spherical representations

We assume that

- G is **split**
- $P = MAN$ is a **minimal** parabolic subgroup of G
- δ is the **trivial** representation of M
- ν is a strictly dominant **real** character of A

and we discuss the unitarity of the Langlands quotient $\overline{X}_P(\delta \otimes \nu)$.

For every K-type μ, we have an intertwining operator

$$R_\mu(\omega, \nu)\colon \operatorname{Hom}_M(E_\mu |_M, V^\delta) \to \operatorname{Hom}_M(E_\mu |_M, V^\delta).$$

Because δ is trivial, we can regard $R_\mu(\omega, \nu)$ as an operator on $(E_\mu^*)^M$. There is a natural representation of the Weyl group on this space, defined by

$$([\sigma] \cdot T)(v) = T(\mu(\sigma^{-1}) \cdot v), \tag{2.1}$$

and when μ is petite, the operator $R_\mu(\omega, \nu)$ depends *only* on this W-representation.

2.1.1 The operator $R_\mu(\omega, \nu)$

Decompose $R_\mu(\omega, \nu)$ in operators corresponding to simple reflections, as in (1.2.1). We need to describe the action of each factor[1]

$$R_\mu(s_\alpha, \gamma)\colon \operatorname{Hom}_M(E_\mu |_M, V^\delta) = (E_\mu^*)^M \to \operatorname{Hom}_M(E_\mu |_M, V^{s_\alpha \cdot \delta = \delta}) = (E_\mu^*)^M.$$

[1] $R_\mu(\omega, \nu)$ acts on $(E_\mu^*)^M$, and so does every factor $R_\mu(s_\alpha, \gamma)$.

For this purpose, consider the decomposition of μ in isotypic components of characters of the $SO(2)$-subgroup K^α attached to α:

$$\mu\mid_M = \bigoplus_{j\in\mathbb{Z}} \phi_j$$

and write

$$(E^*_\mu)^M = \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}_M(\phi_{2n}+\phi_{-2n},\, V^\delta)$$

for the decomposition of $(E^*_\mu)^M$ in MK^α-invariant subspaces.
The operator $R_\mu(s_\alpha,\gamma)$ preserves this decomposition[2], and acts on each MK^α-invariant subspace by a scalar:

$$\frac{1}{D}R_\mu(s_\alpha,\,\gamma):\quad \begin{tikzcd} \underset{\mathrm{Hom}_M(\phi_0,F_\delta)}{\bullet} \arrow[r, dash] \arrow[d, "1"] & \underset{\mathrm{Hom}_M(\phi_2+\phi_{-2},F_\delta)}{\bullet} \arrow[r, dash] \arrow[d, "d_2"] & \underset{Hom(\phi_4+\phi_{-4},F_\delta)}{\bullet} \arrow[r, dash] \arrow[d, "d_4"] & \bullet \arrow[d, "d_6"] & \cdots\cdots \\ \underset{\mathrm{Hom}_M(\phi_0,F_\delta)}{\bullet} \arrow[r, dash] & \underset{\mathrm{Hom}_M(\phi_2+\phi_{-2},F_\delta)}{\bullet} \arrow[r, dash] & \underset{\mathrm{Hom}_M(\phi_4+\phi_{-4},F_\delta)}{\bullet} \arrow[r, dash] & \bullet & \cdots\cdots \end{tikzcd}$$

We have normalized the operator so that it takes the value 1 on a fine K-type.[3] $SL(2)$-calculations show that:

$$d_{2n} = \frac{\Pi_{j=1}^n((2j-1)-\langle\lambda,\,{}^\vee\alpha\rangle)}{\Pi_{j=1}^n((2j-1)+\langle\lambda,\,{}^\vee\alpha\rangle)}$$

for every $n\geq 1$.

Remark 4. *It is clear from the picture that the operator $R_\mu(s_\alpha,\gamma)$ depends on the decomposition of $(E^*_\mu)^M$ in MK^α-invariant subspaces.*

2.1.2 When μ is petite...

If μ is a petite K-type, the space $(E^*_\mu)^M$ reduces to

$$(E^*_\mu)^M = \mathrm{Hom}_M(\phi_0,\,V^\delta)\oplus\mathrm{Hom}_M(\phi_{-2}+\phi_{+2},\,V^\delta). \tag{2.2}$$

We also notice that

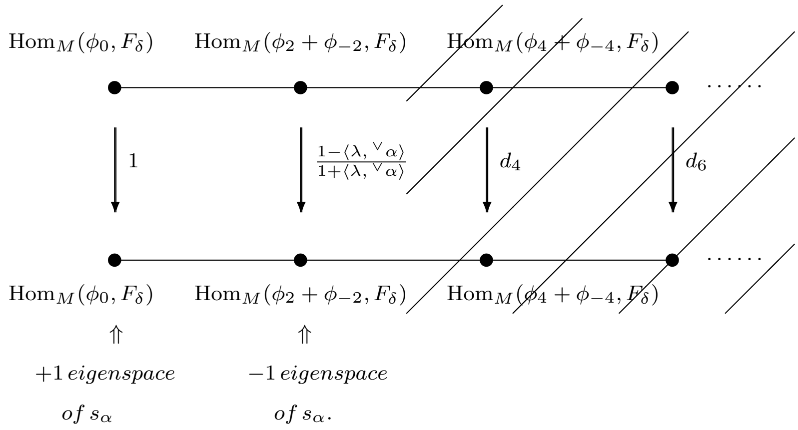
$$\begin{aligned} \mathrm{Hom}_M(\phi_0,\,V^\delta) &\equiv \text{ the } (+1)\text{-eigenspace of } s_\alpha \\ \mathrm{Hom}_M(\phi_{-2}+\phi_{+2},\,V^\delta) &\equiv \text{ the } (-1)\text{-eigenspace of } s_\alpha \end{aligned}$$

in the representation of W on $(E^*_\mu)^M$ defined in (2.1). So we get:

[2]Because it is induced by the corresponding intertwining operator for MG^α

[3]The constant $D = \frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\,\frac{\lambda+1}{2}\;\Gamma\,\frac{\lambda+1}{2}}$ is real and positive.

$$\boxed{\frac{1}{D}R_\mu(s_\alpha,\,\gamma)=\begin{cases} +1 & \textbf{on the } (+1)\textbf{-eigenspace of } s_\alpha \\ \frac{1-\langle\nu,\,\check{\alpha}\rangle}{1+\langle\nu,\,\check{\alpha}\rangle} & \textbf{on the } (-1)\textbf{-eigenspace of } s_\alpha \end{cases}}$$



Remark 5. *When μ is petite, each operator $R_\mu(s_\alpha,\,\gamma)$ can be entirely defined in terms of the representation of the Weyl group on the space of M-fixed vectors. This makes things much simpler, because there is no need to know the decomposition of μ in irreducible representations of $K^\alpha \simeq SO(2)$.*

2.2 Relevant K-types

When μ is a petite K-type, the formula for the operator $R_\mu(\omega,\nu)$ coincides with the formula for the similar operator for a split p-adic group.
To be more precise, results of Barbasch-Moy reduce the problem of the determination of the Iwahori spherical dual of split p-adic group to the problem of determining the unitary dual of finite dimensional representations of the corresponding affine graded Hecke algebra. In this case, for each representation $\tau \in \widehat{W}$, there is an operator $R_\tau(\omega,\gamma)$ with the same formula as the one for the real case. A spherical representation $\overline{X}(\nu)$ is unitary if and only if R_τ is positive definite *for all* τ.
Work of Barbasch for the classical groups, Ciubotaru for F_4, and Barbasch-Ciubotaru for E_6, E_7, and E_8, determine a set of W-representations, called **relevant** with the property that a spherical module $\overline{X}(\nu)$ is unitary, if and only if $R_\tau(\omega,\nu)$ is positive semidefinite *for τ in the relevant set.*

PROBLEM:[4] Find a set of petite K-types so that the $(E_\mu^*)^M$ realize all the relevant W-representations. Call these "the relevant K-types".

[4]This problem has been solved by Dan Barbasch.

If we can solve this problem, then we get powerful necessary conditions for unitarity in the real case: $\overline{X}(triv. \otimes \nu)$ **is unitary only if** $R_\mu(\omega, \nu)$ **is positive semidefinite for** μ **in the relevant set.**

Conjecturally the spherical unitary dual for a split reductive group should be independent of whether the field is real or p-adic. The conjecture is true for the classical groups, so **for classical groups,** $\overline{X}(triv. \otimes \nu)$ **is unitary if and only if** $R_\mu(\omega, \nu)$ **is positive semidefinite for** μ **in the relevant set.**

Next, we list the relevant W-types. For a list of the corresponding relevant K-types, see Barbasch's paper "Relevant and Petite K-types".

Classical groups

For type $\mathbf{A_{n-1}}$, we have $G = SL(n)$, $K = SO(n)$ and $W = S_n$.
The Weyl group representations are parameterized by partitions of n. The relevant W-types correspond to partitions in at most two parts:

$$(n-k, k).$$

For types $\mathbf{B_n}$, and $\mathbf{C_n}$, the Weyl group W consists of permutations and sign changes of of the coordinates of $\mathbb{R}^n$, and the relevant W-types are

$$(n-k, k) \times (0), \qquad (n-k) \times (k).$$

Similarly for $\mathbf{D_n}$.

Exceptional Groups

The relevant W representations are

F_4 $\quad 1_1,\ 2_3,\ 8_1,\ 4_2,\ 9_1,$

E_6 $\quad 1_p,\ 6_p,\ 20_p,\ 30_p,\ 15_q,$

E_7 $\quad 1_a,\ 7'_a,\ 27_a,\ 56'_a,\ 21'_b,\ 35_b,\ 105_b,$

E_8 $\quad 1_x,\ 8_z,\ 35_x,\ 50_x,\ 84_x,\ 112_z,\ 400_z,\ 300_x,\ 210_x.$

The notation is as in Kondo's and Frame's work.

Chapter 3

Non-spherical representations

3.1 What goes wrong?

We assume that

- G is split
- $P = MAN$ is a minimal parabolic subgroup of G
- ν is a strictly dominant real character of A

and we discuss the unitarity of the Langlands quotient $\overline{X}_P(\delta \otimes \nu)$, when δ is a **non-trivial** representation of M.

If we try to apply the same machinery used in the spherical case, we meet two obstacles:

1. The Hermitian operator $R_\mu(\omega, \nu)$ acts on the space $\mathrm{Hom}_M(E\,|_\mu, V^\delta)$, and when δ is non-trivial this space does *not* carry a representation of the Weyl group. Hence, **we can no longer describe the intertwining operators on petite K-types in terms of representations of W.**

2. In the classical case, the factorization of $R_\mu(\omega, \nu)$ as a product of operators corresponding to simple reflections is "easy" to carry out, at least for petite K-types, because there exists a very explicit formula for each factor:

   - Every $R_\mu(s_\alpha, \gamma)$ is an endomorphism of $\mathrm{Hom}_M(E\,|_\mu, V^\delta)$
   - $R_\mu(s_\alpha, \gamma)$ preserves the decomposition of $\mathrm{Hom}_M(E\,|_\mu, V^\delta)$ in MK^α-invariant subspaces[1]

$$\mathrm{Hom}_M(E\,|_\mu, V^\delta) = \bigoplus_{n \in \mathbb{N}} Hom_M(\phi_n + \phi_{-n}, V^\delta)$$

[1] We assume $\mu\,|_{K^\alpha} = \bigoplus_{n \in \mathbb{Z}} \phi_n$.

- $R_\mu(s_\alpha, \gamma)$ acts on each $\mathrm{Hom}_M(\phi_n + \phi_{-n}, V^\delta)$ by a scalar.

The very starting starting point of this construction fails in the non-spherical case: **sometimes, when δ is non-trivial, $R_\mu(s_\alpha, \gamma)$ is not an endomorphism of $\mathrm{Hom}_M(E\,|_\mu, V^\delta)$.**[2]

Overcoming the first obstacle...

In order to overcome the first obstacle, we introduce two important subgroups of W: the stabilizer of δ, W^δ, and the Weyl group of the good co-roots, W_δ^0.

There is a natural action of the Weyl group $W = \frac{N_K(\mathfrak{a})}{Z_K(\mathfrak{a})} = \frac{M'}{M}$ on $\hat{M}$, given by:

$$([\sigma] \cdot \delta)(m) \equiv \delta(\sigma^{-1} m \sigma)$$

for all $\sigma \in M'$, $\delta \in \hat{M}$ and $m \in M$.
Fix an irreducible representation δ of M. The stabilizer of δ in W is the set of all Weyl group elements that stabilize the equivalence class of δ:

$$St_W(\delta) \equiv W^\delta \equiv \{w \in W : w \cdot \delta \simeq \delta\}.^3$$

Next, we define W_δ^0. For every root β, $m_\beta = \sigma_\beta^2$ is a central element of M of order two, so $\delta(m_\beta)$ is equal to plus or minus the identity. A root β in $\Delta(\mathfrak{g}, \mathfrak{a})$ is called a *good* root for δ if $\delta(m_\beta) = +Id$. The set of good co-roots

$$^\vee\Delta_\delta = \{\beta \in {}^\vee\Delta : \delta(m_\beta) = +Id\}$$

forms a root system. We define W_δ^0 to be the Weyl group of $^\vee\Delta_\delta$.
It is a normal subgroup of W^δ, and the quotient

$$R_\delta = \frac{W^\delta}{W_\delta^0}$$

is called "the R-group of δ". When G is split, R_δ is either $\{1\}$, or $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

More details on W^δ and W_δ^0 can be found in chapter A. Chapter B contains many examples, and chapter C describes R_δ as a subgroup of the Dynkin diagram R-group.

The role played by the Weyl group of the good co-roots in the study of non-spherical representations is analogous to the one played by the Weyl group in the study of spherical representations. Indeed, the group W_δ^0 acts on the space $\mathrm{Hom}_M(E_\mu, V^\delta)$

$$\widetilde{\Psi^\mu}[\sigma] \cdot T = T \circ \mu(\sigma^{-1}) \tag{3.1}$$

[2]More precisely, $R_\mu(s_\alpha, \gamma)$ is an endomorphism of $\mathrm{Hom}_M(E\,|_\mu, V^\delta)$ only is the reflection s_α belongs to the stabilizer of δ.

[3]If G is a classical group, then M is abelian and $w \in W^\delta \Leftrightarrow w \cdot \delta = \delta$.

and sometimes the construction of the operator $R_\mu(\omega,\, \nu)$ depends *only* on this representation.
For spherical representations, this happens exactly when the K-type μ is petite (of level at most 3). For non-spherical representations there are stricter conditions. If the decomposition of ω into simple reflections (of W) involves only elements in the stabilizer of δ, then it is enough to assume that μ is petite of level at most 2, and that ω lies in W^0_δ. [4]

3.2 The operator $R_\mu(s_\alpha, \gamma)$

The operator

$$R_\mu(s_\alpha,\, \gamma)\colon\ \mathrm{Hom}_M(E_\mu,\, V^\delta) \to \mathrm{Hom}_M(E_\mu,\, V^{s_\alpha \cdot \delta})$$

is an endomorphism of $\mathrm{Hom}_M(E_\mu,\, V^\delta)$ if and only if s_α stabilizes δ. To describe its action on $\mathrm{Hom}_M(E_\mu,\, V^\delta)$, we need to know whether the root α is good or bad for δ.

When α is a good root, the operator $R_\mu(s_\alpha,\, \gamma)$ behaves just like in the spherical case: it is an endomorphism of

$$\mathrm{Hom}_M(E_\mu \mid_M,\, V^\delta) = \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_M(\phi_{2n} + \phi_{-2n},\, V^\delta)$$

and acts on each of these MK^α-invariant subspaces by a scalar:

$$\begin{tikzcd}
 & \mathrm{Hom}_M(\phi_0, V^\delta) \arrow[r, dash] \arrow[d, "Dd_0"] & \mathrm{Hom}_M(\phi_2 + \phi_{-2}, V^\delta) \arrow[r, dash] \arrow[d, "Dd_2"] & Hom(\phi_4 + \phi_{-4}, V^\delta) \arrow[r, dash] \arrow[d, "Dd_4"] & \bullet \arrow[d, "Dd_6"] & \cdots\cdots \\
R_\mu(s_\alpha,\, \gamma): & \mathrm{Hom}_M(\phi_0, V^\delta) \arrow[r, dash] & \mathrm{Hom}_M(\phi_2 + \phi_{-2}, V^\delta) \arrow[r, dash] & \mathrm{Hom}_M(\phi_4 + \phi_{-4}, V^\delta) \arrow[r, dash] & \bullet & \cdots\cdots
\end{tikzcd}$$

We have set:

$$D = \frac{\pi\, \Gamma(\lambda)}{2^{\lambda-1}\, \Gamma\left(\frac{\lambda+1}{2}\right)\, \Gamma\left(\frac{\lambda+1}{2}\right)}\,,$$

$d_0 = 1$ and

$$d_{2n} = \frac{\Pi_{j=1}^n((2j-1) - \langle \lambda,\, {}^\vee\alpha \rangle)}{\Pi_{j=1}^n((2j-1) + \langle \lambda,\, {}^\vee\alpha \rangle)}$$

[4] In general, we need some extra conditions that guarantee the existence of a matching between the intertwining operator $R_\mu(\omega,\, \nu)$ for G and the intertwining operator $R_\mu(\omega,\, \nu)$ for the split group associated to Δ^0_δ. These operators may be different, because if you regard ω as an element of W^0_δ, then you obtain a different minimal decomposition for ω, and of course a different Gindikin-Karpelevic decomposition for $R_\mu(\omega, nu)$.

for all $n \geq 1$.

When α is a bad root, the reflection s_α does not necessarily stabilize δ. Hence the domain and codomain of the operator

$$R_\mu(s_\alpha, \gamma) \colon \mathrm{Hom}_M(E_\mu, V^\delta) \to \mathrm{Hom}_M(E_\mu, V^{s_\alpha \cdot \delta})$$

may be different.
The decomposition of both $\mathrm{Hom}_M(E_\mu, V^\delta)$ and $\mathrm{Hom}_M(E_\mu, V^{s_\alpha \cdot \delta})$ in MK^α-invariant subspaces involves only odd characters. The operator $R_\mu(s_\alpha, \gamma)$ preserves this decomposition, and carries

$$\mathrm{Hom}_M(\phi_{2n+1} + \phi_{-2n-1}, V^\delta) \to \mathrm{Hom}_M(\phi_{2n+1} + \phi_{-2n-1}, V^{s_\alpha \cdot \delta})$$

for every $n \geq 0$. If T belongs to $\mathrm{Hom}_M(\phi_{2n+1} + \phi_{-2n-1}, V^\delta)$, its image via $R_\mu(s_\alpha, \gamma)$ is the mapping

$$\phi_{2n+1} + \phi_{-2n-1} \to V^{s_\alpha \cdot \delta}, \ (v_+ + v_-) \mapsto D' d_{2n+1} T(v_+ - v_-). \tag{3.2}$$

The operator

$$\Psi_\alpha \colon \mathrm{Hom}_M\left(E_\mu, V^\delta\right) \to \mathrm{Hom}_M\left(E_\mu, V^{s_\alpha \cdot \delta}\right), \, S \mapsto S \circ \mu(\sigma_\alpha^{-1})$$

has a similar effect: if T in $\mathrm{Hom}_M(\phi_{2n+1} + \phi_{-2n-1}, V^\delta)$, then

$$\Psi_\alpha T(v_+ + v_-) = -i(-1)^n T(v_+ - v_-).$$

So we can write

$$R_\mu(s_\alpha, \gamma) \mid_{\mathrm{Hom}_M(\phi_{2n+1}+\phi_{-2n-1}, V^\delta)} = i(-1)^n D' d_{2n+1} \Psi_\alpha = (-1)^n (iD') d_{2n+1} \mu(\sigma_\alpha^{-1}).$$

$$\begin{tikzcd}[column sep=large]
 & \mathrm{Hom}_M(\phi_1+\phi_{-1},V^\delta) & \mathrm{Hom}_M(\phi_3+\phi_{-3},V^\delta) & Hom(\phi_5+\phi_{-5},V^\delta) & & \\
 & \bullet \arrow[r, dash] \arrow[d, "+d_1\mu(\sigma_\alpha^{-1})"] & \bullet \arrow[r, dash] \arrow[d, "-d_3\mu(\sigma_\alpha^{-1})"] & \bullet \arrow[r, dash] \arrow[d, "+d_5\mu(\sigma_\alpha^{-1})"] & \bullet \arrow[d, "-d_7\mu(\sigma_\alpha^{-1})"] & \cdots\cdots \\
\frac{R_\mu(s_\alpha,\gamma)}{iD'} : & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet & \cdots\cdots \\
 & \mathrm{Hom}_M(\phi_1+\phi_{-1},V^\delta) & \mathrm{Hom}_M(\phi_3+\phi_{-3},V^\delta) & \mathrm{Hom}_M(\phi_5+\phi_{-5},V^\delta) & &
\end{tikzcd}$$

We have set:

$$iD' = i \frac{-i\pi \, \Gamma(\lambda)}{2^{\lambda-1} \, \Gamma\left(\frac{\lambda}{2}\right) \, \Gamma\left(\frac{\lambda}{2}+1\right)},$$

$d_1 = 1$, and

$$d_{2n+1} = \frac{(2-\lambda)(4-\lambda)\cdots(2n-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2n+\lambda)}$$

for all $n \geq 1$.

3.3 K-types of level two

Suppose that the decomposition of ω into simple reflections involves only elements in the stabilizer of δ, and assume that μ is a petite K-types of level at most two. Then the intertwining operator

$$R_\mu(\omega,\, \nu)\colon\ \mathrm{Hom}_M(E_\mu,\, V^\delta) \to \mathrm{Hom}_M(E_\mu,\, V^\delta)$$

depends *only* on the representation[5] $\widetilde{\Psi^\mu}$ of W^δ on $\mathrm{Hom}_M(E_\mu,\, V^\delta)$. Each factor $R_\mu(s_\alpha,\, \gamma)$ of the intertwining operator can be constructed in terms of $\widetilde{\Psi^\mu}$, and this construction is independent of the decomposition of μ in isotypic components of K^α-types. We now give the details.

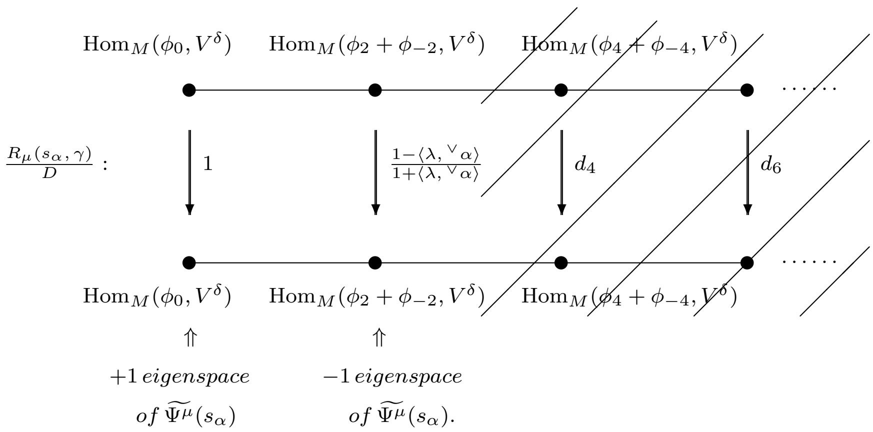
The restriction of μ to the K^α only includes the characters 0, ± 1, ± 2. Hence

$$\mathrm{Hom}_M(E_\mu,\, V^\delta) = \begin{cases} \mathrm{Hom}_M(\phi_0,\, V^\delta) + \mathrm{Hom}_M(\phi_2 + \phi_{-2},\, V^\delta) & \text{if } \alpha \text{ is good for } \delta \\ \mathrm{Hom}_M(\phi_{-1} + \phi_1,\, V^\delta) & \text{if } \alpha \text{ is bad for } \delta. \end{cases}$$

Let's first look at the case in which α is a good root. Because

$$\mathrm{Hom}_M(\phi_0,\, V^\delta) \equiv \ \text{the } (+1)\text{-eigenspace of } \widetilde{\Psi^\mu}(s_\alpha)$$
$$\mathrm{Hom}_M(\phi_{-2} + \phi_{+2},\, V^\delta) \equiv \ \text{the } (-1)\text{-eigenspace of } \widetilde{\Psi^\mu}(s_\alpha)$$

we obtain the following picture:



Just like in the spherical case, we can write:

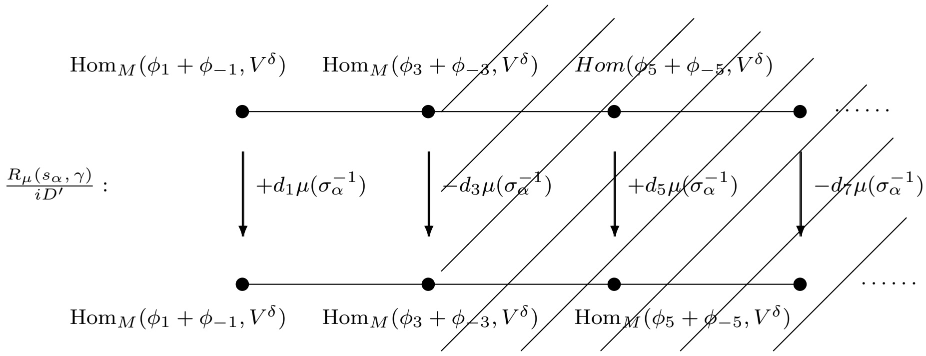
$$R_\mu(s_\alpha,\, \gamma) = \begin{cases} D & \text{on the } (+1)\text{-eigenspace of } \widetilde{\Psi}^\mu(s_\alpha) \\ D\frac{1-\langle\gamma,\,{}^\vee\alpha\rangle}{1+\langle\gamma,\,{}^\vee\alpha\rangle} & \text{on the } (-1)\text{-eigenspace of } \widetilde{\Psi}^\mu(s_\alpha). \end{cases}$$

[5]We recall the definition of $\widetilde{\Psi^\mu}$:

$$\widetilde{\Psi^\mu}[\sigma] \cdot T = T \circ \mu(\sigma^{-1})$$

for all T in $\mathrm{Hom}_M(E_\mu,\, V^\delta)$, and all $[\sigma]$ in W^δ.

Now assume that α is bad. It is clear from the picture



that the operator $R_\mu(s_\alpha,\,\gamma)$ simply acts as a multiple of $\widetilde{\Psi}^\mu(s_\alpha)$:

$$R_\mu(s_\alpha,\,\gamma) = (-iD')\widetilde{\Psi}^\mu(s_\alpha).$$

3.4 A very special case

Suppose that the decomposition of ω into simple reflections involves *only good roots*. This is a very special case, it happens for instance when δ is a genuine representation of M and G is the double cover of E_6 or E_8.[6]
If μ is a petite K-type[7], we can define the intertwining operator $R_\mu(\omega,\,\nu)$ in terms of the representation $\widetilde{\Psi^\mu}$ of W^δ on $\mathrm{Hom}_M(E_\mu, V^\delta)$. Actually, since only good roots are involved, we only need to know the restriction of $\widetilde{\Psi^\mu}$ to the the Weyl group of the good co-roots, W^0_δ.

Because the set of group co-roots forms a root system, there is a real split group attached to it, say $\check{G}$.[8] Let $\check{K}$ be the corresponding maximal compact subgroup and let Θ be the representation of $\check{K}$ with the property that W^0_δ acts on the space of $\check{M}$-fixed vectors in Θ exactly by $\Psi^\mu = \widetilde{\Psi^\mu}\,|_{W^0_\delta}$:

$$\boxed{\begin{array}{c}\text{repr. of } W^0_\delta \text{ on}\\[1ex] \mathrm{Hom}_M(E_\mu, V^\delta)\end{array}} = \boxed{\Psi^\mu} = \boxed{\begin{array}{c}\text{repr. of } W^0_\delta \text{ on}\\[1ex] \mathrm{Hom}_{\check{M}}(E_\Theta, V^{trivial})\end{array}}$$

The intertwining operators $R_\mu(\omega,\nu)$ for G, and $R_\Theta(\omega,\nu)$ for $\check{G}$ have the same Gindikin-Karpelevic decomposition. All the factors agree, because they only depend on ψ^μ, so the full intertwining operators also coincide.

[6] Because $W = W^\delta = W^0_\delta$, every simple root is good.
[7] In this case, μ is allowed to have level three.
[8] In general, $\check{G}$ is not a subgroup of G.

Now consider the p-adic split group attached to ${}^\vee\Delta_\delta$, and call it $\check{\mathbb{H}}$. For spherical principal series, the intertwining operator on petite K-types is independent on the field, so the operator $R_\Theta(\omega,\nu)$ for $\check{G}$ coincides with the p-adic operator R_{Ψ^μ} for $\check{\mathbb{H}}$. It follows that the operator $R_\mu(\omega,\nu)$ for G also coincides with R_{Ψ^μ}.

The unitarity of a spherical principal series of $\check{\mathbb{H}}$ can be detected by looking at the signature of the operator R_τ, for every *relevant* representation τ of $W(\check{\mathbb{H}}) = W_\delta^0$. If we try to match relevant W_δ^0-representations with petite K-types of G containing δ, two possibilities can occur:

1. For every relevant W_δ^0-type τ there is a petite K-types of G such that $\mathrm{Hom}_M(E_\mu, V^\delta) = \tau$, as W_δ^0-representation
2. There is a relevant W_δ^0-type $\bar{\tau}$ that never appears.

Let us discuss the two options separately.

If the matching is complete, we can write:

$$\boxed{X(triv.\otimes\nu)\text{ is not unitary for }\check{\mathbb{H}}}$$
$$\Downarrow$$
$$\boxed{R_\tau\text{ is not positive semi-definite, for some relevant }\tau}$$
$$\Downarrow$$
$$\boxed{R_\mu(\omega,\nu)\text{ is not positive semi-definite, for some }\mu}$$
$$\Downarrow$$
$$\boxed{X(\delta\otimes\nu)\text{ is not unitary for }G}$$

Equivalently,

$$\boxed{X(\delta\otimes\nu)\text{ is unitary for }G}$$
$$\Downarrow$$
$$\boxed{R_\mu(\omega,\nu)\text{ is positive semi-definite, for all }\mu}$$
$$\Downarrow$$
$$\boxed{R_\tau\text{ is positive semi-definite, for all relevant }\tau}$$
$$\Downarrow$$
$$\boxed{X(triv.\otimes\nu)\text{ is unitary for }\check{\mathbb{H}}}$$

If the matching is not complete, then it could happen that the non-spherical principal series $X(\delta\otimes\nu)$ for G is unitary, even if the spherical principal series $X(triv.\otimes\nu)$ for $\check{\mathbb{H}}$ is not unitary. Indeed, the unitarity of $X(triv.\otimes\nu)$ might be ruled out exactly by the W_δ^0-type that we are unable to match.

Remark 6. *When $\check{G}$ is a classical group, the previous considerations apply also if we replace $\check{\mathbb{H}}$ by $\check{G}$.*

3.5 Generalization...

It would be nice to generalize the arguments of the previous section to the case in which $W_\delta^0 \neq W$.

Suppose that μ is a petite K-type and that ω lies in the Weyl group of the good co-roots. If W_δ^0 is not the entire Weyl group, it is very likely that the decomposition of ω in simple reflections in W is *different* from the minimal decomposition of ω as an element of W_δ^0.
Therefore, even if $\mathrm{Hom}_M(E_\mu, V^\delta) = \tau$ as W_δ^0-representations, we cannot expect the operator $R_\mu(\omega, \nu)$ for G to coincide with the p-adic operator $R_\tau(\nu)$ for $\check{\mathbb{H}}$.[9]

In order to generalize the argument described in the previous section, we need the following conditions to be satisfied:

1. For each relevant W_δ^0-type τ, there is a petite K-type μ such that the representation of W_δ^0 on $\mathrm{Hom}_M(E_\mu, V^\delta)$ equals τ.
2. For each pair (μ, τ) as above, the intertwining operators $R_\mu(\omega, \nu)$ for G and $R_\tau(\nu)$ for $\check{\mathbb{H}}$ coincide. This means that if

$$\omega = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_r}$$

is the minimal decomposition of ω as an element of W_δ^0, and

$$\omega = \underbrace{s_{\alpha_1^1} s_{\alpha_2^1} \cdots s_{\alpha_{n_1}^1}}_{s_{\beta_1}} \underbrace{s_{\alpha_1^2} s_{\alpha_2^2} \cdots s_{\alpha_{n_2}^2}}_{s_{\beta_2}} \cdot \underbrace{s_{\alpha_1^r} s_{\alpha_2^r} \cdots s_{\alpha_{n_r}^r}}_{s_{\beta_r}}$$

is the minimal decomposition of ω as an element of W, then you want the "piece" of $R_\mu(\omega, \nu)$ corresponding to $s_{\alpha_1^j} s_{\alpha_2^j} \cdots s_{\alpha_{n_j}^j}$ to match with "the piece" of $R_\tau(\nu)$ corresponding to β_j.[10]

[9] The two operators have different Gindikin-Karpelevic decompositions. Only the factors that correspond to simple good reflections agree.

[10] Recall that $R_\tau(s_{\beta_j}, \gamma_j)$ acts by 1 on the $(+1)$-eigenspace of s_{β_j} in τ, and to act by $\frac{1-\langle {}^\vee\beta_j, \gamma_j\rangle}{1+\langle {}^\vee\beta_j, \gamma_j\rangle}$ on the (-1)-eigenspace.

Appendix A

Good and Bad Roots

A.1 Preliminary remarks

Because G is split, every restricted root β is reduced and every root space $\mathfrak{g}_\beta$ is one-dimensional.
Choose a non-zero element E_β of $\mathfrak{g}_\beta$ that satisfies the normalizing condition[1]

$$B(E_\beta,\, \theta(E_\beta)) = -\frac{2}{\|\beta\|^2}$$

with B the Killing form. Then E_β spans $\mathfrak{g}_\beta$, and $\theta(E_\beta)$ spans $\mathfrak{g}_{-\beta}$.
Denote by H_β the unique element of $\mathfrak{a}$ corresponding to β via the pairing

$$\mathfrak{a} \longleftrightarrow \mathfrak{a}^\star,\ H \longleftrightarrow B(H,\cdot)$$

so that $B(H, H_\beta) = \beta(H)$ for all H in $\mathfrak{a}$. The Lie algebra

$$\mathbb{R}H_\beta \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta} = \mathrm{Span}_{\mathbb{R}}(H_\beta, E_\beta, \theta(E_\beta))$$

is isomorphic to $\mathfrak{sl}(2,\,\mathbb{R})$. An explicit isomorphism is given by:

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &\longrightarrow -\tfrac{2}{\|\beta\|^2} H_\beta \\
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &\longrightarrow E_\beta \\
\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &\longrightarrow -\theta(E_\beta).
\end{aligned}$$

[1] This condition determines E_β up to a sign.

The element $Z_\beta = E_\beta + \theta(E_\beta)$ is fixed by θ (hence it belongs to $\mathfrak{k} = \mathrm{Lie}(K)$) and it generates a subalgebra isomorphic to $\mathfrak{so}(2)$. Set:[2]

$$\sigma_\beta = \exp\left(\frac{\pi}{2} Z_\beta\right)$$

$$m_\beta = \sigma_\beta^2 = \exp(\pi Z_\beta).$$

Then

- σ_β belongs to the normalizer of $\mathfrak{a}$ in K $(= N_K(\mathfrak{a}) = M')$ and it acts on $\mathfrak{a}^\star$ as the root reflection through β
- m_β belongs to the centralizer of $\mathfrak{a}$ in K $(= Z_K(\mathfrak{a}) = M)$ and has order two.

Lemma 1. *Let $\beta(m) = \pm 1$ denote the scalar by which an element m of M acts on the root vector E_β. Then*

(*a*) $\beta(m) = (-\beta)(m)$

(*b*) $\mathrm{Ad}(m)Z_\beta = \beta(m)Z_\beta$

(*c*) $m\sigma_\beta m^{-1} = \sigma_\beta^{\beta(m)}$

(*d*) $\sigma_\beta m \sigma_\beta^{-1} = \begin{cases} m & \text{if } \beta(m) = +1 \\ m_\beta m & \text{if } \beta(m) = -1. \end{cases}$

Proof. We first show that M acts on the vector E_β by a scalar. Since the root space $\mathfrak{g}_\beta$ is one-dimensional, it is enough to show that $\mathrm{Ad}(m)E_\beta$ belongs to $\mathfrak{g}_\beta$, for all m in M. This is easy, because

$$[H,\, \mathrm{Ad}(m)E_\beta] = \mathrm{Ad}(m)[\mathrm{Ad}(m^{-1})H,\, E_\beta] = \mathrm{Ad}(m)[H,\, E_\beta] = \beta(H)\,\mathrm{Ad}(m)E_\beta$$

for all H in $\mathfrak{a}$.
For m in M, define $\beta(m)$ by the equation $\mathrm{Ad}(m)E_\beta = \beta(m)E_\beta$. Because $\mathrm{Ad}(m)$ commutes with the Cartan involution[3], we get

$$\mathrm{Ad}(m)E_{-\beta} = \mathrm{Ad}(m)(-\theta(E_\beta)) = -\theta(\mathrm{Ad}(m)E_\beta) = \beta(m)E_{-\beta}$$

proving that $(-\beta)(m) = \beta(m)$, for all m.
Next, we show that the function $m \mapsto \beta(m)$ only takes the values ± 1 on M, i.e. $\beta(m)^2 \equiv 1$. Let m be any element of M, then

$$B(E_\beta,\, \theta(E_\beta)) = B(\mathrm{Ad}(m)E_\beta,\, \mathrm{Ad}(m)(\theta(E_\beta))) = \beta(m)^2 B(E_\beta,\, \theta(E_\beta)).$$

[2] σ_β is defined only up to inverse, but the action of the operator $\mathrm{Ad}(\sigma_\beta)$ on $\mathfrak{a}$ is completely determined.

[3] Let Θ be the global Cartan involution. Being Θ an involutive automorphism of G which fixes K (hence M), we have

$$mxm^{-1} = \Theta(m\Theta(x)m^{-1}) \qquad \forall\, x \in G \text{ and } m \in M.$$

Differentiating at $x = 1$ we find that $\mathrm{Ad}(m) = \theta\,\mathrm{Ad}(m)\theta$, for all m in M. The results follows from the fact that also θ is an involution.

The scalar $B(E_\beta,\, \theta(E_\beta)) = -\frac{2}{\|\beta\|^2}$ is non-zero, so the claim follows.
Part (b) is trivial:

$$\mathrm{Ad}(m)Z_\beta = \mathrm{Ad}(m)(E_\beta - E_{-\beta}) = \beta(m)E_\beta - (-\beta)(m)E_{-\beta} = \beta(m)Z_\beta.$$

By exponentiating, we find:

$$m\,\exp(tZ_\beta)\,m^{-1} = \exp(t\,\mathrm{Ad}(m)Z_\beta) = \exp(t\beta(m)Z_\beta) = \exp(tZ_\beta)^{\beta(m)}.$$

In particular, for $t = \frac{\pi}{2}$, we get:

$$m\sigma_\beta m^{-1} = \sigma_\beta^{\beta(m)}$$

which is the claim in part (c). Finally,

$$\sigma_\beta m \sigma_\beta^{-1} = \sigma_\beta(m\sigma_\beta m^{-1})^{-1}m = \sigma_\beta\sigma_\beta^{-\beta(m)}m = \begin{cases} m & \text{if } \beta(m) = +1 \\ m_\beta m & \text{if } \beta(m) = -1. \end{cases}$$

The proof of the lemma is now complete. □

Along the lines we have shown that $\beta(m) = \pm 1$ for every m in M. When $m = m_\alpha = \exp(\pi Z_\alpha)$ for some root α, we can be more specific:[4]

$$\beta(m_\alpha) = (-1)^{\frac{2}{\|\alpha\|^2}\langle\alpha,\,\beta\rangle} = (-1)^{\langle{}^\vee\alpha,\,\beta\rangle}. \tag{A.1}$$

A.2 The action of W on $\hat{M}$

In this section we define an action of the Weyl group

$$W = W(G, A) = \frac{N_K(\mathfrak{a})}{Z_K(\mathfrak{a})} = \frac{M'}{M}$$

on the set of equivalence classes of irreducible representations of M. The first step is to define an action of $M' = N_K(\mathfrak{a})$ on $\hat{M}$.
For $\sigma \in M'$, $\delta \in \hat{M}$ and $m \in M$, set:

$$(\sigma \cdot \delta)(m) \equiv \delta(\sigma^{-1}m\sigma). \tag{A.2}$$

It is easy to check that

- $\sigma \cdot \delta$ is a well defined representation of M
- $\sigma \cdot \delta$ is irreducible, because δ is such
- $(\sigma_1\sigma_2) \cdot \delta = \sigma_1 \cdot (\sigma_2 \cdot \delta)$ for all σ_1, σ_2 in M', and $1 \cdot \delta = \delta$

[4] The proof of this formula uses some standard results from the representation theory of $SL(2, \mathbb{C})$ and the fact that, because G is split, every restricted root is the restriction to $\mathfrak{a}$ of one (and only one) root in $\Delta(\mathfrak{g}_0^{\mathbb{C}},\, \mathfrak{a}^{\mathbb{C}})$.

- $\sigma \cdot \delta \simeq \sigma \cdot \delta'$ if $\delta \simeq \delta'$.

Therefore equation $(A.2)$ gives a well defined action of M' on $\hat{M}$. Next we observe that the group M acts trivially: if m_1 belongs to M then

$$(m_1 \cdot \delta)(m) = \delta(m_1^{-1} m m_1) = \delta(m_1)^{-1}\delta(m)\delta(m_1) \tag{A.3}$$

for all m in M, so $(m_1 \cdot \delta)$ is equivalent to δ.[5]
It follows that the action of M' on $\hat{M}$ descends to an action of $W = M'/M$ on the same space. If w belongs to W and σ is any representative of w in M', then

$$(w \cdot \delta)(m) \equiv \delta(\sigma^{-1} m \sigma). \tag{A.4}$$

One final remark. If G is semisimple (hence connected), the Weyl group $W = W(G, A) = M'/M$ coincides with the Weyl group of the restricted root system $\Delta(\mathfrak{g}, \mathfrak{a})$. We should therefore clarify what we mean by a representative of an element τ of $W(\Delta(\mathfrak{g}, \mathfrak{a}))$ inside M'.
If $\tau\colon \mathfrak{a}^\star \to \mathfrak{a}^\star$ belongs to $W(\Delta(\mathfrak{g}, \mathfrak{a}))$ and σ belongs to M', we say that σ represents τ if the restriction to $\mathfrak{a}$ of the adjoint map

$$Ad(\sigma)\colon \mathfrak{a} \to \mathfrak{a}$$

coincides with the dual map to τ. Namely

$$(\tau \cdot T)(H) = T(Ad(\sigma^{-1})H)$$

for all H in $\mathfrak{a}$ and T in $\mathfrak{a}^\star$.

A.3 The stabilizer of δ in W

Fix an irreducible representation δ of M. The stabilizer of δ in W consists of all the elements of the Weyl group that stabilize the equivalence class of δ (with respect to the action of W on $\hat{M}$ defined above):

$$St_W(\delta) \equiv W^\delta \equiv \{w \in W\colon w \cdot \delta \simeq \delta\}.$$

When M is abelian, every irreducible representation of M is one-dimensional and we can say that

$$w \in W^\delta \Leftrightarrow w \cdot \delta = \delta.$$

It's easy to check that W^δ is a subgroup of W, and its index equals the cardinality of the W-orbit of the equivalence class of δ.
How do we check whether a Weyl group element w belongs the stabilizer of δ? Let us consider the case in which w is a root reflection.
If $w = s_\beta = [\sigma_\beta]$ then

$$(w \cdot \delta)(m) = \delta(\sigma_\beta^{-1} m \sigma_\beta) = \delta(\sigma_\beta^{-1}(m\sigma_\beta m^{-1})m) = \delta(\sigma_\beta^{-1}\sigma_\beta^{\beta(m)} m) =$$

[5]Equation $(A.3)$ exhibits $\delta(m_1)$ as an intertwining operator between δ and $(m_1 \cdot \delta)$.

$$= \begin{cases} \delta(m) & \text{if } \beta(m) = +1 \\ \delta(m_\beta)\delta(m) & \text{if } \beta(m) = -1. \end{cases}$$

This shows, in particular, that W^δ contains s_β for every β such that $\delta(m_\beta) = 1$. These reflections generate a very special subgroup of W^δ, that will discuss shortly.

Remark. *For every root β, the map $\delta(m_\beta)$ is either plus or minus the identity.*

Proof. Because m_β has order two, $\delta(m_\beta)^2$ is equal to the identity. So, in order to conclude that $\delta(m_\beta) = \pm Id$, it is enough to prove that $\delta(m_\beta)$ is a scalar. This result will follow by Shur's lemma[6] once we show that $\delta(m_\beta)$ is a self-intertwining operator for δ. By lemma (1),

$$m_\beta m m_\beta^{-1} = \sigma_\beta(\sigma_\beta m \sigma_\beta^{-1})\sigma_\beta^{-1} = \sigma_\beta m \sigma_\beta^{-1} = m$$

for all m in M such that $\beta(m) = +1$. Similarly, if $\beta(m) = -1$, then

$$m_\beta m m_\beta^{-1} = \sigma_\beta(\sigma_\beta m \sigma_\beta^{-1})\sigma_\beta^{-1} = \sigma_\beta(m_\beta m)\sigma_\beta^{-1} =$$

$$= m_\beta(\sigma_\beta m \sigma_\beta^{-1}) = m_\beta(m_\beta m) = m.$$

This shows that every m_β is central in M, so $\delta(m_\beta)$ is central in $\delta(M)$. □

A.4 The set of good co-roots

Let δ be an irreducible representation of M. A root β in $\Delta(\mathfrak{g}, \mathfrak{a})$ is called a *good* root for δ if $\delta(m_\beta) = +Id$.

Definition. *The set*

$$^\vee\Delta_\delta = \{\beta \in {}^\vee\Delta \colon \delta(m_\beta) = +Id\}$$

is called the set of good co-roots.

It follows from previous considerations that the stabilizer of δ contains the reflections through good roots.
The main properties of $^\vee\Delta_\delta$ are described in the following lemma.

Lemma 2. *Let δ be an irreducible representation of M. Then*

(a) *$^\vee\Delta_\delta$ is a root system*

(b) *If the sum of two good co-roots is a co-root, then it is a good co-root.*

(c) *If the sum of two bad co-roots is a co-root, then it is a good co-root.*

[6] Because M is finite, the irreducible representation δ is also finite-dimensional.

Proof. Because ${}^\vee\Delta_\delta$ is included in ${}^\vee\Delta$, we only need to prove that it is closed under reflection. Let ${}^\vee\alpha$ and ${}^\vee\beta$ be good co-roots, i.e. assume that

$$\delta(m_\alpha) = \delta(m_\beta) = +Id.$$

We can write:

$$\delta(m_{s_\beta(\alpha)}) = \delta(\sigma_\beta m_\alpha \sigma_\beta^{-1}) = (s_\beta^{-1} \cdot \delta)(m_\alpha).$$

By assumption, β is a good root, so s_β (together with its inverse) stabilizes the equivalence class of δ. This gives:[7].

$$(s_\beta^{-1} \cdot \delta)(m_\alpha) = T \circ \delta(m_\alpha) \circ T^{-1} = T \circ (+Id) \circ T^{-1} = +Id.$$

Hence $s_\beta(\alpha)$ is a good root, and $s_{{}^\vee\beta}({}^\vee\alpha) = {}^\vee s_\beta(\alpha)$ is a good co-root.
Parts (b) and (c) of the lemma follow from the fact that if ${}^\vee\alpha$, ${}^\vee\beta$ and ${}^\vee\gamma$ are co-roots and

$$^\vee\gamma =^\vee \alpha +^\vee \beta \tag{A.5}$$

then $m_\gamma = m_\alpha \cdot m_\beta$, and of course

$$\delta(m_\gamma) = \delta(m_\alpha) \cdot \delta(m_\beta).$$

For brevity reasons, we only sketch the proof of this fact. Without loss of generality, we can assume that $\|\alpha\| \leq \|\beta\|$. Because the restricted roots for a split group form a reduced root system, there are severe limits to the possible angles between pairs of roots. Taking ($A.5$) into account, we see that only two possibilities can occur:

(i) $\langle {}^\vee\alpha, \beta\rangle = -1$

(ii) $\langle {}^\vee\beta, \gamma\rangle = +1$.

Condition (i) implies that $s_\beta(\alpha) = \gamma$ and that[8]

$$\beta(m_\alpha) = (-1)^{\langle {}^\vee\alpha,\beta\rangle} = (-1)^{-1} = -1$$

Then, by lemma 1,

$$m_\gamma = m_{s_\beta(\alpha)} = \sigma_\beta m_\alpha \sigma_\beta^{-1} = m_\beta m_\alpha = m_\alpha m_\beta.$$

After re-naming the roots, we can use this result to show that

$$m_\alpha = m_{-\beta} m_\gamma$$

if case (ii) holds. Then, because $m_{-\beta} = m_\beta$ and $m_\beta^2 = 1$, we also get:

$$m_\gamma = m_\beta m_\alpha = m_\alpha m_\beta.$$

□

[7]Call T the intertwining operator between δ and $s_\beta^{-1} \cdot \delta$.
[8]By equation ($A.1$).

A.5 The Weyl group of the good co-roots

Let δ be an irreducible representation of M, and let ${}^\vee\Delta_\delta$ be the root system of good co-roots. Define W^0_δ to be the Weyl group of ${}^\vee\Delta_\delta$.
It is a subgroup of

$$W({}^\vee\Delta(\mathfrak{g},\mathfrak{a})) = W(\Delta(\mathfrak{g},\mathfrak{a})) = W$$

but not necessarily Levi. For instance, for $E8$ the Weyl group of the good co-roots can be of type $E8$, $D8$ or $E7 \times A1$.
It is also a subgroup of W^δ, because reflections through good roots stabilize the equivalence class of δ. We can say more:

Remark. *W^0_δ is a normal subgroup of W^δ.*

Proof. It is enough to prove that

$$ws_\alpha w^{-1} = s_{w(\alpha)} \in W^0_\delta$$

for all w in W^δ and α in Δ_δ. This follows easily from the fact that $w(\alpha)$ is a good root:

$$\delta(m_{w(\alpha)}) = \delta(\sigma m_\alpha \sigma^{-1}) = (w^{-1}\cdot\delta)(m_\alpha) = T\circ\delta(m_\alpha)\circ T^{-1} = T\circ(+Id)\circ T^{-1} = +Id.$$

We have denoted by σ a representative for w in $M' = N_K(\mathfrak{a})$, and by T an intertwining operator between δ and $w^{-1}\cdot\delta$. □

A.6 The R-group R_δ

Let δ be an irreducible representation of M. The Weyl group of the good co-roots W^0_δ is a normal subgroup of the stabilizer of δ, so the quotient

$$R_\delta = \frac{W^\delta}{W^0_\delta}$$

is well defined. We call this quotient "the R-group of δ".

Lemma 3. *Let δ be an irreducible representation of M and let R_δ be the R-group of δ. Then*

(*a*) *R_δ is a finite group*

(*b*) *Every element of R_δ has order two*

(*c*) *R_δ is an abelian group.*

We will only sketch the proof of this lemma, for brevity reasons.

Proof. Part (c) is an immediate consequence of (b); parts (a) and (b) follow from the fact that R_δ is isomorphic to the group

$$R_\delta^c = \{w \in W^\delta \colon w({}^\vee\Delta_\delta^+) = {}^\vee\Delta_\delta^+\}. \tag{A.6}$$

We can regard R_δ^c as the stabilizer of ρ_δ (the semi-sum of the positive good co-roots) inside W. Hence, by Chevalley's lemma, R_δ^c is generated by reflections through simple co-roots orthogonal to ρ_δ.

The set of all the co-roots orthogonal to ρ_δ forms a root-system, that we denote by Δ_S. It can be shown that Δ_S consists of bad strongly orthogonal[9] simple roots, together with their negatives. Reflections through simply orthogonal roots commute, so the Weyl group of Δ_S is abelian, and every element has order two. By construction, R_δ^c is included in $W(\Delta_S)$ so it has the same properties. □

Lemma 4. *Let δ be an irreducible representation of M. The stabilizer of δ is the semidirect product of the Weyl group of the good co-roots and the group R_δ^c defined in* (A.6)*:*

$$W^\delta = W_\delta^0 \rtimes R_\delta^c.$$

In this decomposition W_δ^0 is normal, and the quotient W^δ/W_δ^0 is isomorphic to R_δ^c, hence to R-group R_δ.

Lemma 5. *If G is connected, semi-simple and has a complexification, then M is a finite abelian group and is generated by the m_αs. It follows that*

$$W^\delta = \{w \in W \colon w({}^\vee\Delta_\delta) = {}^\vee\Delta_\delta\}$$

$$R_\delta^c = \{w \in W \colon w({}^\vee\Delta_\delta^+) = {}^\vee\Delta_\delta^+\}.$$

for every irreducible representation δ of M.

[9]The roots α and β are said to be strongly orthogonal if they are orthogonal, and neither $\alpha+\beta$ nor $\alpha-\beta$ is a root.

Appendix B

Examples of R-groups$\cdots$

B.1 The example of $SL(n, \mathbb{R})$

The data for $SL(n, \mathbb{R})$

We briefly recall the data for the group $SL(n, \mathbb{R})$ and fix the notations that we will be using throughout the section.

- $G = SL(n, \mathbb{R})$
- $K = SO(n)$
- $\mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{R}) = \mathfrak{a}_0 \oplus_{\alpha\in\Delta} (\mathfrak{g}_0)_\alpha$, with

  $\mathfrak{a}_0 = \{$ diagonal matrices with trace $0\}$
  $\Delta = \{\epsilon_i - \epsilon_j : i,\, j = 1 \ldots n,\, i \neq j\}$
  $\Delta^+ = \{\epsilon_i - \epsilon_j : i,\, j = 1 \ldots n,\, i < j\}$

  where for each $l = 1 \ldots n$, we have denoted by ϵ_l the linear functional

  $$\epsilon_l\colon \mathfrak{a}_0 \to \mathbb{R},\ \mathrm{diag}(h_1,\, h_2, \ldots,\, h_n) \mapsto h_l$$

- $A = \{$diagonal matrices, with positive entries and det. $1\}$
- $M = \left\{\mathrm{diag}\,(c_1,\, c_2, \ldots,\, c_n) : c_j = \pm 1,\, \Pi_{j=1}^{n} c_j = +1\right\} \simeq \mathbb{Z}_2^{n-1}$
- $\widehat{M} = \{\delta_S\colon S \subset \{1, \ldots,\, n\}$ s.t. $\mid S \mid < \left[\frac{n}{2}\right]\}$, with

  $$\delta_S\colon\ \mathrm{diag}\,(c_1,\, c_2, \ldots,\, c_n) \mapsto \Pi_{j\in S}\, c_j.$$

  For all subsets S of $\{1, \ldots,\, n\}$, δ_S is a well defined representation of M. We notice that[1] $\Pi_{j\in S}\, c_j = \Pi_{j\in (S^C)}\, c_j$, so δ_S is equivalent to δ_{S^C}, and we obtain a total of $2^{n-1} =\mid M \mid$ inequivalent representations.

- The Weyl group W acts as the group of all permutations of the set $\{\epsilon_1,\, \epsilon_2 \ldots \epsilon_n\}$, so W is isomorphic to the symmetric group $\mathcal{S}_n$.

[1] Because every element of M has determinant one.

The *good* roots for δ_S

When S is the empty set, δ_S is the trivial representation of M and all the roots are good.
Assume that $S = \{j_1, j_2, \ldots, j_p\} \subset \{1, \ldots, n\}$, with $1 < p \leq \left[\frac{n}{2}\right]$. To identify the good roots for δ_S, we need to construct the element m_α for every positive restricted root α, and to evaluate δ_S at m_α.
Set $\alpha = \epsilon_i - \epsilon_j$, with $i < j$. Then

$$H_\alpha = \mathrm{diag}(d_1, d_2, \ldots, d_n)$$

with $d_i = -d_j = 1$ and $d_l = 0$ otherwise. Therefore

$$m_\alpha = \exp\left(\frac{2\pi i}{\|\alpha\|^2} H_\alpha\right) = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1})$$

with $\lambda_i = \lambda_j = -1$ and $\lambda_l = +1$ otherwise. We notice that

$$\delta_S(m_{\epsilon_i - \epsilon_j}) = \begin{cases} +1, & \text{if either } \{i, j\} \subseteq S \text{ or } \{i, j\} \subseteq S^C \\ -1, & \text{otherwise.} \end{cases}$$

So $\epsilon_i - \epsilon_j$ is a good root if and only if both indices i and j lie in S, or none of them does. We obtain:

$$\Delta_{\delta_S} = \{\pm(\epsilon_i - \epsilon_j)\}_{i<j,\, i,j \in S} \sqcup \{\pm(\epsilon_i - \epsilon_j)\}_{i<j,\, i,j \in S^C}$$

and

$$\boxed{{}^\vee\Delta_{\delta_S} = \{\pm(\epsilon_i - \epsilon_j)\}_{i<j,\, i,j \in S} \sqcup \{\pm(\epsilon_i - \epsilon_j)\}_{i<j,\, i,j \in S^C}}\,.$$

Remark. It is a root system of type $A_{p-1} \times A_{q-1}$, with $p = \#S$ and $q = \#S^C$.

The Weyl group of the good co-roots for δ_S

If $p = \#S \geqq 1$, then ${}^\vee\Delta_{\delta_S} = A_{p-1} \times A_{q-1}$ and

$$\boxed{W^0_{\delta_S} = W(A_{p-1}) \times W(A_{q-1}) \simeq \mathcal{S}_p \times \mathcal{S}_q}\,.$$

We notice that $W^0_{\delta_S}$ acts on the set $\{\epsilon_1, \epsilon_2 \ldots \epsilon_n\}$ by

- permuting the ϵ_is, with i in S
- permuting the ϵ_js, with j in S^C.

It is a subgroup of W of order $p!q!$ and index $[W : W^0_{\delta_S}] = \frac{n!}{p!q!} = \begin{pmatrix} n \\ p \end{pmatrix}$.

If S is the empty set, then $\Delta_{\delta_S} = \Delta$ and of course $W^0_{\delta_S} = W$ (it has order $n!$ and index 1).

The stabilizer of the δ_S

Because $G = SL(n)$ is connected, semisimple and has a complexification, we can identify W^{δ_S} with the set of of Weyl group elements preserving the good roots for δ_S.[2]

If S is the empty set, then every root is good and $W^{\delta_S} = W$. If S is not empty, then we must look for Weyl group elements that stabilize the set:

$$\Delta_{\delta_S} = \{\pm(\epsilon_i - \epsilon_j)\}_{i<j,\, i,j\in S} \sqcup \{\pm(\epsilon_i - \epsilon_j)\}_{i<j,\, i,j\in S^C}.$$

It is not hard to see that

- Δ_{δ_S} is stable under any permutation of the set $\{\epsilon_i : i \in S\}$, as well as any permutation of the set $\{\epsilon_j : j \in S^C\}$
- If $p \neq \frac{n}{2}$, there are no other Weyl group elements that preserve Δ_{δ_S}
- If $p = \frac{n}{2}$ (and n is of course even), then there are other Weyl group elements that preserve Δ_{δ_S}, namely all the permutations of the form:

  $$\pi = (i_1\, j_1)(i_2\, j_2)\cdots(i_{\frac{n}{2}}\, j_{\frac{n}{2}})$$

  with $i_1, i_2, \dots i_{\frac{n}{2}}$ in S and $j_1, j_2, \dots j_{\frac{n}{2}}$ in S^C.
  WLOG we can assume that $n = 2m$, and that $S = \{1, 2, \dots, m\}$. Then any permutation π as above can be decomposed as a product $\sigma_2\tilde{\pi}\sigma_1$, with:

  - $\sigma_1 \in \mathcal{S}_{\{1,\dots,m\}}$
  - $\tilde{\pi} = (1\, m+1)(2\, m+2)\cdots(mn)$
  - $\sigma_2 \in \mathcal{S}_{\{m+1,\dots,n\}}$.

  It follows that $\tilde{\pi}$ is a generator for the R-group, of order two.

Here is a synopsis of the results: if $0 < p \neq \frac{n}{2}$, then

$$\boxed{W^{\delta_S} = W(A_{p-1}) \times W(A_{q-1}) = W_{\delta_S^0}}\,.$$

[2] General remark: If G is a connected semisimple Lie group and has a complexification $G^{\mathbb{C}}$, then the group M is generated by M_0 (the identity component of M) and by the elements $\{m_\alpha\}_{\alpha \text{ real}}$.[3] If G is also split, then M is discrete (so M_0 is trivial) and every root is real, so M is generated by all the m_α's. We can therefore write:

$$\begin{aligned} W^\delta &= \{w \in W : (w\cdot\delta)(m) = \delta(m) \quad \forall m \in M\} \\ &= \{w \in W : (w\cdot\delta)(m_\alpha) = \delta(m_\alpha) \quad \forall \alpha \in \Delta\} \\ &= \{w \in W : \delta(m_{w\cdot\alpha}) = \delta(m_\alpha) \quad \forall \alpha \in \Delta\} \\ &= \{w \in W : w \text{ preserves } \Delta_\delta\}. \end{aligned}$$

If $p = \frac{n}{2}$ (and n is even), then

$$\boxed{W^{\delta_S} = (W(A_{p-1}) \times W(A_{q-1})) \ltimes \mathbb{Z}_2 = W_{\delta_S^0} \ltimes \mathbb{Z}_2}.$$

Remark. $W^0_{\delta_S}$ is a (normal) subgroup of W^{δ_S} of index 1 if $p \neq \frac{n}{2}$, and index 2 if $p = \frac{n}{2}$ (and n is even).

The R-group

By definition, the R-group R_{δ_S} is the quotient $W^{\delta_S}/W^0_{\delta_S}$. It follows from the previous considerations that:

- If S is the empty set, then $W^0_{\delta_S} = W^{\delta_S} = W$ and the R-group R_{δ_S} is trivial.

- If $0 < \#S \neq \frac{n}{2}$, then $W^0_{\delta_S} = W^{\delta_S} = W(A_{p-1}) \times W(A_{q-1})$ and again the R-group R_{δ_S} is trivial. We have set $p = \#S$ and $q = n - p$.

- If $\#S = \frac{n}{2}$, then R_{δ_S} has order two and is isomorphic to $\mathbb{Z}_2$. We can pick the permutation $\tilde{\pi}$ as a generator.

B.2 The example of $SP(2n, \mathbb{R})$

The data for $SP(2n, \mathbb{R})$

- $G = SP(2n, \mathbb{R})$
- $K = \left\{ \begin{pmatrix} A & -C \\ C & A \end{pmatrix} \in SL(2n,\mathbb{R})\colon AA^T + CC^T = I,\ AC^T - CA^T = O,\ \det(k) = 1 \right\}$
  We notice that K is isomorphic to $U(n)$ via the mapping
  $$\begin{pmatrix} A & -C \\ C & A \end{pmatrix} \mapsto A + i\,C.$$
- $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{a}_0 \oplus_{\alpha\in\Delta} (\mathfrak{g}_0)_\alpha$, with
  $$\mathfrak{a}_0 = \left\{ \begin{pmatrix} H & O \\ O & -H \end{pmatrix} : H \text{ diagonal matrix} \right\}$$
  $$\Delta = \{\pm\epsilon_i \pm \epsilon_j : i,\, j = 1\ldots n,\, i < j\} \sqcup \{\pm 2\epsilon_l : l = 1\ldots n\}$$
  $$\Delta^+ = \{\epsilon_i \pm \epsilon_j : i,\, j = 1\ldots n,\, i < j\} \sqcup \{2\epsilon_l : l = 1\ldots n\}$$
  where for each $l = 1\ldots n$, we have denoted by ϵ_l the linear functional
  $$\epsilon_l\colon \mathfrak{a}_0 \to \mathbb{R},\ diag(h_1,\, h_2, \ldots,\, h_n, -h_1,\, -h_2, \ldots,\, -h_n) \mapsto h_l$$
- $A = \left\{ \begin{pmatrix} D & O \\ O & D^{-1} \end{pmatrix} : D \text{ diagonal matrix, with positive entries} \right\}$

- $M = \left\{ \begin{pmatrix} \Lambda & O \\ O & \Lambda \end{pmatrix} : \Lambda \text{ diagonal matrix, with entries } \pm 1 \right\} \simeq \mathbb{Z}_2^n$

- $\widehat{M} = \{\delta_S \colon S \subset \{1, \dots, n\}$, with

$$\delta_S \colon \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_1, \lambda_2, \dots, \lambda_n) \mapsto \Pi_{j \in S} \lambda_j.$$

  For all subsets S of $\{1, \dots, n\}$, δ_S is a well defined representation of M. Because there are no equivalences, we obtain a total of $2^n = | M |$ (inequivalent) representations.

- The Weyl group W acts as the group of all permutations and sign changes of the set $\{\epsilon_1, \epsilon_2 \dots \epsilon_n\}$, so W is isomorphic to the semidirect product of $\mathbb{Z}_2^n$ and $\mathcal{S}_n$ (with $\mathcal{S}_n$ acting on $\mathbb{Z}_2^n$).

The *good* roots for δ_S: Δ_{δ_S}

When S is the empty set, δ_S is the trivial representation of M and all the roots are good. So it is enough to consider the case $S = \{j_1, j_2, \dots, j_p\} \subset \{1, \dots, n\}$, with $p > 1$.
To identify the good roots for δ_S, we need to construct the element m_α for every positive restricted root α, and to evaluate δ_S at m_α.[4]

- If $\alpha = \epsilon_i + \epsilon_j$, then

$$H_\alpha = \operatorname{diag}(d_1, d_2, \dots, d_n, -d_1, -d_2, \dots, -d_n)$$

  with $d_i = d_j = 1/2$ and $d_l = 0$ otherwise. Therefore

$$m_\alpha = \exp\left(\frac{2\pi i}{\|\alpha\|^2} H_\alpha\right) = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_n^{-1})$$

  with $\lambda_i = \lambda_j = -1$ and $\lambda_l = +1$ otherwise. We notice that

$$\delta_S(m_{\epsilon_i + \epsilon_j}) = \begin{cases} +1, & \text{if either } \{i, j\} \subseteq S \text{ or } \{i, j\} \subseteq S^C \\ -1, & \text{otherwise.} \end{cases}$$

  So $\epsilon_i + \epsilon_j$ is a good root if and only if both indices i and j lie in S, or none of them does.

[4]By definition, a root α is *good* for δ_S if and only if $\delta_S(m_\alpha) = 1$. Recall that if G has a complexification, and α is a real root, we can construct m_α by the formula:

$$\boxed{m_\alpha = \exp \quad \frac{2\pi i}{\|\alpha\|^2} H_\alpha}\,.$$

For details, please refer to Knapp's book "Lie groups beyond an introduction", chapter seven, section 8.

- If $\alpha = \epsilon_i - \epsilon_j$, then
$$H_\alpha = \mathrm{diag}(d_1,\, d_2, \dots, d_n,\, -d_1,\, -d_2, \dots, -d_n)$$
with $d_i = -d_j = 1/2$ and $d_l = 0$ otherwise, and again
$$m_\alpha = \mathrm{diag}(\lambda_1,\, \lambda_2, \dots, \lambda_n,\, \lambda_1^{-1},\, \lambda_2^{-1}, \dots, \lambda_n^{-1})$$
with $\lambda_i = \lambda_j = -1$ and $\lambda_l = +1$ otherwise. Because $m_{\epsilon_i - \epsilon_j} = m_{\epsilon_i + \epsilon_j}$, we deduce that $\epsilon_i - \epsilon_j$ is a good root if and only if $\epsilon_i + \epsilon_j$ is a good root.

- Finally, if $\alpha = 2\epsilon_k$, then
$$H_\alpha = \mathrm{diag}(d_1,\, d_2, \dots, d_n,\, -d_1,\, -d_2, \dots, -d_n)$$
with $d_k = 1$ and $d_l = 0$ otherwise. Therefore
$$m_\alpha = \exp\left(\frac{2\pi i}{\|\alpha\|^2} H_\alpha\right) = \mathrm{diag}(\lambda_1,\, \lambda_2, \dots, \lambda_n,\, \lambda_1^{-1},\, \lambda_2^{-1}, \dots, \lambda_n^{-1})$$
with $\lambda_k = -1$ and $\lambda_l = +1$ otherwise. We notice that
$$\delta_S(m_{2\epsilon_k}) = \begin{cases} +1, & \text{if } k \in S^C \\ -1, & \text{if } k \in S. \end{cases}$$
Therefore $2\epsilon_k$ is a good root if and only if k is *not* in S.

We conclude that for every not-empty $S \subset \{1, \dots, n\}$, the set of good roots
$$\Delta_{\delta_S} = \left(\{\pm\epsilon_i \pm \epsilon_j\}_{i \neq j,\, i,j \in S^C} \sqcup \{\pm 2\epsilon_k\}_{k \in S^C}\right) \sqcup \{\pm\epsilon_i \pm \epsilon_j\}_{i \neq j,\, i,j \in S}.$$
is of type $C_q \times D_p$. The set of good co-roots
$$\boxed{{}^\vee\Delta_{\delta_S} = \left(\{\pm\epsilon_i \pm \epsilon_j\}_{i \neq j,\, i,j \in S^C} \sqcup \{\pm \epsilon_k\}_{k \in S^C}\right) \sqcup \{\pm\epsilon_i \pm \epsilon_j\}_{i \neq j,\, i,j \in S}}$$
is a root system of type $B_q \times D_p$. Here $p = \#S$ and $q = \#S^C = n - p$.

If S is the empty set, then ${}^\vee\Delta_{\delta_S} = {}^\vee\Delta$ (of type B_n).

The Weyl group of the good co-roots for δ_S: $W^0_{\delta_S}$

If $p = \#S \geqq 1$, then ${}^\vee\Delta_{\delta_S} = B_q \times D_p$ and $\boxed{W^0_{\delta_S} = W(B_q) \times W(D_p) = W(C_q) \times W(D_p)}$. It has order
$$\mid W^0_{\delta_S} \mid = \mid W(C_q) \mid \cdot \mid W(D_p) \mid = (2^q q!)(2^{p-1} p!) = 2^{n-1} q! p!$$
and index
$$[W : W^0_{\delta_S}] = \frac{2^n n!}{2^{n-1} q! p!} = 2 \begin{pmatrix} n \\ p \end{pmatrix}.$$
$W^0_{\delta_S}$ acts on the set $\{\epsilon_1,\, \epsilon_2 \dots \epsilon_n\}$ by

- permuting the ϵ_is, with i in S

- permuting the ϵ_js, with j in S^C

- changing sign to an *even* number of ϵ_is, with i in S

- changing sign to an *arbitrary* number of ϵ_js, with j in S^C.

If S is the empty set, then ${}^\vee\Delta_{\delta_S} = {}^\vee\Delta$ and of course $W^0_{\delta_S} = W$ (it has order $2^n n!$ and index 1).

The stabilizer of the δ_S: W^{δ_S}

Because $G = Sp(2n)$ is connected, semisimple and has a complexification, we can identify W^{δ_S} with the set of of Weyl group elements preserving the good roots for δ_S.

If S is the empty set, then every root is good and $W^{\delta_S} = W$. The case $S \neq \emptyset$ is more interesting, indeed we must look for Weyl group elements that stabilize the set:

$$\Delta_{\delta_S} = \left(\{\pm\epsilon_i \pm \epsilon_j\}_{i\neq j,\, i,j\in S^C} \sqcup \{\pm 2\epsilon_k\}_{k\in S^C}\right) \sqcup \{\pm\epsilon_i \pm \epsilon_j\}_{i\neq j,\, i,j\in S}.$$

It is not hard to see that

- Δ_{δ_S} is stable under the following operations
  - all permutations and sign changes of the set $\{\epsilon_i : i \in S\}$
  - all permutations and sign changes of the set $\{\epsilon_j : j \in S^C\}$.
- There are no other Weyl group elements that preserve the set Δ_{δ_S}.

Therefore:

$$\boxed{W_{\delta_S} = W(C_q) \times W(C_p)}.$$

This group has order

$$\mid W^{\delta_S} \mid = \mid W(C_q) \mid \cdot \mid W(C_p) \mid = (2^q q!)(2^p p!) = 2^n q! p!$$

and index

$$[W : W^{\delta_S}] = \frac{2^n n!}{2^n q! p!} = \begin{pmatrix} n \\ p \end{pmatrix}.$$

Remark. $W^0_{\delta_S}$ is a (normal) subgroup of W^{δ_S} of index 2.

The R-group

By definition, the R-group R_{δ_S} is the quotient $W^{\delta_S}/W_{\delta_S^0}$. It follows from the previous considerations that:

- If S is the empty set, then $W^0_{\delta_S} = W^{\delta_S} = W$ and the R-group R_{δ_S} is trivial.

- If S is not empty, then R_{δ_S} has order two, and is isomorphic to $\mathbb{Z}_2$. We can choose as generator any sign change $\epsilon_i \mapsto -\epsilon_i$, with i in S.

Appendix C

The Dynkin diagram R-group

The main reference for this chapter is Dana Pascovici's paper, "The Dynkin diagram R-group".[1]

Let DD be a connected Dynkin diagram. We denote by Δ the corresponding irreducible root system, and by Π a choice of simple roots for Δ. The set Π is in one-one correspondence with the set of vertices of the Dynkin diagram. In this correspondence, disjoint vertices correspond to simply orthogonal simple roots.[2]
To every connected Dynkin diagram DD we attach a finite abelian group R_{DD}, that can be easily computed by looking at DD. We call this group "the R-group of the Dynkin diagram DD". It plays a role in our discussion on the R_δ- groups, because for a simple split real group and a minimal principal series the R-group R_δ is always a subgroup of R_{DD}. This implies that the order of R_δ can only be at most four.

C.1 Preliminary definition (the simply laced case)

The definition of R_{DD} is particularly easy when the Dynkin diagram DD is simply laced, so we start with this case.

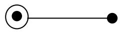
An element of R_{DD} is a set S of mutually disjoint vertices of DD, s.t. any vertex $x \notin S$ is connected to an ***even*** *number of elements of S. R_{DD} is made into a group with the operation of symmetric difference of sets.*

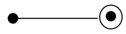
[1] Representation Theory 5 (2001), 1-16.

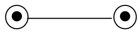
[2] Two roots α and β are "simply-orthogonal" if $<\alpha, \beta>= 0$ and $\alpha \pm \beta$ is *not* a root. For simple roots, this the usual notion of orthogonality. Indeed, the difference of two simple roots is never a root, and $\alpha + \beta$ is a root if and only if the α-string through β has length strictly greater than one, and this happens exactly when the two roots are not orthogonal.

Let us make some examples.

The Dynkin diagram of is trivial. Indeed we notice that

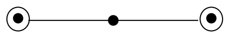
$S =$  is not in R_{DD}, because the second vertex is an element of S^C which is connected to *one* element of S.

$S =$  is not in R_{DD}, for similar reasons.

$S =$  is not in R_{DD}, because the two vertices are not disjoint.

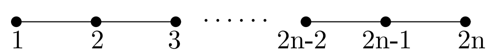
Therefore, the only element of R_{DD} is the empty set: $S =$.

A similar argument shows that the Dynkin diagram of is isomorphic to $\mathbb{Z}_2$, the non trivial element of R_{DD} being

.

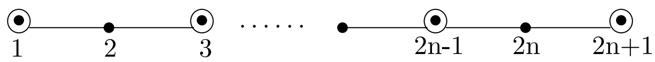
We now list the R-group of every connected simply laced Dynkin diagram. If the group is non trivial, we give the non trivial elements.[3]

If $l = 2n$ is even, the Dynkin diagram of A_l is trivial:



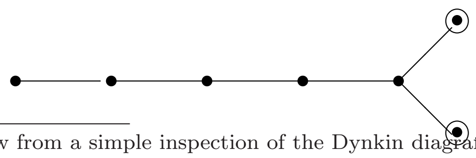
.

If $l = 2n + 1$ is odd, the Dynkin diagram of A_l is isomorphic to $\mathbb{Z}_2$. The non trivial element is:



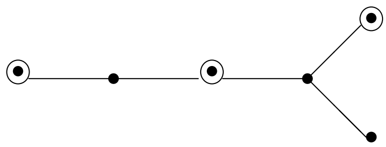
.

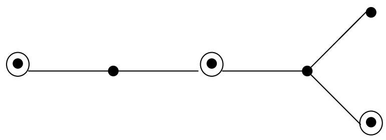
If $l = 2n + 1$ is odd, the Dynkin diagram of D_l is isomorphic to $\mathbb{Z}_2$. The non trivial element is:

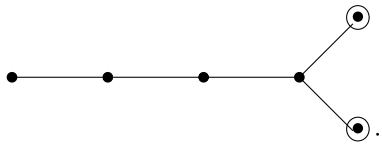


[3] All the results follow from a simple inspection of the Dynkin diagram.

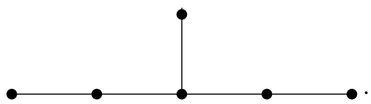
If $l = 2n$ is even, the Dynkin diagram of D_l is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The non trivial element are:



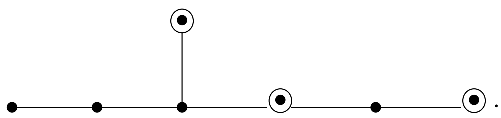




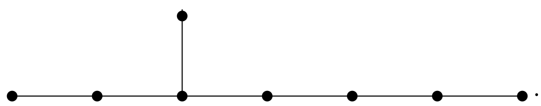
The Dynkin diagram of E_6 is trivial:



The Dynkin diagram of E_7 is isomorphic to $\mathbb{Z}_2$. The non trivial element is:



The Dynkin diagram of E_8 is trivial:



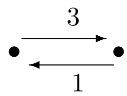
C.2 General definition

We now give the general definition of Dynkin diagram R-group, which is valid also in the not-simply laced case. The first step is to associate to any Dynkin diagram DD a labelled directed graph Γ_{DD}:

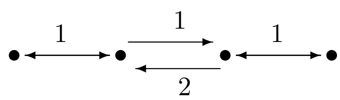
The vertices of Γ_{DD} are the same as the vertices of DD (hence they are in one-one correspondence with the set of simple roots).
Two vertices α and β of Γ_{DD} are connected by an arrow (pointing from α to β) labelled with the integer $n_{\alpha,\beta} = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$.

In the simply laced case, the labelled directed graph Γ_{DD} is just the Dynkin diagram DD, with all the edges labelled with one. In the non-simply laced cases, Γ_{DD} is given by:

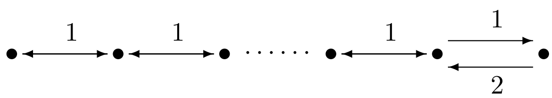
G_2



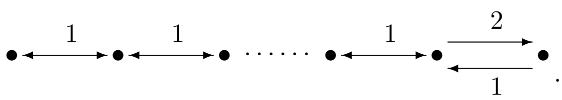
F_4



B_n



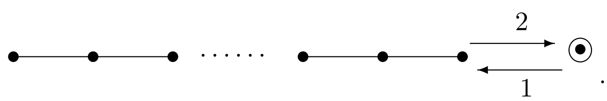
C_n



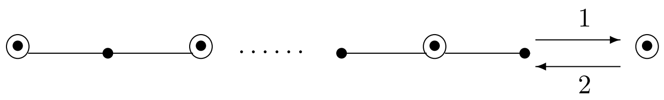
Next, we define the Dynkin diagram R-group R_{DD}:

An element of R_{DD} is a set of mutually disjoint vertices of Γ_{DD}, s.t. for any vertex $\gamma \notin S$ the sum of the labels on arrows going out of γ and into elements of S is ***even****, i.e. $\sum_{\alpha\in S} n_{\gamma,\alpha} \equiv 0 \mod 2$.*
R_{DD} is made into a group with the operation of symmetric difference of sets.

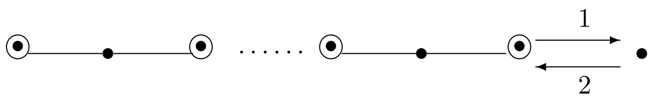
It is easy to check that the R-group is trivial for DD of type G_2 and F_4.
For DD of type C_n, the R-group is isomorphic to $\mathbb{Z}_2$, and is generated by



Finally, we discuss the case in which DD is of type B_n.
The R-group of B_n is always of order two, but the non-trivial element depends on the parity of n. More precisely, we can take



as a generator the R-group of B_{2m+1}, and



as a generator for the R-group of B_{2m}.

C.3 The relation between R_{DD} and R_δ

Let G be a simple split real group, and let DD be its Dynkin diagram. Let $P = MAN$ be the Langlands decomposition of a minimal parabolic subgroup of G. For any representation δ of M, we consider the R-group R_δ associated to δ.

Theorem 1. *R_δ is always a subgroup of the Dynkin diagram R_{DD}.*
In particular, R_δ equals R_{DD} when δ is maximally bad.

We recall that a representation of M is called "maximally bad" if all the *simple* roots are bad. For instance, the representation $\delta_{\{1,3,5,7\}} = \delta_{\{2,4,6,8\}}$ of $M \subset SL(8)$ is maximally bad. So is the representation $\delta_{\{1,3,5\}}$ of $M \subset SP(10)$. If the group G is connected, then M has *at most one* maximally bad representation (because M is generated by the m_αs, for α simple). Sometimes M has *no* maximally bad representations at all.[4]

In the previous section we have shown that the R-group Dynkin diagram is trivial for DD of type A_{2n}, E_6, E_8, F_4, G_2. In any other case, R_{DD} is a finite abelian group of order two or four. As an immediate consequence, we obtain:

Corollary. *For SL_{2n+1}, E_6, E_8, F_4, and G_2, R_δ is always trivial.*[5]
For types A_{2n+1}, B_n, C_n, D_{2n+1} and E7, R_δ has cardinality at most two.
For type D_{2n}, R_δ is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

[4] This is the case for $SO_0(n+1,n)$.
[5] For the other split groups, it has order at most four.

Appendix D

Minimal Principal Series for Split Groups

Let G be a real split semisimple Lie group.

D.1 Minimal Principal Series

Fix a Cartan involution θ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}(G)$. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and set:

- $M = Z_K(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in K
- $A = \exp_G(\mathfrak{a})$ the vector subgroup of G with Lie algebra $\mathfrak{a}$
- $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ the set of restricted roots.

Notice that M is finite when G is split, and is abelian when G is linear.
By construction, $MA = Z_G(\mathfrak{a})$ is the Levi factor of a minimal parabolic subgroup of G. Suppose that $(\delta, \mathbb{C}_\delta)$ is an irreducible (tempered unitary) representation of M, and that ν is a character of A. Choose a minimal parabolic subgroup $P = MAN$ so that $\Re(\nu)$ is weakly dominant for the roots of A in N.[1]
You can of course regard $\delta \otimes \nu$ as a representation of P, with N acting trivially. The induced representation

$$X_P(\delta, \nu) = \mathrm{Ind}_P^G(\delta \otimes \nu)$$

is called **a minimal principal series for** G.
$X_P(\delta, \nu)$ is the representation of G by left translation on the space of functions

$$\mathcal{H}^P_{\delta\otimes\nu} = \{F \colon G \to \mathbb{C}_\delta \colon \mathrm{Res}_K(F) \in L^2(K, \mathbb{C}_\delta) \text{ and}$$

[1] Equivalently, choose a positive system Δ^+ in Δ such that

$$\langle \Re(\nu), \alpha \rangle \geq 0 \quad \forall\, \alpha \in \Delta^+$$

and set $N = \exp_G(\bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha)$.

$$F(gman) = e^{-(\nu+\rho)\log(a)}\delta(m)^{-1}F(g),\ \forall\, man \in P = MAN,\ \forall\, g \in G\}.$$

Remark. *The choice of P is unique only when $Re(\nu)$ is non singular (i.e. $\Re(\nu)$ is strictly dominant for the roots of A in N). The induced representation $Ind_P^G(\delta \otimes \nu)$ is independent of this choice.*

Proof. The fist step is to identify all the minimal parabolic subgroups of G with Levi factor MA for which $\Re(\nu)$ is weakly dominant.
Partition the restricted roots according to their inner product with $\Re(\nu)$: $\Delta = \Delta_L \sqcup \Delta_U^+ \sqcup \Delta_U^-$, with

$$\Delta_L = \{\alpha \in \Delta \colon \langle \Re(\nu),\, \alpha\rangle = 0\}$$

$$\Delta_U^+ = \{\alpha \in \Delta \colon \langle \Re(\nu),\, \alpha\rangle > 0\}$$

$$\Delta_U^- = \{\alpha \in \Delta \colon \langle \Re(\nu),\, \alpha\rangle < 0\}.$$

The set Δ_L is a root system, and every positive system Δ^+ (in Δ) making $\Re(\nu)$ weakly dominant is of the form

$$\Delta^+ = \Delta_L^+ \sqcup \Delta_U^+$$

for some choice of a positive system Δ_L^+ (in Δ_L).
Denote by L the centralizer of $\Re(\nu)$ in G. Then L contains MA and has Lie algebra

$$\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Delta_L} \mathfrak{g}_\alpha$$

($\mathfrak{m} = \{0\}$ in the split case).
Any choice of Δ_L^+ determines a minimal parabolic subgroup of L containing MA, say $P_L = MAN_L$, and the map

$$P_L = MAN_L \longleftrightarrow P = P_L U = MA(N_L U)$$

gives a one-one correspondence between the set of arbitrary minimal parabolics in L containing MA and the set of minimal parabolics in G making $\Re(\nu)$ weakly dominant.
Let us continue with the proof of the claim. By induction by stages,

$$\mathrm{Ind}_P^G(\delta \otimes \nu) = \mathrm{Ind}_Q^G\left(\mathrm{Ind}_{P_L}^Q(\delta \otimes \nu)\right)$$

where $Q = LU$ is the (non-minimal) parabolic subgroup of G defined by $\Re(\nu)$.[2]
Then, because Q is canonically attached to ν, in order to prove that the minimal principal series $X_P(\delta, \nu)$ is independent of the choice of P, we only have to show

[2] The Lie algebra of Q is

$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\colon\, \langle\alpha,\Re(\nu)\rangle \geq 0} \mathfrak{g}_\alpha = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Delta_L} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\Delta_U^+} \mathfrak{g}_\alpha.$$

that $\mathrm{Ind}_{P_L}^{Q}(\delta\otimes\nu)$ is independent of P_L. This is easier to do, because $\mathrm{Ind}_{P_L}^{Q}(\delta\otimes\nu)$ is unitarily induced.[3]
An explicit computation shows that the character of the unitary representation $\mathrm{Ind}_{P_L}^{Q}(\delta\otimes\nu)$ is independent of the choice of P_L, then the result follows from the fact that two unitary representations with the same character are isomorphic. $\square$

D.2 Langlands quotient

For simplicity, assume ν to be **real**.
Let MA be the Levi factor of a minimal parabolic subgroup of G, let δ be an irreducible tempered unitary representation of M and let ν be a character of A. Choose *any* minimal parabolic subgroup $P = MAN$ making ν weakly dominant, and let $\bar{P} = MA\bar{N}$ be the opposite parabolic. The representation $\delta\otimes\nu$ of MA can be regarded as a representation of both P and $\bar{P}$. Let us denote by

$$X_{quo}(\delta,\nu) = \mathrm{Ind}_{P}^{G}(\delta\otimes\nu)$$

and

$$X_{sub}(\delta,\nu) = \mathrm{Ind}_{\bar{P}}^{G}(\delta\otimes\nu)$$

the corresponding induced representations of G.
When ν is strictly dominant, there is an intertwining operator

$$A = A(\bar{P}\colon P\colon \delta\colon \nu)\colon X_{quo}(\delta,\nu) \longrightarrow X_{sub}(\delta,\nu)$$

defined by the convergent integral:

$$[A(\bar{P}\colon P\colon \delta\colon \nu)F](x) = \int_{\bar{N}} F(x\bar{n})\,d\bar{n}. \tag{D.1}$$

When ν is weakly dominant, the integral in $(D.1)$ does not necessarily converge. To obtain a convergent integral we need to integrate on the smaller subgroup

$$\bar{U} = \exp\left(\bigoplus_{\alpha\in\Delta_U^-}\mathfrak{g}_\alpha\right) \subseteq \bar{N}.$$

The integral

$$[A(\bar{P}\colon P\colon \delta\colon \nu)F](x) = \int_{\bar{U}} F(x\bar{n})\,d\bar{n} \tag{D.2}$$

converges absolutely for all continuous functions F in $\mathcal{H}^{P}_{\delta\otimes\nu}$, so we still have an intertwining operator from $X_{quo}(\delta\otimes\nu)$ to $X_{sub}(\delta\otimes\nu)$.
Define **the Langlands quotient representation** to be the closure of the image of this operator:

$$\bar{X}(\delta,\nu) = \overline{\mathrm{Im}(A(\bar{P}\colon P\colon \delta\colon \nu))}.$$

[3] By construction, $\langle\Re(\nu),\alpha\rangle = 0$ for the roots of A in N_L, so ν is imaginary and $\mathrm{Ind}_{P_L}^{Q}(\delta\otimes\nu)$ is unitary.

It is clear that $\bar{X}(\delta,\nu)$ is a subrepresentation of $X_{sub}(\delta,\nu)$ and a quotient of $X_{quo}(\delta,\nu)$. According to Langlands and Milicic, it is actually the largest completely reducible subrepresentation of $X_{sub}(\delta,\nu)$ and the largest completely reducible quotient of $X_{quo}(\delta,\nu)$.

D.3 Reducibility

Remark. *The Langlands quotient $\bar{X}(\delta,\nu)$ may be reducible.*

Proof. Because $\bar{X}(\delta,\nu)$ is the closure of the image of

$$X_{quo}(\delta,\nu) = \mathrm{Ind}_P^G(\delta\otimes\nu) = \mathrm{Ind}_Q^G\left(\mathrm{Ind}_{P_L}^Q(\delta\otimes\nu)\right)$$

via the long intertwining operator, we start by discussing the reducibility of the unitary representation $\mathrm{Ind}_{P_L}^Q(\delta\otimes\nu)$.
By deep results of Harish-Chandra and Knapp-Stein, the number of irreducible constituents of $\mathrm{Ind}_{P_L}^Q(\delta\otimes\nu)$ is equal to the order of the R-group $R(\delta,\nu)$, and these constituents are all distinct. Let

$$\mathrm{Ind}_{P_L}^Q(\delta\otimes\nu) = \bigoplus_{i=1}^{|R(\delta,\nu)|} X_L^i$$

be the decomposition of $\mathrm{Ind}_{P_L}^Q(\delta\otimes\nu)$ as a direct sum of irreducible representations. We get:

$$X_{quo}(\delta,\nu) = \bigoplus_{i=1}^{|R(\delta,\nu)|} \mathrm{Ind}_Q^G\left(X_L^i\right)$$

and

$$\bar{X}(\delta,\nu) = \frac{X_{quo}(\delta,\nu)}{\ker(A)} = \bigoplus_{i=1}^{|R(\delta,\nu)|} \frac{\mathrm{Ind}_Q^G\left(X_L^i\right)}{\ker(A)\cap\mathrm{Ind}_Q^G\left(X_L^i\right)}.$$

By construction the space

$$\bar{X}^i(\delta,\nu) = \frac{\mathrm{Ind}_Q^G\left(X_L^i\right)}{\ker(A)\cap\mathrm{Ind}_Q^G\left(X_L^i\right)}$$

is the largest completely reducible quotient of $\mathrm{Ind}_Q^G\left(X_L^i\right)$.
Langlands has proved that each $\bar{X}^i(\delta,\nu)$ is irreducible, so the decomposition

$$\bar{X}(\delta,\nu) = \bigoplus_{i=1}^{|R(\delta,\nu)|} \bar{X}^i(\delta,\nu) \tag{D.3}$$

exhibits $\bar{X}(\delta,\nu)$ as a direct sum of irreducible representations.
Equation ($D.3$) also shows that the reducibility of the Langlands quotient $\bar{X}(\delta,\nu)$ comes entirely from the reducibility of the unitarily induced representation $\mathrm{Ind}_{P_L}^Q(\delta\otimes\nu)$. □

Remark 7. *When G is split, the number of irreducible summands of the unitarily induced representation* $\mathrm{Ind}_{P_L}^{Q}(\delta\otimes\nu)$ *is equal to the cardinality of the R-group $R_\delta(\nu)$. We define*

$$R_\delta(\nu) = \frac{W^\delta(\nu)}{W^0_\delta(\nu)} = \frac{\{w \in W^\delta : w\cdot\nu = \nu\}}{\{w \in W^0_\delta : w\cdot\nu = \nu\}}.$$

For an explicit example of how to compute the number of Langlands quotients, see section $(F.2)$.

How about the reducibility of the (minimal) principal series $X_{quo}(\delta,\nu) = \mathrm{Ind}_P^G(\delta\otimes\nu)$? Because $\bar{X}(\delta,\nu) = \frac{X_{quo}(\delta,\nu)}{\ker(A)}$, reducibility can occur if and only if

(i) the Langlands quotient is reducible

(ii) the intertwining operator A has a kernel.

Of course these two conditions can happen at the same time.
For minimal principal series in split groups these conditions are equivalent to:

$(i)'$ the R-group $R_\delta(\nu)$ is non-trivial

$(ii)'$ there is a root α such that the inner product $\langle\alpha^\vee,\nu\rangle$ is a non-zero integer k, and

$$(-1)^{k+1} = \delta(m_\alpha).$$

This parity condition means that $\langle\alpha^\vee,\nu\rangle$ should be an odd integer if α is a good root for δ (i.e. $\delta(m_\alpha) = +1$), and an even integer if α is a bad.[4]

[4] For a motivation of this reducibility condition, see section 4.2 in Vogan's book "Representations of Real Reductive Lie Groups".

Appendix E

The operator $R_\mu(s_\alpha,\, \gamma)$

E.1 Preliminary remarks

Lemma 6. *Let α be a restricted root and let σ_α be a representative in $M' = N_K(\mathfrak{a})$ for the root reflection s_α. For (μ, E_μ) in $\hat{K}$ and (δ, V^δ) in $\hat{M}$, consider the operator*

$$\Psi_\alpha\colon\ \mathrm{Hom}_M(E_\mu\mid_M,\, V^\delta) \to \mathrm{Hom}_M(E_\mu\mid_M,\, V^{s_\alpha\cdot\delta}),\ T \mapsto T\circ\mu(\sigma_\alpha^{-1}).$$

Ψ_α is well defined and can be computed as follows. Let K^α be the $SO(2)$ sugbroup attached to α, and let

$$E_\mu = \bigoplus_{n\in\mathbb{Z}} \phi_n$$

be the decomposition of μ in isotypic components of irreducible representations of K^α. Then

$$(\Psi_\alpha T)\mid_{\phi_n} = (-i)^n T\mid_{\phi_n}$$

for all T in $\mathrm{Hom}_M(E_\mu\mid_M,\, V^\delta)$ and n in $\mathbb{Z}$.

Proof. In order to show that Ψ_α is well defined, we prove that the homomorphism

$$T\circ\mu(\sigma_\alpha^{-1})\colon E_\mu \longrightarrow V^{s_\alpha\cdot\delta}$$

is invariant under the action of M.[1]
By assumption, T is a map from E_μ to V^δ with the property that

$$\delta(m_1)\cdot T(\mu(m_1^{-1})v) = T(v)$$

for all v in E_μ and all m_1 in M. Then

[1] An element m of M acts on the domain by $\mu(m^{-1})$ and on the codomain by $(s_\alpha\cdot\delta)(m)$. So $T\circ\mu(\sigma_\alpha^{-1})$ is invariant under M if and only if

$$(s_\alpha\cdot\delta)(m)\cdot(T\circ\mu(\sigma_\alpha^{-1}))(\mu(m^{-1})v) = (T\circ\mu(\sigma_\alpha^{-1}))(v)$$

for all m in M and all v in E_μ.

$$(s_\alpha \cdot \delta)(m) \cdot (T \circ \mu(\sigma_\alpha^{-1}))(\mu(m^{-1})v) =$$

$$= \delta(\underbrace{\sigma_\alpha^{-1} m \sigma_\alpha}_{m_1}) T(\underbrace{\mu(\sigma_\alpha^{-1})\mu(m^{-1})\mu(\sigma_\alpha)}_{\mu(m_1^{-1})} \mu(\sigma_\alpha^{-1})v) =$$

$$= \delta(m_1) \cdot T(\mu(m_1^{-1})\mu(\sigma_\alpha^{-1})v) = T(\mu(\sigma_\alpha^{-1})v) = (T \circ \mu(\sigma_\alpha^{-1}))(v). \quad \checkmark$$

The rest of the lemma follows from the fact that $\mu(\sigma_\alpha^{-1}) = \mu\left(\exp(-\frac{\pi}{2} Z_\alpha)\right)$ acts by

$$\exp(-n\frac{\pi}{2}i) = i^{-n} = (-i)^n$$

on Φ_n. Indeed, Φ_n is the isotypic component of the character χ_n of $SO(2)$. □

Corollary. *Let Ψ_α be the operator defined above, and let T be an element of* $\mathrm{Hom}_M(E_\mu\,|_M, V^\delta)$.

- *If v belongs to $\phi_{2k} + \phi_{-2k}$, then $(\Psi_\alpha T)(v) = (-1)^k T(v)$*
- *If $v = v_+ + v_-$ belongs to $\phi_{(2k+1)} + \phi_{-(2k+1)}$, then*

$$(\Psi_\alpha T)(v_+ + v_-) = i(-1)^{k+1} T(v_+ - v_-)$$

Remark 8. *The operator Ψ_α is not uniquely defined when α is a bad root for δ.*

Proof. Indeed, the element $\sigma_\alpha = \exp(\frac{\pi}{2} Z_\alpha) = \exp(\frac{\pi}{2}(E_\alpha + \theta E_\alpha))$ depends on the choice of E_α. Here E_α is any non-zero element of the α-root space satisfying the normalizing condition

$$B(E_\alpha, \theta E_\alpha) = -2/\|\alpha\|^2. \qquad (\diamondsuit)$$

Because G is assumed to be split, the α-root space is one-dimensional and the condition ($\diamondsuit$) determines E_α uniquely, up to a sign. The element σ_α is therefore defined only up inverse.
When α is a good root for δ, this ambiguity does not affect the operator Ψ_α, because

$$\mathrm{Hom}_M(E_\mu\,|_M, V^\delta) = \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_M(\phi_{2n} + \phi_{-2n}, V^\delta)$$

and the elements σ_α and σ_α^{-1} act in the same way on the even character of $SO(2)$. On the contrary, when α is bad for δ, the decomposition of $\mathrm{Hom}_M(E_\mu\,|_M, V^\delta)$ only involves odd characters of $SO(2)$ and the elements σ_α and σ_α^{-1} act with opposite sign on the odd characters of $SO(2)$. Therefore, when α is a bad root, choosing $-E_\alpha$ instead of E_α has the effect of replacing Ψ_α with $-\Psi_\alpha$. □

The following pictures are meant to illustrate the action of the operator Ψ_α on the space $Hom_M(\phi_0, V^\delta)$.
If α is a good root for δ, then

$$\mathrm{Hom}_M(E_\mu\,|_M,\, V^\delta) = \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}_M(\phi_{2n} + \phi_{-2n},\, V^\delta)$$

and Ψ_α acts by:

$$\begin{tikzcd}
Hom_M(\phi_0, V^\delta) \arrow[r, dash] \arrow[d, "+1"] & Hom_M(\phi_2 + \phi_{-2}, V^\delta) \arrow[r, dash] \arrow[d, "-1"] & Hom_M(\phi_4 + \phi_{-4}, V^\delta) \arrow[r, dash] \arrow[d, "+1"] & \bullet \arrow[d, "-1"] & \cdots\cdots \\
Hom_M(\phi_0, V^\delta) \arrow[r, dash] & Hom_M(\phi_2 + \phi_{-2}, V^\delta) \arrow[r, dash] & Hom_M(\phi_4 + \phi_{-4}, V^\delta) \arrow[r, dash] & \bullet & \cdots\cdots
\end{tikzcd}$$

If α is a bad root for δ, then we can write

$$\mathrm{Hom}_M(E_\mu\,|_M,\, V^\delta) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_M(\phi_{2n+1},\, V^\delta)$$

with Ψ_α acting by:

$$\begin{tikzcd}
\cdots\cdots & T\,|_{\varphi_{-3}} \arrow[r, dash] \arrow[d, "-i"] & T\,|_{\varphi_{-1}} \arrow[r, dash] \arrow[d, "+i"] & T\,|_{\varphi_{+1}} \arrow[r, dash] \arrow[d, "-i"] & T\,|_{\varphi_{+3}} \arrow[d, "+i"] & \cdots\cdots \\
\cdots\cdots & \Psi_\alpha T\,|_{\varphi_{-3}} \arrow[r, dash] & \Psi_\alpha T\,|_{\varphi_{-1}} \arrow[r, dash] & \Psi_\alpha T\,|_{\varphi_{+1}} \arrow[r, dash] & \Psi_\alpha T\,|_{\varphi_{+3}} & \cdots\cdots
\end{tikzcd}$$

Lemma 7. *The mapping $[\sigma_\alpha] \mapsto \Psi_\alpha$ defines a a representation of the Weyl group of the good roots on the space* $\mathrm{Hom}_M(E_\mu\,|_M,\, V^\delta)$.

Proof. The key point here is that if α is a good root, then $s_\alpha \cdot \delta = \delta$ and Ψ_α becomes an automorphism of $\mathrm{Hom}_M(E_\mu\,|_M,\, V^\delta)$.
Because

$$[\sigma_\alpha\sigma_\beta] \cdot T = T \circ \mu((\sigma_\alpha\sigma_\beta)^{-1}) = (T \circ \mu(\sigma_\beta^{-1})) \circ \mu(\sigma_\alpha^{-1}) = \Psi_\alpha(\Psi_\beta T)$$

for every pair of good root, the mapping $[\sigma_\alpha] \mapsto \Psi_\alpha$ extends to a homomorphism of W_δ^0 into $\mathrm{Aut}(\mathrm{Hom}_M(E_\mu\mid_M, V^\delta))$. The result is a representation of W_δ^0 on $\mathrm{Hom}_M(E_\mu\mid_M, V^\delta)$, defined by the formula:

$$([\sigma]\cdot T)(v) = T(\mu(\sigma^{-1})v) \tag{E.1}$$

for all σ in W_δ, all T in $\mathrm{Hom}_M(E_\mu\mid_M, V^\delta)$ and all v in E_μ. □

Remark 9. *Equation (E.1) also defines a representation of the stabilizer of δ on the space* $\mathrm{Hom}_M(E_\mu\mid_M, V^\delta)$.

E.2 The operator $R_\mu(s_\alpha, \gamma)$ for α simple

We introduce some notations:

- $K^\alpha = \exp(\mathbb{R}Z_\alpha)$ is the $SO(2)$ subgroup attached to α
- $\chi_l\colon \exp(tZ_\alpha) \mapsto e^{ilt}$ is the l^{th} character of K^α
- ϕ_l is the isotypic component of χ_l inside μ, so that $E_\mu = \bigoplus_{l\in\mathbb{Z}} \phi_l$ is the decomposition of E_μ in K^α-stable subspaces.

In this section we compute the operator $R_\mu(s_\alpha, \gamma)$, for every α simple. For T in $\mathrm{Hom}_M\left(E_\mu\mid_M, V^\delta\right)$ and v in ϕ_l, we have:

$$\begin{aligned}
(R_\mu(s_\alpha, \gamma)T)(v) &= \textstyle\int_{\bar N^\alpha} e^{-(\rho^\alpha+\gamma|_{\mathfrak{a}^\alpha})(H^\alpha(\bar n))} T((\sigma_\alpha\kappa^\alpha(\bar n))^{-1}\cdot v)\, d\bar n = \\
&= \textstyle\int_{\bar N^\alpha} e^{-(\rho^\alpha+\gamma|_{\mathfrak{a}^\alpha})(H^\alpha(\bar n))} T\left(\chi_{+l}(\sigma_\alpha\kappa^\alpha(\bar n))^{-1} v\right)\, d\bar n = \\
&= \left[\textstyle\int_{\bar N^\alpha} e^{-(\rho^\alpha+\gamma|_{\mathfrak{a}^\alpha})(H^\alpha(\bar n))} \chi_{+l}(\sigma_\alpha\kappa^\alpha(\bar n))^{-1} d\bar n\right] T(v).
\end{aligned}$$

To proceed we need to understand the Iwasawa decomposition of an element $\bar n$ of $\bar N$. Let us compute such decomposition. Because G is split, the space

$$\mathfrak{g}^\alpha \equiv \mathbb{R}H_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

is three-dimensional. Let E_α be a non-trivial element of $\mathfrak{g}_\alpha$ subject to the normalizing condition $B(E_\alpha, \theta E_\alpha) = \frac{+2}{\|\alpha\|^2} H_\alpha$. Then θE_α is a generator of $\mathfrak{g}_{-\alpha}$ and the mapping

$$\psi_\alpha\colon \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}^\alpha = \mathbb{R}H_\alpha + \mathbb{R}E_\alpha + \mathbb{R}\theta(E_\alpha)$$

defined by:

$$\underline{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto E_\alpha, \quad \underline{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto -\theta(E_\alpha), \quad \underline{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \frac{+2}{\|\alpha\|^2} H_\alpha$$

is an isomorphism between $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{g}^\alpha$.
When G has a complexification[2], ψ_α lifts to a group homomorphism

$$\Psi_\alpha\colon SL(2,\mathbb{R}) \to G^\alpha .$$

We can therefore "induce" the Iwasawa decomposition from $SL(2,\mathbb{R})$ to G^α:

$$\bar n = \exp(t\,\theta(E_\alpha)) = \exp(-t\,\psi_\alpha(\underline{f})) = \Psi_\alpha(\exp(-t\,\underline{f})) = \Psi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -t & 1\end{pmatrix}\right) =$$

$$= \Psi_\alpha\left(\begin{pmatrix} \cos(\arctan(t)) & \sin(\arctan(t)) \\ -\sin(\arctan(t)) & \cos(\arctan(t))\end{pmatrix}\begin{pmatrix} \sqrt{1+t^2} & 0 \\ 0 & 1/\sqrt{1+t^2}\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\right) =$$

$$= \Psi_\alpha\left(\exp(\arctan(t)\,(\underline{e}-\underline{f})\right)\,\Psi_\alpha\left(\exp(\ln(\sqrt{1+t^2})\,\underline{h})\right)\,\Psi_\alpha\left(\exp(x\,\underline{e})\right) =$$

$$= \exp\left(\psi_\alpha(\arctan(t)\,(\underline{e}-\underline{f}))\right)\,\exp\left(\psi_\alpha(\ln(\sqrt{1+t^2})\,\underline{h})\right)\,\exp\left(\psi_\alpha(x\,\underline{e})\right) =$$

$$= \underbrace{\exp\left(\arctan(t)\,Z_\alpha\right)}_{\kappa^\alpha(\bar n)}\,\exp\left(\underbrace{\ln(\sqrt{1+t^2})\,\frac{2}{\|\alpha\|^2}H_\alpha}_{H^\alpha(\bar n)}\right)\,\exp\left(x\,E_\alpha\right).$$

Therefore

- $\rho^\alpha(H^\alpha(\bar n)) = \frac{1}{2}\alpha\left(\ln(\sqrt{1+t^2})\,\frac{2}{\|\alpha\|^2}H_\alpha\right) = \ln(\sqrt{1+t^2})$

- $\gamma\,|_{\mathfrak{a}^\alpha}\,(H^\alpha(\bar n)) = \gamma\left(\ln(\sqrt{1+t^2})\,\frac{2}{\|\alpha\|^2}H_\alpha\right) = \ln(\sqrt{1+t^2})\,\langle\gamma,\,\frac{2}{\|\alpha\|^2}\alpha\rangle =$
  $= \ln(\sqrt{1+t^2})\,\langle\gamma,\,{}^\vee\alpha\rangle$

---

[2] This is always the case if G is semi-simple. Indeed every adjoint group has a complexification: if $G = \mathrm{Ad}\,\mathfrak{g}$, you can take $G^{\mathbb{C}}$ to be $\mathrm{Ad}(\mathfrak{g}^{\mathbb{C}})$.
It also true, more generally, if the group G is real reductive and satisfies the condition

$$Z(G)\cap K = \{1\}.$$

Indeed, if $G = K\,\exp(\mathfrak{p}_0)$ is the Cartan decomposition of G, and ζ is the center of the Lie algebra of G (so that $\mathfrak{g} = [\mathfrak{g},\,\mathfrak{g}]\oplus\zeta$), then we can write

$$G = \underbrace{K\,\exp(\mathfrak{p}_0\cap[\mathfrak{g},\,\mathfrak{g}])}_{G^1}\,\underbrace{\exp(\mathfrak{p}_0\cap\zeta)}_{Z^1}. \qquad (\circledast)$$

with G^1 real reductive (of the same rank as G) and Z^1 a vector group included in the center. Because $Z(G^1) = Z(G)\cap K = \{1\}$, the group G^1 is actually semi-simple. So $(\circledast)$ is a decomposition of G as a direct product of an adjoint group and a vector group, both of which have a complexification. As a result, we obtain a complexification for G.
Finally, we notice that $Z(G)$ acts by scalars on any irreducible representation of G (this is Schur's lemma), and that $Z(G)\cap K$ acts trivially on the trivial K-type included in any irreducible spherical representation ρ of G (hence on the whole representation space E_ρ). So, when dealing with spherical representations, we can assume w.l.o.g. that the condition $Z(G)\cap K = \{1\}$ is satisfied.

- $\chi_l(\sigma_\alpha \kappa^\alpha(\bar{n}))^{-1} = \chi_l\left(\exp\left(-(\frac{\pi}{2} + \arctan(t))Z_\alpha\right)\right) = e^{-l\,i\,\frac{\pi}{2}} e^{-l\,i\,[\arctan(t)]} =$
  $= e^{-l\,i\,\frac{\pi}{2}} \left(\frac{1+i\,t}{\sqrt{1+t^2}}\right)^{-l}$, for all l in $\mathbb{Z}$.

Let us go back to the computation of $R_\mu(s_\alpha,\, \gamma)T(v)$.

$R_\mu(s_\alpha,\, \gamma)T(v) = \left[\int_{\bar{N}^\alpha} e^{-(\rho^\alpha + \gamma|_{\mathfrak{a}^\alpha})(H^\alpha(\bar{n}))} \chi_{+l}(\sigma_\alpha \kappa^\alpha(\bar{n}))^{-1} d\bar{n}\right] T(v) \;=$

$= \left[e^{-l\,i\,\frac{\pi}{2}} \int_{\mathbb{R}} (\sqrt{1+t^2})^{-(1+\langle\gamma,\,{}^\vee\alpha\rangle)} \left(\frac{1+i\,t}{\sqrt{1+t^2}}\right)^{-l} dt\right] T(v) =^3$

$= \left[e^{-l\,i\,\frac{\pi}{2}} \int_{\mathbb{R}} (\sqrt{1+t^2})^{-(1+\langle\gamma,\,{}^\vee\alpha\rangle)} \left(\frac{1-i\,t}{\sqrt{1+t^2}}\right)^{-l} dt\right] T(v) =^4$

$= \left[e^{-l\,i\,\frac{\pi}{2}} \int_{-\pi/2}^{\pi/2} (\cos\theta)^{1+\lambda}\, e^{l\,i\,\theta}\, \frac{1}{(\cos\theta)^2}\, d\,\theta\right] T(v)$

$= \left[e^{-l\,i\,\frac{\pi}{2}} \int_{-\pi/2}^{\pi/2} (\cos\theta)^{\lambda-1}\, e^{l\,i\,\theta}\, d\,\theta\right] T(v) =^5$

$= \left[e^{-l\,i\,\pi} \int_0^{\pi} (\sin\, x)^{\lambda-1}\, e^{l\,i\,x}\, d\,x\right] T(v)$

$= \left[\frac{\pi\,\Gamma(\lambda)\, e^{-il\frac{\pi}{2}}}{2^{\lambda-1}\,\Gamma\left(1+\frac{\lambda+l-1}{2}\right)\Gamma\left(1+\frac{\lambda-l-1}{2}\right)}\right] T(v).$

The last equality follows from the following result:

$$\int_0^{\pi} (\sin\, t)^a\, e^{i\,b\,t}\, d\,t = \frac{\pi\,\Gamma(1+a)\, e^{i\,\pi\,b/2}}{2^a\,\Gamma(1+\frac{a+b}{2})\,\Gamma(1+\frac{a-b}{2})}$$

for each b in $\mathbb{R}$, and for each a in $\mathbb{C}$ such that $\mathrm{Re}(a) > -1$.[6]

For brevity of notations, we set

$$d_l = \left[\frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(1+\frac{\lambda-l-1}{2}\right)\,\Gamma\left(1+\frac{\lambda+l-1}{2}\right)}\right].$$

Then

$$R_\mu(s_\alpha,\, \gamma)T(v) = (-i)^l d_l\, T(v)$$

[3] Perform the change of variable ($t \mapsto -t$).

[4] Apply the change variable $\theta \to x = \theta + \pi/2$, which gives:

$\sqrt{1+t^2} = \frac{1}{\cos\theta}$

$\frac{1-i\,t}{\sqrt{1+t^2}} = \cos\theta + i\,\tan\theta\,\cos\theta = e^{i\,\theta}$

$d\,t = -\frac{1}{(\cos\theta)^2}\, d\,\theta.$

[5] Another change of variable $\theta \to x = \theta + \pi/2$.

[6] See e.g. [?].

for all T in $\mathrm{Hom}_M\,(E_\mu\,|_M,\,\mathbb{C})$ and all v in ϕ_l.
The next task is to give a more explicit description of d_l, and to do so we must distinguish between the even and the odd case.[7]

The case $l = 2n$, $n \geq 0$

It is convenient to introduce the constant

$$D = d_0 = \frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda+1}{2}\right)\,\Gamma\left(\frac{\lambda+1}{2}\right)}$$

and to look at the normalized coefficients:

$$\frac{1}{D}\,d_{2n} = \frac{\Gamma\left(\frac{\lambda+1}{2}\right)\,\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}+n\right)\,\Gamma\left(\frac{\lambda+1}{2}-n\right)}.$$

To simplify this expression we recall the factorization property of the Γ function

$$\Gamma(z+1) = z\,\Gamma(z)$$

and we introduce the notation

$$(z)_n = z(z+1)(z+2)\cdots(z+n-1)$$

for each z in $\mathbb{C}$, and for every positive integer n. Then

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(z+n)\,\Gamma(z-n)} = \frac{\Gamma(z)\,(z-n)_n\,\Gamma(z-n)}{(z)_n\,\Gamma(z)\,\Gamma(z-n)} = \frac{(z-n)_n}{(z)_n} = \frac{(z-1)(z-2)\cdots(z-n)}{z(z+1)\cdots(z+n-1)}.$$

Setting $z = \frac{\lambda+1}{2}$, we find:

$$\frac{1}{D}\,d_{2n} = \frac{(\lambda-1)(\lambda-3)\cdots(\lambda-2n+1)}{(\lambda+1)(\lambda+3)\cdots(\lambda+2n-1)} = (-1)^n\frac{(1-\lambda)(3-\lambda)\cdots(2n-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2n-1+\lambda)}.$$

It follows that

$$R_\mu(s_\alpha,\,\gamma)T(v) = (-i)^{2n}d_{2n}\,T(v) = \frac{(1-\lambda)(3-\lambda)\cdots(2n-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2n-1+\lambda)}\,T(v)$$

for all T in $\mathrm{Hom}_M\,(E_\mu\,|_M,\,\mathbb{C})$ and all v in ϕ_{2n}. Same result for $v \in \phi_{2n}$, because $(-i)^{-2n}d_{-2n} = (-1)^n d_{-2n} = (-1)^n d_{2n} = (-i)^{2n}d_{2n}$.

The case $l = 2n+1$, $n \geq 0$

We introduce the constant

$$D' = d_1 = (-i)\frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda}{2}\right)\,\Gamma\left(\frac{\lambda}{2}+1\right)},$$

[7] Because $d_{+l} = d_{-l}$, we assume $l \geq 0$.

and consider the normalized coefficients:

$$\frac{1}{D'}\,d_{2n+1}=(+i)\frac{\Gamma\left(\frac{\lambda}{2}\right)\,\Gamma\left(\frac{\lambda}{2}+1\right)}{\Gamma\left(\frac{\lambda}{2}-n\right)\,\Gamma\left(\frac{\lambda}{2}+1+n\right)}.$$

Using the formulas

$$\frac{\Gamma(z)}{\Gamma(z-n)}=\frac{(z-n)_n\,\Gamma(z-n)}{\Gamma(z-n)}=(z-n)_n=(z-1)(z-2)\cdots(z-n)$$

$$\frac{\Gamma(z')}{\Gamma(z'+n)}=\frac{\Gamma(z')}{(z')_n\,\Gamma(z')}=\frac{1}{z'(z'+1)\cdots(z'+n-1)}$$

for $z=\frac{\lambda}{2}$ and $z'=\frac{\lambda}{2}+1$, we can write:

$$\frac{1}{D'}\,d_{2n+1}=(+i)\frac{(\lambda-2)(\lambda-4)\cdots(\lambda-2n)}{(\lambda+2)(\lambda+4)\cdots(\lambda+2n)}=(-1)^n(+i)\frac{(2-\lambda)(4-\lambda)\cdots(2n-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2n+\lambda)}.$$

Therefore:

$$(R_\mu(s_\alpha,\gamma)T)\mid_{\phi_{2n+1}}=(-i)^{2n+1}d_{2n+1}T\mid_{\phi_{2n+1}}=$$

$$=(-i)^{2n+1}(-1)^n(+i)D'\frac{(2-\lambda)(4-\lambda)\cdots(2n-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2n+\lambda)}\,T\mid_{\phi_{2n+1}}=$$

$$=+D'\frac{(2-\lambda)(4-\lambda)\cdots(2n-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2n+\lambda)}\,T\mid_{\phi_{2n+1}}$$

and

$$(R_\mu(s_\alpha,\gamma)T)\mid_{\phi_{-2n-1}}=(-i)^{-2n-1}d_{-2n-1}T\mid_{\phi_{2n+1}}=(-i)^{-2n-1}d_{2n+1}T\mid_{\phi_{-2n-1}}=$$

$$=(-i)^{-2n-1}(-1)^n(+i)D'\frac{(2-\lambda)(4-\lambda)\cdots(2n-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2n+\lambda)}\,T\mid_{\phi_{-2n-1}}$$

$$=-D'\frac{(2-\lambda)(4-\lambda)\cdots(2n-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2n+\lambda)}\,T\mid_{\phi_{-2n-1}}.$$

Conclusions

Write $E_\mu=\bigoplus_{l\in\mathbb{Z}}\phi_l$ for a decomposition of μ in isotypic components of irreducible representations of the $SO(2)$-subgroup attached to α, and denote by F the (common) vector space for the representations δ and $s_\alpha\cdot\delta$ of M. The intertwining operator

$$R_\mu(s_\alpha,\gamma)\colon\ \mathrm{Hom}_M(E_\mu,\,V^\delta=F)\to\mathrm{Hom}_M(E_\mu,\,V^{s_\alpha\cdot\delta}=F)$$

acts as follows: for every $T\colon E_\mu\to F$ in the domain, $R_\mu(s_\alpha,\gamma)T$ is the unique homomorphism $E_\mu\to F$ such that

- $(R_\mu(s_\alpha,\gamma)T)\mid_{\phi_{2n}}=D\frac{(1-\lambda)(3-\lambda)\cdots(2n-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2n-1+\lambda)}\,T\mid_{\phi_{2n}}$

- $(R_\mu(s_\alpha,\gamma)T)\mid_{\phi_{-2n}}= D\frac{(1-\lambda)(3-\lambda)\cdots(2n-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2n-1+\lambda)}\,T\mid_{\phi_{-2n}}$
- $(R_\mu(s_\alpha,\gamma)T)\mid_{\phi_{2n+1}}= D'\frac{(2-\lambda)(4-\lambda)\cdots(2n-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2n+\lambda)}\,T\mid_{\phi_{2n+1}}$
- $(R_\mu(s_\alpha,\gamma)T)\mid_{\phi_{-2n-1}}= -D'\frac{(2-\lambda)(4-\lambda)\cdots(2n-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2n+\lambda)}\,T\mid_{\phi_{-2n-1}}$

for every integer $n\geq 0$. The constants

$$D=\frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda+1}{2}\right)\,\Gamma\left(\frac{\lambda+1}{2}\right)}$$

and

$$D'=(-i)\frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda}{2}\right)\,\Gamma\left(\frac{\lambda}{2}+1\right)}$$

have been chosen so that

$$(R_\mu(s_\alpha,\gamma)T)\mid_{\phi_0}= D\cdot T\mid_{\phi_0}$$

and

$$(R_\mu(s_\alpha,\gamma)T)\mid_{\phi_1}= D'\cdot T\mid_{\phi_1}\ .$$

For brevity of notations, set:

$$c_{2n}=D\,\frac{(1-\lambda)(3-\lambda)\cdots(2n-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2n-1+\lambda)}\ ;\ c_{-2n}=c_{2n}$$

and

$$c_{2n+1}=D'\frac{(2-\lambda)(4-\lambda)\cdots(2n-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2n+\lambda)}\ ;\ c_{-2n-1}=c_{2n+1}.$$

Then we have the following picture:

$$\begin{tikzcd}
\cdots\cdots & T\mid_{\phi_{-2}} \arrow[r, dash] \arrow[d, "+c_2"] & T\mid_{\phi_{-1}} \arrow[r, dash] \arrow[d, "-c_1"] & T\mid_{\phi_0} \arrow[r, dash] \arrow[d, "+c_0"] & T\mid_{\phi_{+1}} \arrow[r, dash] \arrow[d, "+c_1"] & T\mid_{\phi_{+2}} \arrow[d, "+c_2"] & \cdots\cdots \\
\cdots\cdots & R_\mu(s_\alpha)T\mid_{\phi_{-2}} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{-1}} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_0} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{+1}} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{+2}} & \cdots\cdots
\end{tikzcd}$$

Remark 10. *It is possible to give an even simpler description of the operator $R_\mu(s_\alpha,\gamma)$, if we know whether the root α is good or bad for δ. Indeed, these conditions force an element of $\mathrm{Hom}_M(E_\mu,V^\delta)$ to be trivial on all the ϕ_l with l odd, or on all the ϕ_l with l even respectively.*

E.2.1 The operator $R_\mu(s_\alpha,\, \gamma)$ for α simple and good

For every good root α, the operator

$$R_\mu(s_\alpha,\, \gamma)\colon\, \mathrm{Hom}_M(E_\mu,\, V^\delta) \to \mathrm{Hom}_M(E_\mu,\, V^{s_\alpha\cdot\delta=\delta})$$

is an endomorphism of $\mathrm{Hom}_M(E_\mu,\, V^\delta)$.
Moreover, the decomposition of $\mathrm{Hom}_M(E_\mu,\, V^\delta)$ in MK^α-stable subspaces involves only even characters.
This is the content of the next two lemmas.

Lemma 8. *If α is good for δ, then*

$$\mathrm{Hom}_M(E_\mu,\, V^\delta) = \mathrm{Hom}_M(E_\mu,\, V^{s_\alpha\cdot\delta}).$$

Proof. For every good root α, the reflection s_α stabilizes δ. Indeed the Weyl group of the good co-roots is a (normal) subgroup of the stabilizer of δ. $\square$

Lemma 9. *If α is good for δ, then*

$$\mathrm{Hom}_M(E_\mu,\, V^\delta) = \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}_M(\phi_{2n} + \phi_{-2n},\, V^\delta).$$

Proof. Let n be any integer and let T be an element of $\mathrm{Hom}_M(E_\mu,\, V^\delta)$. We show that the restriction of T to every "odd isotypic" ϕ_{2n+1} is trivial.
Pick v in ϕ_{2n+1}, then

$$T(v) = \delta(m_\alpha)T(\mu(m_\alpha^{-1})v) = \underbrace{\delta(m_\alpha)}_{+Id}T(\underbrace{\chi_{2n+1}(m_\alpha^{-1})v}_{=e^{\pi(2n+1)i}v=-v}) = -T(v)$$

so $T(v)$ must be equal to zero. $\square$

The domain and codomain of $R_\mu(s_\alpha,\, \gamma)$ are now understood.
We already know that

$$(R_\mu(s_\alpha,\, \gamma)T)\mid_{\phi_{\pm 2n}} = c_{2n}\, T\mid_{\phi_{\pm 2n}}$$

for all T in $\mathrm{Hom}_M(\bigoplus_{n\in\mathbb{Z}} \phi_{2n},\, V^\delta)$, and all $n \geq 0$. So the action of $R_\mu(s_\alpha,\, \gamma)$ is given by:

$$\begin{tikzcd}
\cdots\cdots \arrow[r, dash] & T\mid_{\phi_{-4}} \arrow[r, dash] \arrow[d, "+c_4"] & T\mid_{\phi_{-2}} \arrow[r, dash] \arrow[d, "+c_2"] & T\mid_{\phi_{0}} \arrow[r, dash] \arrow[d, "+c_0"] & T\mid_{\phi_{+2}} \arrow[r, dash] \arrow[d, "+c_2"] & T\mid_{\phi_{+4}} \arrow[r, dash] \arrow[d, "+c_4"] & \cdots\cdots \\
\cdots\cdots \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{-4}} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{-2}} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{0}} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{+2}} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{+4}} \arrow[r, dash] & \cdots\cdots
\end{tikzcd}$$

It is clear from this picture that $R_\mu(s_\alpha,\, \gamma)$ preserves the decomposition of

$\mathrm{Hom}_M(E_\mu,\,V^\delta)$ in MK^α-invariant subspaces:

$$\mathrm{Hom}_M(E_\mu,\,V^\delta)=\bigoplus_{n\in\mathbb{N}}\mathrm{Hom}_M(\phi_{2n}+\phi_{-2n},\,V^\delta).$$

More precisely, $R_\mu(s_\alpha,\,\gamma)$ acts on $\mathrm{Hom}_M(\phi_{2n}+\phi_{-2n},\,V^\delta)$ as scalar multiplication by

$$c_{2n}=D\,\frac{\Pi_{j=1}^n((2j-1)-\langle\lambda,\,{}^\vee\alpha\rangle)}{\Pi_{j=1}^n((2j-1)+\langle\lambda,\,{}^\vee\alpha\rangle)}$$

for all $n>0$, and it acts on $\mathrm{Hom}_M(\phi_0,\,V^\delta)$ as scalar multiplication D.
We can normalize the intertwining operator so that it takes the value $+1$ on the lowest K-type. This corresponds to dividing $R_\mu(s_\alpha,\,\gamma)$ by D.[8]
The normalized operator acts trivially on $\mathrm{Hom}_M(\phi_0,\,V^\delta)$, and it acts on each subspace $\mathrm{Hom}_M(\phi_{2n}+\phi_{-2n},\,V^\delta)$ by the scalar d_n:

$$\frac{1}{D}R_\mu(s_\alpha,\,\gamma):\quad
\begin{tikzcd}
\mathrm{Hom}_M(\phi_0,V^\delta) \arrow[r, dash] \arrow[d, "1"] & \mathrm{Hom}_M(\phi_2+\phi_{-2},V^\delta) \arrow[r, dash] \arrow[d, "d_2"] & Hom(\phi_4+\phi_{-4},V^\delta) \arrow[r, dash] \arrow[d, "d_4"] & \bullet \arrow[d, "d_6"] & \cdots\cdots \\
\mathrm{Hom}_M(\phi_0,V^\delta) \arrow[r, dash] & \mathrm{Hom}_M(\phi_2+\phi_{-2},V^\delta) \arrow[r, dash] & \mathrm{Hom}_M(\phi_4+\phi_{-4},V^\delta) \arrow[r, dash] & \bullet & \cdots\cdots
\end{tikzcd}$$

We have set $d_0=1$ and

$$\boxed{d_{2n}=\frac{\Pi_{j=1}^n((2j-1)-\langle\lambda,\,{}^\vee\alpha\rangle)}{\Pi_{j=1}^n((2j-1)+\langle\lambda,\,{}^\vee\alpha\rangle)}}$$

for all $n\geq 1$.

E.2.2 The operator $R_\mu(s_\alpha,\,\gamma)$ for α simple and bad

When α is a bad root, the reflection s_α does not necessarily stabilize δ. Hence the operator $R_\mu(s_\alpha,\,\gamma)$ may fail to be an endomorphism of $\mathrm{Hom}_M(E_\mu,\,V^\delta)$.
Moreover, the decomposition of $\mathrm{Hom}_M(E_\mu,\,V^\delta)$ in MK^α-stable subspaces involves only odd characters.
We give the details in the next two lemmas.

Lemma 10. *The reflection s_α may fail to stabilize δ when α is a bad root.*

Proof. Suppose that there exists a positive root β for which the Cartan integer $\langle\alpha,\,{}^\vee\beta\rangle$ is odd. Then

$$\sigma_\alpha^{-1}m_\beta\sigma_\alpha=m_\beta m_\alpha^{\frac{1}{2}[1-(-1)^{\langle\alpha,\,{}^\vee\beta\rangle}]}=m_\beta m_\alpha$$

[8]The constant D is real and positive, so this normalization does not affect the signature.

and

$$(s_\alpha \cdot \delta)(m_\beta) = \delta(\sigma_\alpha^{-1} m_\beta \sigma_\alpha) = \delta(m_\beta m_\alpha) = \delta(m_\beta)\underbrace{\delta(m_\alpha)}_{-Id} = -\delta(m_\beta).$$

□

Remark 11. *For classical split groups, the reflection s_α across a bad root is in the stabilizer of δ if and only if there are no positive roots β for which the Cartan integer $\langle \alpha, {}^\vee\beta \rangle$ is odd.*

Proof. If G is a classical split group, then M is abelian and is generated by all the elements $m_\beta = \exp(\pi Z_\beta)$. So s_α stabilizes δ if and only if

$$(s_\alpha \cdot \delta)(m_\beta) = \delta(m_\beta)$$

for every positive root β. Because

$$(s_\alpha \cdot \delta)(m_\beta) = \delta(\sigma_\alpha^{-1} m_\beta \sigma_\alpha) = \delta(m_\beta m_\alpha^{\frac{1}{2}[1-(-1)^{\langle \alpha, {}^\vee\beta \rangle}]})$$

we only need

$$\delta(m_\alpha^{\frac{1}{2}[1-(-1)^{\langle \alpha, {}^\vee\beta \rangle}]}) = +Id.$$

When $\langle \alpha, {}^\vee\beta \rangle$ is even, this condition is automatically satisfied because m_α^0 is the identity of M and $\delta(e) = +Id$. When $\langle \alpha, {}^\vee\beta \rangle$ is odd, this condition always fails because $\delta(m_\alpha) = -Id$. □

Example: let G be $SL(2)$ and let δ be the sign representation of M. The root $\alpha = \epsilon_1 - \epsilon_2$ is a bad root for δ (because $m_\alpha = \mathrm{diag}(-1, -1)$). There are no other positive roots, and in particular there are no positive roots β for which the Cartan integer $\langle \alpha, {}^\vee\beta \rangle$ is odd. Hence s_α to stabilize δ.
Now let G be $SL(3)$ and let δ be the representation of M that picks up the first diagonal entry of an element of M. The root $\alpha = \epsilon_1 - \epsilon_2$ is a bad root for δ (because $m_\alpha = \mathrm{diag}(-1, -1, +1)$ so $\delta(m_\alpha) = -1$). We notice that the Cartan integer $\langle \epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3 \rangle = +1$ is odd, so s_α does not stabilize δ.

Lemma 11. *If α is bad for δ, then*

$$\mathrm{Hom}_M(E_\mu, V^\delta) = \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_M(\phi_{2n+1} + \phi_{-2n-1}, V^\delta).$$

Proof. We prove that for all T of $\mathrm{Hom}_M(E_\mu|_M, V^\delta)$ and all n in $\mathbb{Z}$, the restriction of T to the "even isotypic" ϕ_{2n} is trivial.
Pick v in ϕ_{2n}, then

$$T(v) = \delta(m_\alpha)T(\mu(m_\alpha^{-1})v) = \underbrace{\delta(m_\alpha)}_{-1} T(\underbrace{\chi_{2n}(m_\alpha^{-1})v}_{=e^{\pi(2n)i}v=+v}) = -T(v)$$

so $T(v)$ must be zero. □

Now we discuss the action of the operator

$$R_\mu(s_\alpha,\,\gamma)\colon\ \mathrm{Hom}_M(E_\mu,\,V^\delta) \to \mathrm{Hom}_M(E_\mu,\,V^{s_\alpha\cdot\delta}).$$

Because

$$(R_\mu(s_\alpha,\,\gamma)T)\mid_{\phi_{\pm(2n+1)}} = \pm\, c_{2n+1}T\mid_{\phi_{\pm(2n+1)}}$$

for all T in $\mathrm{Hom}_M(E_\mu,\,V^\delta)$, and all $n \geq 0$, we obtain the following picture:

$$\begin{tikzcd}
\cdots\cdots \arrow[r, dash] & T\mid_{\phi_{-3}} \arrow[r, dash] \arrow[d, "-c_3"] & T\mid_{\phi_{-1}} \arrow[r, dash] \arrow[d, "-c_1"] & T\mid_{\phi_{+1}} \arrow[r, dash] \arrow[d, "+c_1"] & T\mid_{\phi_{+3}} \arrow[r, dash] \arrow[d, "+c_3"] & \cdots\cdots \\
\cdots\cdots \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{-3}} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{-1}} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{+1}} \arrow[r, dash] & R_\mu(s_\alpha)T\mid_{\phi_{+3}} \arrow[r, dash] & \cdots\cdots
\end{tikzcd}$$

We notice that $R_\mu(s_\alpha,\,\gamma)$ preserves the MK^α-invariant subspaces, and it carries

$$\mathrm{Hom}_M(\phi_{2n+1}+\phi_{-2n-1},\,V^\delta) \longrightarrow \mathrm{Hom}_M(\phi_{2n+1}+\phi_{-2n-1},\,V^{s_\alpha\cdot\delta}).$$

If T belongs to $\mathrm{Hom}_M(\phi_{2n+1}+\phi_{-2n-1},\,V^\delta)$, its image via $R_\mu(s_\alpha,\,\gamma)$ is the mapping

$$\phi_{2n+1}+\phi_{-2n-1} \longrightarrow V^{s_\alpha\cdot\delta},\ (v_+ +\, v_-) \longmapsto c_{2n+1}\,T(v_+ -\, v_-). \tag{E.2}$$

It is interesting to compare the action of $R_\mu(s_\alpha,\,\gamma)$ with that one of the operator

$$\Psi_\alpha\colon\ \mathrm{Hom}_M\left(E_\mu,\,V^\delta\right) \to \mathrm{Hom}_M\left(E_\mu,\,V^{s_\alpha\cdot\delta}\right),\ S \mapsto S\circ\mu(\sigma_\alpha^{-1})$$

considered in section $(E.1)$.
For all $n \geq 0$, and all T in $\mathrm{Hom}_M(\phi_{2n+1}+\phi_{-2n-1},\,V^\delta)$, we have

$$\Psi_\alpha T(v_+ +\, v_-) = -i(-1)^n T(v_+ -\, v_-)$$

so we can write

$$R_\mu(s_\alpha,\,\gamma)\mid_{\mathrm{Hom}_M(\phi_{2n+1}+\phi_{-2n-1},\,V^\delta)} = i(-1)^n c_{2n+1}\Psi_\alpha.$$

The composition $R_\mu(s_\alpha,\,\gamma)\circ(\Psi_\alpha)^{-1}$ is an endomorphism of $\mathrm{Hom}_M\left(E_\mu,\,V^\delta\right)$.
It acts on each $\mathrm{Hom}_M\left(\phi_{2n+1}+\phi_{-(2n+1)},\,V^\delta\right)$ as scalar multiplication by

$$i(-1)^n c_{2n+1} = (-1)^n(iD')d_{2n+1}.$$

The constant $iD' = i\frac{-i\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda}{2}+1\right)}$ is real and positive, and the normalized operator

$$\frac{R_\mu(s_\alpha,\,\gamma)\circ(\Psi_\alpha)^{-1}}{iD'}$$

acts by:

$$\begin{array}{ccccccccc}
 & \mathrm{Hom}_M(\phi_1+\phi_{-1},V^\delta) & & \mathrm{Hom}_M(\phi_3+\phi_{-3},V^\delta) & & Hom(\phi_5+\phi_{-5},V^\delta) & & & \\
 & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \cdots\cdots \\
\frac{R_\mu(s_\alpha,\,\gamma)\circ(\Psi_\alpha)^{-1}}{iD'}: & \Big\downarrow d_1 & & \Big\downarrow -d_3 & & \Big\downarrow +d_5 & & \Big\downarrow -d_7 & \\
 & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \cdots\cdots \\
 & \mathrm{Hom}_M(\phi_1+\phi_{-1},V^\delta) & & \mathrm{Hom}_M(\phi_3+\phi_{-3},V^\delta) & & \mathrm{Hom}_M(\phi_5+\phi_{-5},V^\delta) & & &
\end{array}$$

Finally, we recall that $d_1 = 1$ and

$$\boxed{d_{2n+1} = \frac{(2-\lambda)(4-\lambda)\cdots(2n-\lambda)}{(2+\lambda)(4+\lambda)\cdots(2n+\lambda)}}$$

for all $n \geq 1$.

Remark 12. *The operator (Ψ_α) is defined up to a minus sign, but the composition $(\Psi_\alpha)^{-1}\circ R_\mu(s_\alpha,\,\gamma)$ is not affected by this choice.*

Proof. By definition, $\sigma_\alpha = \exp\left(\frac{\pi}{2}Z_\alpha\right) = \exp\left(\frac{\pi}{2}(E_\alpha+\theta E_\alpha)\right)$. As noticed in section $(E.1)$ there is an ambiguity of sign in the choice of E_α. This implies that σ_α is defined up to inverse and Ψ_α up to a minus sign.
We notice that replacing σ_α with σ_α^{-1} has also the effect of switching ϕ_{2n+1} with ϕ_{-2n-1}, and c_{2n+1} with $c_{-2n-1} = -c_{2n+1}$. It follows that the operator $\frac{R_\mu(s_\alpha,\,\gamma)\circ(\Psi_\alpha)^{-1}}{iD'}$ still acts by $(-1)^n d_{2n+1}$ on $\mathrm{Hom}_M\left(\phi_{2n+1}+\phi_{-(2n+1)},\,V^\delta\right)$. $\square$

E.3 The operator $R_\mu(s_\alpha,\,\gamma)$ on petite K-types

For an explicit example, see section $(F.3)$.
When the K-type μ is petite, the restriction of μ to the $SO(2)$-subgroup attached to α can only include the characters 0, ± 1, ± 2, ± 3. Therefore

$$\mathrm{Hom}_M(E_\mu,\,V^\delta) = \begin{cases} \mathrm{Hom}_M(\phi_{-2}+\phi_0+\phi_2,\,V^\delta) & \text{if } \alpha \text{ is good for } \delta \\ \mathrm{Hom}_M(\phi_{-3}+\phi_{-1}+\phi_1+\phi_3,\,V^\delta) & \text{if } \alpha \text{ is bad for } \delta. \end{cases}$$

We analyze the two cases separately.

If α is good $\cdots$

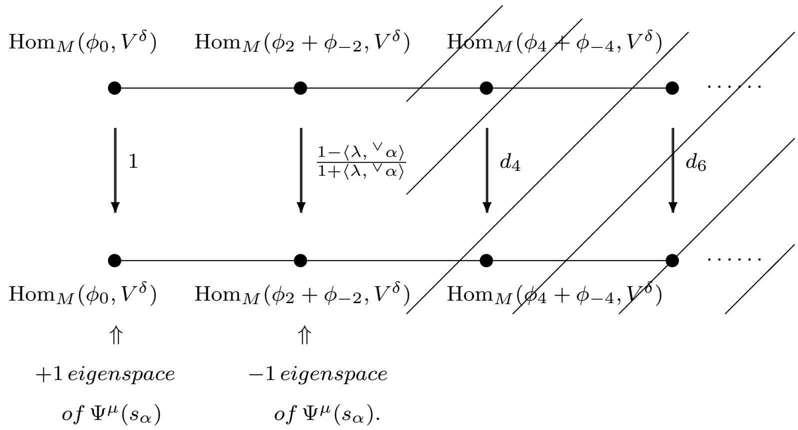
If α is a good root for δ, the operator $R_\mu(s_\alpha,\,\gamma)$ is an endomorphism of

$$\mathrm{Hom}_M(E_\mu\mid_M,\,V^\delta) = \mathrm{Hom}_M(\phi_0,\,V^\delta) \oplus \mathrm{Hom}_M(\phi_{-2}+\phi_{+2},\,V^\delta). \qquad \text{(E.3)}$$

It acts on $\mathrm{Hom}_M(\phi_0,\,V^\delta)$ by D, and on $\mathrm{Hom}_M(\phi_{-2}+\phi_{+2},\,V^\delta)$ by $Dd_2{=}D\,\frac{1-\langle\lambda,\,^\vee\alpha\rangle}{1+\langle\lambda,\,^\vee\alpha\rangle}$.
Let Ψ^μ be the representation of the Weyl group of the good co-roots W^0_δ on $(E.3)$ defined by $([\sigma]\cdot T)(v) = T(\mu(\sigma^{-1})v)$.
The reflection $s_\alpha = [\sigma_\alpha]$ belongs to W^0_δ, so it acts on $\mathrm{Hom}_M(E_\mu,\,V^\delta)$. We have:

$$\begin{aligned}\mathrm{Hom}_M(\phi_0,\,V^\delta) &\equiv \text{ the } (+1)\text{-eigenspace of } s_\alpha\\ \mathrm{Hom}_M(\phi_{-2}+\phi_{+2},\,V^\delta) &\equiv \text{ the } (-1)\text{-eigenspace of } s_\alpha.\end{aligned}$$

Therefore, we obtain the following picture:



We can write:

$$\boxed{\frac{1}{D}R_\mu(s_\alpha,\,\gamma) = \begin{cases} +1 & \textbf{on the } (+1)\textbf{-eigenspace of } \Psi^\mu(s_\alpha)\\ \frac{1-\langle\gamma,\,^\vee\alpha\rangle}{1+\langle\gamma,\,^\vee\alpha\rangle} & \textbf{on the } (-1)\textbf{-eigenspace of } \Psi^\mu(s_\alpha)\,.\end{cases}}$$

Remark 13. *When μ is petite and α is good, the operator $R_\mu(s_\alpha,\,\gamma)$ can be defined in terms of the representation Ψ^μ of W^0_δ on the space $\mathrm{Hom}_M(E_\mu\mid_M,\,V^\delta)$.*
There is no need to know the decomposition of μ in irreducible representations of $K^\alpha \simeq SO(2)$.

If α is bad $\cdots$

If α is a bad root for δ, then

$$\mathrm{Hom}_M(E_\mu,\,V^\delta) = \mathrm{Hom}_M(\phi_1+\phi_{-1},\,V^\delta) + \mathrm{Hom}_M(\phi_3+\phi_{-3},\,V^\delta)$$

and of course

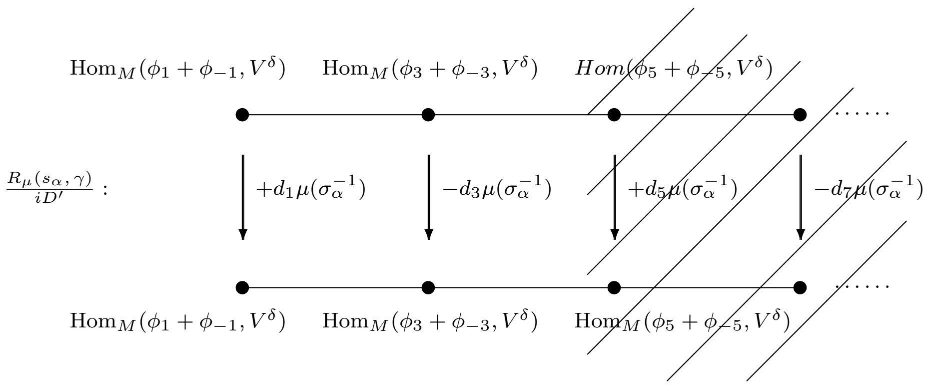
$$\mathrm{Hom}_M(E_\mu,\,V^{s_\alpha\cdot\delta}) = \mathrm{Hom}_M(\phi_1+\phi_{-1},\,V^{s_\alpha\cdot\delta}) + \mathrm{Hom}_M(\phi_3+\phi_{-3},\,V^{s_\alpha\cdot\delta}).$$

The normalized operator $\frac{R_\mu(s_\alpha,\,\gamma)}{iD'}$ acts on $\mathrm{Hom}_M(\phi_1+\phi_{-1},\,V^\delta)$ as $d_1\Psi^\alpha = \Psi^\alpha$:

$$\frac{1}{iD'}\,R_\mu(s_\alpha,\,\gamma)\cdot T = \Psi_\alpha\cdot T = T\circ\mu(\sigma_\alpha^{-1})$$

and on $\mathrm{Hom}_M(\phi_3+\phi_{-3},\,V^\delta)$ as the operator $-d_3\Psi^\alpha$:

$$\frac{1}{iD'}\,R_\mu(s_\alpha,\,\gamma)\cdot T = -d_3\Psi_\alpha\cdot T = -d_3 T\circ\mu(\sigma_\alpha^{-1}).$$



The representation Ψ^μ of W^0_δ on $\mathrm{Hom}_M(E_\mu,\,V^\delta)$ extends to a representation $\widetilde{\Psi^\mu}$ of W^δ on the same space:

$$\widetilde{\Psi^\mu}[\sigma]\cdot T = T\circ\mu(\sigma^{-1})$$

for all T in $\mathrm{Hom}_M(E_\mu,\,V^\delta)$.
If s_α belongs to the stabilizer of δ, we can interpret the operator $R_\mu(s_\alpha,\,\gamma)$ in terms of this Weyl group representation:

$$\frac{1}{D}\,R_\mu(s_\alpha,\,\gamma) = \widetilde{\Psi^\mu}(s_\alpha)$$

(as noticed in section $(E.1)$ there is an ambiguity of sign. The choice of this sign is independent of γ and μ).
If s_α does not belong to W^δ, then there is no similar interpretation.

Remark 14. *If μ has level 3, we still need to know the decomposition of μ in irreducible representations of $K^\alpha \simeq SO(2)$.*[9]
If μ has level at most 2, then we can construct $R_\mu(s_\alpha,\,\gamma)$ only in terms of the representation $\widetilde{\Psi^\mu}$ of W^δ on the $\mathrm{Hom}_M(E_\mu,\,V^\delta)$.

[9] The $(+i)$-eigenspace of $\mu(\sigma_\alpha)$ is the union of ϕ_1 and ϕ_{-3}.

E.3.1 The operator $R_\mu(s_\alpha,\gamma)$ on K-types of level at most two

If the K-type μ has level two, the restriction of μ to the $SO(2)$-subgroup attached to α can only include the characters 0, ± 1, ± 2. Therefore

$$\mathrm{Hom}_M(E_\mu, V^\delta) = \begin{cases} \mathrm{Hom}_M(\phi_0, V^\delta) + \mathrm{Hom}_M(\phi_2+\phi_{-2}, V^\delta) & \text{if } \alpha \text{ is good for } \delta \\ \mathrm{Hom}_M(\phi_{-1}+\phi_1, V^\delta) & \text{if } \alpha \text{ is bad for } \delta. \end{cases}$$

For every simple root α such that s_α belongs to W^δ, we can define the operator $R_\mu(s_\alpha,\gamma)$ *only* in terms of the representation $\widetilde{\Psi^\mu}$ of W^δ on $\mathrm{Hom}_M(E_\mu, V^\delta)$: $R_\mu(s_\alpha,\gamma) = (-iD')\widetilde{\Psi}^\mu(s_\alpha)$ is α is bad, and

$$R_\mu(s_\alpha,\gamma) = \begin{cases} D & \text{on the } (+1)\text{-eigenspace of } \widetilde{\Psi}^\mu(s_\alpha) \\ D\frac{1-\langle\gamma,{}^\vee\alpha\rangle}{1+\langle\gamma,{}^\vee\alpha\rangle} & \text{on the } (-1)\text{-eigenspace of } \widetilde{\Psi}^\mu(s_\alpha) \end{cases}$$

if α is good.

E.3.2 The operator $R_\mu(s_\alpha,\gamma)$ on fine K-types

Finally, we discuss the case in which the K-type μ is fine.
The restriction of μ to the $SO(2)$-subgroup attached to α can only include the characters 0, ± 1. Therefore

$$\mathrm{Hom}_M(E_\mu, V^\delta) = \begin{cases} \mathrm{Hom}_M(\phi_0, V^\delta) & \text{if } \alpha \text{ is good for } \delta \\ \mathrm{Hom}_M(\phi_{-1}+\phi_1, V^\delta) & \text{if } \alpha \text{ is bad for } \delta. \end{cases}$$

When α is good, the operator $R_\mu(s_\alpha,\gamma)$ is a scalar operator, equal to D. We notice that the operator $\widetilde{\Psi^\mu}(s_\alpha)$ is trivial. So we can write: $R_\mu(s_\alpha,\gamma) = D\widetilde{\Psi^\mu}(s_\alpha)$. When α is bad and the root reflection s_α stabilizes δ, the operator $R_\mu(s_\alpha,\gamma)$ acts as $-iD'\widetilde{\Psi^\mu}(s_\alpha)$.

Remark 15. *If μ is fine, the operator $R_\mu(s_\alpha,\gamma)$ is a multiple of $\widetilde{\Psi^\mu}(s_\alpha)$, for every root α such that s_α is in the stabilizer of δ.*

Corollary. *$R_\mu(\omega,\nu)$ is a multiple of $\widetilde{\Psi^\mu}(\omega) = \mu(\sigma_\alpha^{-1})$, when ω belongs to the stabilizer of δ.*

One final remark. If $\#R_\mu > \#R_\mu(\nu)$, then there is at least one Langlands quotient that contains more than one fine K-type. The operator $R_\mu(\omega,\nu)$ may separate the two fine K-types or act with the same sign. Only in the latter case we can hope for unitarity.
To conclude the section, we give example of these two possible behaviors.

1. Consider the minimal principal series for $SL(2)$ induced from the sign representation, with parameter $\nu = a\epsilon_1 - a\epsilon_2$, $a > 0$.
   There are two lowest K-types, χ_1 and χ_{-1}, and indeed the R-group R_δ

has cardinality two.[10] These K-types lie in the same Langlands quotient ($R_\delta(\nu)$ is trivial). We also notice that the element ω is the reflection through the (unique positive) root and it belongs to W^δ, so we are in the right setting. ω acts by $+i$ on χ_1 and by $-i$ on χ_{-1}, so it separates the two fine K-types, and there is no hope for unitarity.

2. Consider the minimal principal series for $SL(4)$ induced from the representation $\delta = \delta_{2,3}$ of M, with parameter $\nu = a\epsilon_1 + b\epsilon_2 - b\epsilon_3 - a\epsilon_4$, with $a > b > 0$.[11] There are two fine representations of $SO(4)$ containing δ,[12] $\psi_1 + \psi_2$ and $\psi_1 - \psi_2$, and indeed the R-group R_δ has cardinality two.[13] The two lowest K-types lie in the same Langlands quotient, because $R_\delta(\nu)$ is trivial.[14] The element $\omega = (14)(23)$ belongs to W^0_δ, hence to W^δ.
The setting is similar to the one of the previous example, but in this case the intertwining operator does *not* separate the two lowest K-types (the signs are the same).

[10] There are no good roots, but there is one positive root stabilizing δ.

[11] The notations are the same used in section $(B.1)$.

[12] They are the two irreducible summands of $\bigwedge^2(\mathbb{C}^4)$.

[13] The Weyl group of the good co-roots is

$$W^0_\delta = (symm.\, group\ on\, \{2,3\}) \times (symm.\, group\, on\ \{1,4\}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

but the stabilizer of δ also contains the permutation $(12)(34)$.

[14] Because $W^0_\delta(\nu) = W^\delta(\nu) = \{Id\}$.

Appendix F

Non-spherical representations of $SP(4)$

F.1 Preliminary remarks

The data for $SP(4)$

We recall the data for $SP(4)$, mainly to fix the notations:

- $G = SP(4) = \{x \in GL(4) \colon x^T J x = J\}$
  with J the skew-symmetric matrix $J = \begin{pmatrix} O & I_2 \\ -I_2 & O \end{pmatrix}$. We can also write:
  $G = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : A^T C - C^T A = O = B^T D - D^T B;\ A^T D - C^T B = I\}$

- $\mathfrak{g} = \mathfrak{sp}(4) = \{X \in \mathfrak{gl}(4) \colon x^T J + JX = O\} = \{\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} : B\,C \text{ symmetric}\}$

- $\theta : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto -X^T$

- $K = SP(4) \cap SO(4) \simeq U(2)$ via the mapping
$$\begin{pmatrix} A & -C \\ C & A \end{pmatrix} \mapsto A + iC$$

- $\mathfrak{k} = \{$skew-symmetric matrices in $\mathfrak{g}\} \simeq \mathfrak{u}(2)$ via the mapping
$$\begin{pmatrix} A & C \\ -C^T & -A^T \end{pmatrix} \mapsto A + iC$$

- $\mathfrak{p} = \{$symmetric matrices in $\mathfrak{g}\} = \{\begin{pmatrix} A & C \\ C^T & -A^T \end{pmatrix} : A$ and C are symmetric$\}$

- $\mathfrak{a}$ = maximal abelian subspace in $\mathfrak{g} = \{\begin{pmatrix} \Lambda & O \\ O & -\Lambda \end{pmatrix} : \Lambda$ is diagonal$\}$
- $A = \{\begin{pmatrix} D & O \\ O & D^{-1} \end{pmatrix} : D$ is diagonal, with positive entries$\}$
- $M = \{\begin{pmatrix} D & O \\ O & D \end{pmatrix} : D$ is diagonal, with entries $\pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm(\epsilon_1 \pm \epsilon_2), \pm 2\epsilon_1, \pm 2\epsilon_2\}$. We notice that:
  - If $\alpha = \epsilon_1 - \epsilon_2$, then (in the $U(2)$-picture)
  $$\sigma_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad m_\alpha = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
  - If $\alpha = \epsilon_1 + \epsilon_2$, then
  $$\sigma_\alpha = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad \text{and} \qquad m_\alpha = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
  - If $\alpha = 2\epsilon_1$, then
  $$\sigma_\alpha = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad m_\alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
  - If $\alpha = 2\epsilon_2$, then
  $$\sigma_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \qquad \text{and} \qquad m_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
- For the simple root $\epsilon_1 - \epsilon_2$, we find $MG^\alpha \simeq SL^\pm(2)$ and $MG^\alpha \cap K \simeq O(2)$
- For the simple root $2\epsilon_2$, we find $MG^\alpha \simeq O(1) \times SL(2)$ and $MG^\alpha \cap K \simeq O(1) \times U(1)$ (in the $U(2)$-picture).

Irreducible representations of M

The group M is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, so it has four characters

- $\delta_0 : \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mapsto 1$
- $\delta_1 : \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mapsto a_1$
- $\delta_2 : \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mapsto a_1 a_2$

- $\delta_3\colon \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mapsto a_2.$

The Weyl group fixes both the trivial representation δ_0 and the determinant representation δ_2, but switches δ_1 and δ_3.

Irreducible representations of K

In this subsection we describe the reducible representations of K, and their restriction to the subgroups M, and K^α for α simple.

Classification

$$\hat{K} = \{a\epsilon_1 + b\epsilon_2\colon a,\, b \in \mathbb{Z},\, a \geq b\}$$

We notice that $a\epsilon_1 + b\epsilon_2$ has dimension $a - b + 1$.

ϵ_1 is the standard representation;

$-\epsilon_1$ is the dual of the standard representation;

$\epsilon_1 + \epsilon_2$ is representation $\bigwedge^2(\mathbb{C}^2)$;

$-\epsilon_1 - \epsilon_2$ is the dual of $\bigwedge^2(\mathbb{C}^2)$.

Remark 16. *Using the isomorphism $U(2) \simeq \frac{S^1 \otimes SU(2)}{\pm(1,I)}$, we can give another classification of the irreducible representations of $K = U(2)$:*

$$\hat{K} = \{(m,n)\colon m \in \mathbb{Z},\, n \in \mathbb{N} \text{ and } m + n \equiv 0\,(mod\,2)\}$$

Here m stands for the m^{th} character of S^1, and n stands for the irreducible representation of $SU(2)$ on the space of homogeneous polynomials of degree n. The tensor product (m,n) has dimension $1 \cdot (n+1) = n+1$ and is trivial on $\pm(1,I)$ if and only if $(-1)^{m+n} = +1$.
The equivalence between the two classifications is given by:

$$(m,n) \mapsto \frac{m+n}{2}\epsilon_1 + \frac{m-n}{2}\epsilon_2$$
$$a\epsilon_1 + b\epsilon_2 \mapsto (a+b, a-b).$$

Restriction from K to M

$$Res_M(a\epsilon_1 + b\epsilon_2) = \begin{cases} \left(\frac{a-b}{2}+1\right)\delta_0 + \left(\frac{a-b}{2}\right)\delta_2 & \text{if } a \text{ and } b \text{ are both even} \\ \left(\frac{a-b}{2}\right)\delta_0 + \left(\frac{a-b}{2}+1\right)\delta_2 & \text{if } a \text{ and } b \text{ are both odd} \\ \left(\frac{a-b+1}{2}\right)\delta_1 + \left(\frac{a-b+1}{2}\right)\delta_3 & \text{if } a \text{ and } b \text{ have different parity.} \end{cases}$$

Restriction from $K \simeq U(2)$ to $O(2)$

Recall that $\widehat{O(2)} = \{\sigma_0^+, \sigma_0^-\} \cup \{\sigma_j : j \geq 1\}$. The notations have been chosen so that

$$Ind_{SO(2)}^{O(2)}(\chi_0) = \sigma_0^+ + \sigma_0^- \quad \text{and} \quad Ind_{SO(2)}^{O(2)}(\chi_j) = \sigma_j \ \ \forall j \geq 1.$$

We have:

$$Res_{O(2)}(a\epsilon_1 + b\epsilon_2) = \begin{cases} \sigma_0^+ + \bigoplus_{j \ even=1\ldots a-b} \sigma_j & \text{if } a \text{ and } b \text{ are both even} \\ \sigma_0^- + \bigoplus_{j \ even=1\ldots a-b} \sigma_j & \text{if } a \text{ and } b \text{ are both odd} \\ \bigoplus_{j \ odd=1\ldots a-b} \sigma_j & \text{if } a \text{ and } b \text{ have different parity.} \end{cases}$$

Restriction from $K \simeq U(2)$ to $U(1) \times U(1)$

$$Res_{U(1)\times U(1)}(a\epsilon_1 + b\epsilon_2) = \sum_{k=0\ldots(a-b)} (\chi_{a-k}) \times (\chi_{b+k}).$$

Fine and petite K-types

Let μ be the irreducible representation of $U(2)$ with highest weight $a\epsilon_1 + b\epsilon_2$. The eigenvalues of $\mu(iZ_\alpha)$ are

$0, \pm 2, \ldots, \pm(a-b)$ if $\alpha = \epsilon_1 \pm \epsilon_2$ and $a-b$ is even;

$\pm 1, \pm 3, \ldots, \pm(a-b)$ if $\alpha = \epsilon_1 \pm \epsilon_2$ and $a-b$ is odd;

$b, b+1, \ldots, a$ if $\alpha = 2\epsilon_1$ or $\alpha = 2\epsilon_2$.

Therefore we conclude that $a\epsilon_1 + b\epsilon_2$ is fine if and only if

$$\mid a \mid \leq 1 \quad \mid b \mid \leq 1 \quad \mid a-b \mid \leq 1$$

and is petite if and only if

$$\mid a \mid \leq 3 \quad \mid b \mid \leq 3 \quad \mid a-b \mid \leq 3.$$

We obtain the following list:

Level 1 (fine)

0	the trivial representation
ϵ_1	the standard representation
$-\epsilon_2$	the dual of the standard representation
$\epsilon_1 + \epsilon_2$	representation $\bigwedge^2(\mathbb{C}^2)$
$-\epsilon_1 - \epsilon_2$	the dual of $\bigwedge^2(\mathbb{C}^2)$.

Level 2

$$\epsilon_1-\epsilon_2 \qquad -\epsilon_1-2\epsilon_2 \qquad 2\epsilon_1+\epsilon_2 \qquad 2\epsilon_1 \qquad -2\epsilon_2 \qquad 2\epsilon_1+2\epsilon_2 \qquad -2\epsilon_1-2\epsilon_2.$$

Level 3

$$2\epsilon_1-\epsilon_2 \qquad \epsilon_1-2\epsilon_2 \qquad 3\epsilon_1 \qquad -3\epsilon_2.$$

F.2 Number of Langlands quotients of $X_P(\delta \otimes a\epsilon_1)$

Let $P = MAN$ be the minimal parabolic subgroup introduced in section $F.1$ and let ν be weakly dominant character of A:

$$\nu = a\epsilon_1 \quad a > 0.$$

For every non trivial representation δ of M, we discuss the number of Langlands quotients of the principal series $X_P(\delta \otimes a\epsilon_1)$.

The representation δ_1 of M is included in two fine K-types (ϵ_1 and $-\epsilon_2$), so $X_P(\delta_1 \otimes \nu)$ contains two lowest K-types. To understand whether they belong to the same Langlands quotient, we look at the group $R_{\delta_1}(\nu)$.
The only positive root that is good for δ_1 is $2\epsilon_2$, so $W^0_{\delta_1} = \{Id,\, s_{2\epsilon_2}\} = \mathbb{Z}_2$, while

$$W^{\delta_1} = \{Id,\, s_{2\epsilon_1},\, s_{2\epsilon_2},\, s_{2\epsilon_1}\cdot s_{2\epsilon_2}\} = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

The R-group $R_{\delta_1} = W^{\delta_1}/W^0_{\delta_1}$ has order two, as expected.
To find $R_{\delta_1}(\nu)$, we look for elements of $W^0_{\delta_1}$ and W^{δ_1} that stabilize ν:

$$W^0_{\delta_1}(\nu) = W^{\delta_1}(\nu) = \{Id,\, s_{2\epsilon_2}\}.$$

The R-group $R_{\delta_1}(\nu) = W^{\delta_1}(\nu)/W^0_{\delta_1}(\nu)$ is trivial, hence there is a unique Langlands quotient.
Next, we consider the principal series $X_P(\delta_3 \otimes a\epsilon_1)$. There are two lowest K-types, because there are exactly two fine K-types containing δ_3 (ϵ_1 and $-\epsilon_2$). We have:

- $W^0_{\delta_3} = \{Id,\, s_{2\epsilon_1}\} = \mathbb{Z}_2$
- $W^{\delta_3} = \{Id,\, s_{2\epsilon_1},\, s_{2\epsilon_2},\, s_{2\epsilon_1}\cdot s_{2\epsilon_2}\} = \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\#R_{\delta_3} = \#(W^{\delta_3}/W^0_{\delta_3}) = 2$, as expected
- $W^0_{\delta_3}(\nu) = \{Id\}$
- $W^{\delta_3}(\nu) = \{Id,\, s_{2\epsilon_2}\}$

- $\#R_{\delta_3}(\nu) = \#(W^{\delta_3}(\nu)/W^0_{\delta_3}(\nu)) = 2$.

Hence there are two Langlands quotients.
Finally, we look at the principal series $X_P(\delta_2 \otimes a\epsilon_1)$. There are two lowest K-types, because there are exactly two fine K-types containing δ_2 $(\pm(\epsilon_1 + \epsilon_2))$. The good roots for δ_2 are $\pm(\epsilon_1 + \epsilon_2)$ and $\pm(\epsilon_1 - \epsilon_2)$. We have:

- $W^0_{\delta_2} = \{Id,\, s_{\epsilon_1+\epsilon_2},\, s_{\epsilon_1-\epsilon_2},\, s_{\epsilon_1+\epsilon_2} \cdot s_{\epsilon_1-\epsilon_2}\} = \mathbb{Z}_2 \times \mathbb{Z}_2$
- $W^{\delta_2} = W$ (order 8)
- $\#R_{\delta_2} = \#(W^{\delta_2}/W^0_{\delta_2}) = 2$, as expected
- $W^0_{\delta_2}(\nu) = \{Id\}$
- $W^{\delta_2}(\nu) = \{Id,\, s_{2\epsilon_2}\}$
- $\#R_{\delta_2}(\nu) = \#(W^{\delta_2}(\nu)/W^0_{\delta_2}(\nu)) = 2$.

Again, there are two Langlands quotients.

To motivate these results, we give a different argument for computing the number of Langlands quotient of $X_P(\delta \otimes a\epsilon_1)$.

If $\nu = a\epsilon_1$ then there is exactly one positive root orthogonal to ν, namely $2\epsilon_2$. Let $P^1 = M^1A^1N^1$ be the parabolic subgroup containing P determined by ν.[1] By the principle of double induction

$$X_P(\delta_3 \otimes \nu) = Ind^G_P(\delta_3 \otimes \nu) = Ind^G_{P^1}(\delta^1_3 \otimes \nu^1)$$

where

- $\delta^1_3 = Ind^{M^1}_{M^1 \cap P = M(A\cap M^1)(N \cap M^1)}(\delta_3 \otimes \nu\mid_{A\cap M^1})$
- $\nu^1 = \nu\mid_{A^1}$.

Here $\nu\mid_{A\cap M^1} = 0$ and $\nu^1 = \nu\mid_{A^1} = \nu$. If we write

$$M^1 = MG^{2\epsilon_2} = O(1) \times SL(2)$$

then

- $M = O(1) \times O(1)$ (the second copy comes from scalar matrices in $SL(2)$)

[1] We have:
- $\mathrm{Lie}(P^1) = \mathfrak{m} + \mathfrak{a} + \mathfrak{g}_{-2\epsilon_2} \oplus \mathfrak{g}_{2\epsilon_2} \oplus \mathfrak{g}_{\epsilon_1+\epsilon_2} \oplus \mathfrak{g}_{\epsilon_1-\epsilon_2} \oplus \mathfrak{g}_{2\epsilon_1}$
- $\mathrm{Lie}(M^1) = \mathbb{R}H_{2\epsilon_2} \oplus \mathfrak{g}_{2\epsilon_2} \oplus \mathfrak{g}_{-2\epsilon_2} \simeq \mathfrak{sl}(2)$
- $\mathrm{Lie}(A^1) = \mathrm{Ker}(2\epsilon_2)$
- $\mathrm{Lie}(N^1) = \mathfrak{g}_{\epsilon_1+\epsilon_2} \oplus \mathfrak{g}_{\epsilon_1-\epsilon_2} \oplus \mathfrak{g}_{2\epsilon_1}$.

- $\delta_3 = tr \otimes sign$.

Therefore

$$\delta_3^1 = (triv.\, of\, O(1)) \times Ind^{SL(2)}_{min.\, parab.\, of\, SL(2)}(sign \otimes 0).$$

This representation is reducible, and has two irreducible components.
Hence $X_P(\delta_3 \otimes \nu)$ has two Langlands quotients.
Using the same argument, we find that $\delta_2 = sign \otimes sign$ hence

$$\delta_2^1 = (sign\, of\, O(1)) \times Ind^{SL(2)}_{min.\, parab.\, of\, SL(2)}(sign \otimes 0)$$

is reducible with two components, while $\delta_1 = sign \otimes triv$, so

$$\delta_1^1 = (sign\, of\, O(1)) \times Ind^{SL(2)}_{min.\, parab.\, of\, SL(2)}(triv. \otimes 0)$$

is irreducible.

F.3 Intertwining operators for $X_P(\delta_3 \otimes a\epsilon_1)$

We dedicate this section to the construction of intertwining operators for the minimal principal series $X_P(\delta_3 \otimes a\epsilon_1)$.

For brevity, set $\delta_3 = \delta$ and $a\epsilon_1 = \nu$. The element $\omega = s_{2\epsilon_1}$ stabilizes δ_1 and carries ν into $-\nu$, so we have an intertwining operator

$$A(\omega,\, \delta,\, \nu)\colon X_P(\delta \otimes \nu) \to X_P(\delta \otimes -\nu),\, F \mapsto [x \mapsto \textstyle\int_{\bar{N}^1} F(x\omega\bar{n})\, d\bar{n}].$$

We decompose ω as a product of simple reflections:

$$\omega = s_{2\epsilon_1} = s_{\epsilon_1-\epsilon_2} s_{2\epsilon_2} s_{\epsilon_1-\epsilon_2}$$

and we look at the corresponding decomposition of the operator $A(\omega,\, \delta,\, \nu)$:

$$A(\omega,\, \delta,\, \nu) = A(s_{\epsilon_1-\epsilon_2},\, s_{2\epsilon_2}s_{\epsilon_1-\epsilon_2}\cdot\delta,\, s_{2\epsilon_2}s_{\epsilon_1-\epsilon_2}\cdot\nu) \circ A(s_{2\epsilon_2},\, s_{\epsilon_1-\epsilon_2}\cdot\delta,\, s_{\epsilon_1-\epsilon_2}\cdot\nu) \circ A(s_{\epsilon_1-\epsilon_2},\, \delta,\, \nu).$$

Because $\delta_3 = \delta$ and $a\epsilon_1 = \nu$, we get:

$$A(\omega,\, \delta,\, \nu) = A(s_{\epsilon_1-\epsilon_2},\, \delta_1,\, -a\epsilon_2) \circ A(s_{2\epsilon_2},\, \delta_1,\, a\epsilon_2) \circ A(s_{\epsilon_1-\epsilon_2},\, \delta_3,\, a\epsilon_1).$$

We notice that

- $A(s_{\epsilon_1-\epsilon_2},\, \delta_3,\, a\epsilon_1)\colon X_P(\delta_3 \otimes a\epsilon_1) \to X_P(\delta_1 \otimes a\epsilon_2)$

- $A(s_{2\epsilon_2},\, \delta_1,\, a\epsilon_2)\colon X_P(\delta_1 \otimes a\epsilon_2) \to X_P(\delta_1 \otimes -a\epsilon_2)$

- $A(s_{\epsilon_1-\epsilon_2},\, \delta_1,\, -a\epsilon_2)\colon X_P(\delta_1 \otimes -a\epsilon_2) \to X_P(\delta_3 \otimes -a\epsilon_1)$.

For every K-type μ, we obtain an operator

$$R_\mu(\omega,\, \delta_3,\, a\epsilon_1)\colon\ \mathrm{Hom}_M(E_\mu,\, V^{\delta_3}) \to \mathrm{Hom}_M(E_\mu,\, V^{\delta_3}) \tag{F.1}$$

that factorizes as the product of three factors:

- $R_\mu(s_{\epsilon_1-\epsilon_2},\, \delta_3,\, a\epsilon_1)\colon\ \mathrm{Hom}_M(E_\mu,\, V^{\delta_3}) \to \mathrm{Hom}_M(E_\mu,\, V^{\delta_1})$
- $R_\mu(s_{2\epsilon_2},\, \delta_1,\, a\epsilon_2)\colon\ \mathrm{Hom}_M(E_\mu,\, V^{\delta_1}) \to \mathrm{Hom}_M(E_\mu,\, V^{\delta_1})$
- $R_\mu(s_{\epsilon_1-\epsilon_2},\, \delta_1,\, -a\epsilon_2)\colon\ \mathrm{Hom}_M(E_\mu,\, V^{\delta_1}) \to \mathrm{Hom}_M(E_\mu,\, V^{\delta_3})$.

We need to construct the intertwining operator $(F.1)$ for every *petite* K-type μ. It is convenient to work simultaneously with[2]

$$R_\mu(\omega,\, \delta_1,\, a\epsilon_1)\colon\ \mathrm{Hom}_M(E_\mu,\, V^{\delta_1}) \to \mathrm{Hom}_M(E_\mu,\, V^{\delta_1})$$

because *each factor* of the operator

$$R_\mu(\omega,\, \delta_1,\, a\epsilon_1) \oplus R_\mu(\omega,\, \delta_1,\, a\epsilon_3)$$

is an endomorphism of $\mathrm{Hom}_M(E_\mu,\, V^{\delta_1}) \oplus \mathrm{Hom}_M(E_\mu,\, V^{\delta_3})$.

Remark 17. *At the moment we are looking at operators defined on the full principal series. Later we will worry about how the various K-types split between the two Langlands quotients.*

$\mu = \epsilon_1$

The irreducible representation of $U(2)$ with highest weight ϵ_1 is the standard representation, and has dimension two. In order to compute the various factors of the intertwining operator, we need to know the restriction of μ to M and to the $SO(2)$-subgroups attached to the simple roots.

Explicit description of μ

There exists a basis $\{x,\, y\}$ of E_μ with the following properties:

---

[2] This operator factorizes as the product of three factors:

- $R_\mu(s_{\epsilon_1-\epsilon_2},\, \delta_1,\, a\epsilon_1)\colon\ \mathrm{Hom}_M(E_\mu,\, V^{\delta_1}) \to \mathrm{Hom}_M(E_\mu,\, V^{\delta_3})$
- $R_\mu(s_{2\epsilon_2},\, \delta_3,\, a\epsilon_2)\colon\ \mathrm{Hom}_M(E_\mu,\, V^{\delta_3}) \to \mathrm{Hom}_M(E_\mu,\, V^{\delta_3})$
- $R_\mu(s_{\epsilon_1-\epsilon_2},\, \delta_3,\, -a\epsilon_2)\colon\ \mathrm{Hom}_M(E_\mu,\, V^{\delta_3}) \to \mathrm{Hom}_M(E_\mu,\, V^{\delta_1})$.

- $\mu \mid_M = \delta_1 + \delta_3$, with $V_\mu(\delta_1) = \mathbb{C}x$ and $V_\mu(\delta_3) = \mathbb{C}y$

- $\mu \mid_{K^{\epsilon_1-\epsilon_2}} = \chi_{-1} + \chi_1$, and $V_\mu(\chi_1) = \mathbb{C}(x+iy)$ and $V_\mu(\chi_{-1}) = \mathbb{C}(x-iy)$

- $\mu \mid_{K^{2\epsilon_2}} = \xi_0 + \xi_1$, and $V_\mu(\xi_0) = \mathbb{C}x$ and $V_\mu(\xi_1) = \mathbb{C}(y)$.

We choose the basis $T\colon ax+by \mapsto a$ in $\mathrm{Hom}_M(E_\mu, V^{\delta_1})$, and $T'\colon ax+by \mapsto b$ in $\mathrm{Hom}_M(E_\mu, V^{\delta_3})$. Notice that $T = T' \circ \mu(\sigma_{\epsilon_1-\epsilon_2}^{-1})$.

The various factors$\cdots$

Having set the notations, we describe the action of the various factors:

- $R_\mu(s_{\epsilon_1-\epsilon_2}, \delta_1, a\epsilon_1) \oplus R_\mu(s_{\epsilon_1-\epsilon_2}, \delta_3, a\epsilon_1) = \begin{pmatrix} O & C_1 \\ -C_1 & 0 \end{pmatrix}$
- $R_\mu(s_{2\epsilon_2}, \delta_3, a\epsilon_2) \oplus R_\mu(s_{2\epsilon_2}, \delta_1, a\epsilon_2) = \begin{pmatrix} C_0 & 0 \\ 0 & -iC_1 \end{pmatrix}$
- $R_\mu(s_{\epsilon_1-\epsilon_2}, \delta_3, -a\epsilon_2) \oplus R_\mu(s_{\epsilon_1-\epsilon_2}, \delta_1, -a\epsilon_2) = \begin{pmatrix} O & C_1 \\ -C_1 & 0 \end{pmatrix}$.

This gives:

$$R_\mu(\omega, \delta_1, a\epsilon_1) \oplus R_\mu(\omega, \delta_3, a\epsilon_1) = C_1^2 \begin{pmatrix} iC_1 & 0 \\ 0 & -C_0 \end{pmatrix}.$$

We have set:[3]

$$C_0 = \frac{\pi\,\Gamma(a)}{2^{a-1}\,\Gamma\left(\frac{a+1}{2}\right)\,\Gamma\left(\frac{a+1}{2}\right)}$$

$$C_1 = \frac{\pi\,\Gamma(a)}{2^{a-1}\,\Gamma\left(\frac{a}{2}\right)\,\Gamma\left(\frac{a}{2}+1\right)}.$$

$\mu = -\epsilon_2$

The irreducible representation of $U(2)$ with highest weight $-\epsilon_2$ is the dual of the standard representation, and has dimension two.

Explicit description of μ

There exists a basis $\{x, y\}$ of E_μ with the following properties:

[3] To compute these constants, we must know that:

- $\langle a\epsilon_1, {}^\vee(\epsilon_1-\epsilon_2)\rangle = a$
- $\langle a\epsilon_2, {}^\vee(2\epsilon_2)\rangle = a$
- $\langle -a\epsilon_2, {}^\vee(\epsilon_1-\epsilon_2)\rangle = a$

- $\mu\mid_M = \delta_1 + \delta_3$, with $V_\mu(\delta_1) = \mathbb{C}y$ and $V_\mu(\delta_3) = \mathbb{C}x$

- $\mu\mid_{K^{\epsilon_1-\epsilon_2}} = \chi_{-1} + \chi_1$, and $V_\mu(\chi_1) = \mathbb{C}(x+iy)$ and $V_\mu(\chi_{-1}) = \mathbb{C}(x-iy)$

- $\mu\mid_{K^{2\epsilon_2}} = \xi_{-1} + \xi_0$, and $V_\mu(\xi_{-1}) = \mathbb{C}x$ and $V_\mu(\xi_0) = \mathbb{C}(y)$.

An argument similar to the one used before shows that [4]

$$R_\mu(\omega,\,\delta_1,\,a\epsilon_1) \oplus R_\mu(\omega,\,\delta_3,\,a\epsilon_1) = C_1^2 \begin{pmatrix} -iC_1 & 0 \\ 0 & -C_0 \end{pmatrix}.$$

$\mu = 2\epsilon_1 + \epsilon_2$

The irreducible representation of $U(2)$ with highest weight $2\epsilon_1 + \epsilon_2$ has dimension two.

Explicit description of μ

There exists a basis $\{x,\,y\}$ of E_μ with the following properties:

- $\mu\mid_M = \delta_1 + \delta_3$, with $V_\mu(\delta_1) = \mathbb{C}y$ and $V_\mu(\delta_3) = \mathbb{C}x$

- $\mu\mid_{K^{\epsilon_1-\epsilon_2}} = \chi_{-1} + \chi_1$, and $V_\mu(\chi_1) = \mathbb{C}(x+iy)$ and $V_\mu(\chi_{-1}) = \mathbb{C}(x-iy)$

- $\mu\mid_{K^{2\epsilon_2}} = \xi_1 + \xi_2$, and $V_\mu(\xi_1) = \mathbb{C}x$ and $V_\mu(\xi_2) = \mathbb{C}(y)$.

Then we get[5]

$$R_\mu(\omega,\,\delta_1,\,a\epsilon_1) \oplus R_\mu(\omega,\,\delta_3,\,a\epsilon_1) = C_1^2 \begin{pmatrix} +iC_1 & 0 \\ 0 & -C_0\frac{1-a}{1+a} \end{pmatrix}.$$

$\mu = -\epsilon_1 - 2\epsilon_2$

This the dual of the previous representation.

Explicit description of μ

There exists a basis $\{x,\,y\}$ of E_μ with the following properties:

- $\mu\mid_M = \delta_1 + \delta_3$, with $V_\mu(\delta_1) = \mathbb{C}x$ and $V_\mu(\delta_3) = \mathbb{C}y$

[4] This matrix is with respect to the basis $\{T',\,T\}$. We have inverted the basis elements, because we want to get an endomorphism of $\mathrm{Hom}_M(E_\mu, V^{\delta_1}) + \mathrm{Hom}_M(E_\mu, V^{\delta_3})$, with δ_1 coming first.

[5] Again with respect to the basis $\{T',\,T\}$.

- $\mu\,|_{K^{\epsilon_1-\epsilon_2}} = \chi_{-1} + \chi_1$, and $V_\mu(\chi_1) = \mathbb{C}(x+iy)$ and $V_\mu(\chi_{-1}) = \mathbb{C}(x-iy)$

- $\mu\,|_{K^{2\epsilon_2}} = \xi_{-2} + \xi_{-1}$, and $V_\mu(\xi_{-2}) = \mathbb{C}x$ and $V_\mu(\xi_{-1}) = \mathbb{C}(y)$.

Then we get[6]

$$R_\mu(\omega,\, \delta_1,\, a\epsilon_1) \oplus R_\mu(\omega,\, \delta_3,\, a\epsilon_1) = C_1^2 \begin{pmatrix} -iC_1 & 0 \\ 0 & -C_0\frac{1-a}{1+a} \end{pmatrix}.$$

Remark 18. *An explicit computation shows that, when looking at petite K-types for detecting unitarity, it is enough to stop at level 2. Indeed, the petite K-types of level 3 do not give rise to any additional restriction on the values of the parameters.*
Therefore we omit the construction for the intertwining operators corresponding to $\mu = 2\epsilon_1 - \epsilon_2$ $\mu = \epsilon_1 - 2\epsilon_2$, $\mu = 3\epsilon_1$ and $\mu = -3\epsilon_2$.

F.4 The Langlands quotients of $X_P(\delta_3 \otimes a\epsilon_1)$

In this section, we discuss the unitarity of the Langlands quotients of $X_P(\delta_3 \otimes a\epsilon_1)$.

In section $(F.2)$ we have proved that the minimal principal series

$$X_P(\delta_3 \otimes a\epsilon_1) = Ind_{P=MAN}^G(\delta_3 \otimes \nu) = Ind_{P^1=M^1A^1N^1}^G(\delta_3^1 \otimes \nu^1)$$

is reducible. We can write:

- $M^1 = MG^{2\epsilon_2} = O(1) \times SL(2)$
- $M = O(1) \times O(1) \subset M^1$
- $\delta_3 = tr \otimes sign.$
- $\delta_3^1 = Ind_{M^1\cap P}^{M^1}(\delta_3 \otimes \nu\,|_{A\cap M^1}) = Ind_{M^1\cap P}^{M^1}(\delta_3 \otimes 0) =$

  $= (triv.\, of\, O(1)) \times Ind_{min.\,parab.\,of\,SL(2)}^{SL(2)}(sign \otimes 0) = (\delta_3^1)_+ + (\delta_3^1)_-.$

We obtain:[7]

$$X_P(\delta_3 \otimes a\epsilon_1) = Ind_{P^1}^G((\delta_3^1)_+ \otimes \nu^1) + Ind_{P^1}^G((\delta_3^1)_- \otimes \nu^1).$$

The easiest way to distinguish between these two summands is to look at the action of the $SO(2)$-subgroup attached to $2\epsilon_2$: the first summand contains every K-type whose restriction to $K^{2\epsilon_2}$ includes a positive odd character, the second summand contains every K-type whose restriction to $K^{2\epsilon_2}$ includes a negative

[6] With respect to the basis $\{T,\, T'\}$.
[7] This may not be a decomposition in irreducible subspaces. The intertwining operator A can in fact have a Kernel.

odd character.[8]

Next we describe the petite K-types included in each summand.
By Frobenious reciprocity, the multiplicity of μ in $Res_K(X_P(\delta_3 \otimes a\epsilon_1))$ equals the multiplicity of δ_3 in $Res_M(\mu)$. It is easy to check that:

- $Res_M(0) = \delta_0$
- $Res_M(\epsilon_1) = Res_M(-\epsilon_2) = \delta_1 + \delta_3$
- $Res_M(\epsilon_1 + \epsilon_2) = Res_M(-\epsilon_1 - \epsilon_2) = \delta_2$
- $Res_M(\epsilon_1 - \epsilon_2) = \delta_0 + 2\delta_2$
- $Res_M(2\epsilon_1 + \epsilon_2) = Res_M(-\epsilon_1 - 2\epsilon_2) = \delta_1 + \delta_3$
- $Res_M(2\epsilon_1) = Res_M(-2\epsilon_2) = 2\delta_0 + \delta_2$
- $Res_M(2\epsilon_1 + 2\epsilon_2) = Res_M(-2\epsilon_1 - 2\epsilon_2) = \delta_0$
- $Res_M(2\epsilon_1 - \epsilon_2) = Res_M(\epsilon_1 - 2\epsilon_2) = 2\delta_1 + 2\delta_3$
- $Res_M(3\epsilon_1) = Res_M(-3\epsilon_2) = 2\delta_1 + 2\delta_3$.

So $X_P(\delta_3 \otimes a\epsilon_1)$ contains one copy of ϵ_1, $-\epsilon_2$, $2\epsilon_1 + \epsilon_2$ $-\epsilon_1 - 2\epsilon_2$, and two copies of $2\epsilon_1 - \epsilon_2$, $\epsilon_1 - 2\epsilon_2$, $3\epsilon_1$, $-3\epsilon_2$. We can say more:

- $Ind^G_{P^1}((\delta^1_3)_+ \otimes \nu^1)$ contains one copy of ϵ_1, $2\epsilon_1 + \epsilon_2$, $2\epsilon_1 - \epsilon_2$, $\epsilon_1 - 2\epsilon_2$, and two copies of $3\epsilon_1$.
- $Ind^G_{P^1}((\delta^1_3)_+ \otimes \nu^1)$ contains one copy of $-\epsilon_2$, $-\epsilon_1 - 2\epsilon_2$, $2\epsilon_1 - \epsilon_2$, $\epsilon_1 - 2\epsilon_2$, and two copies of $-3\epsilon_2$.

The corresponding Langlands quotients

- $\bar{X}_P(\delta_3 \otimes a\epsilon_1)_+ = \frac{Ind^G_{P^1}((\delta^1_3)_+ \otimes \nu^1)}{\mathrm{Ker}(A)}$
- $\bar{X}_P(\delta_3 \otimes a\epsilon_1)_- = \frac{Ind^G_{P^1}((\delta^1_3)_- \otimes \nu^1)}{\mathrm{Ker}(A)}$

are unitary only if the intertwining operator $R_\mu(\omega,\, \delta_3,\, a\epsilon_1)$ is semi-definite for every petite K-type included in the quotient, that has level two or less.
More explicitly:

- $\bar{X}_P(\delta_3 \otimes a\epsilon_1)_+$ is unitary only if $R_\mu(\omega,\, \delta_3,\, a\epsilon_1)$ is semi-definite for $\mu = \epsilon_1$ and $2\epsilon_1 + \epsilon_2$.
- $\bar{X}_P(\delta_3 \otimes a\epsilon_1)_-$ is unitary only if $R_\mu(\omega,\, \delta_3,\, a\epsilon_1)$ is semi-definite for $\mu = -\epsilon_2$ and $-\epsilon_1 - 2\epsilon_2$.

[8] The restriction of a K-type μ to $K^{2\epsilon_2}$ contains both even and odd characters. Look at the odd ones: if they are all negative, μ belongs only to $Ind^G_{P^1}((\delta^1_3)_- \otimes \nu^1)$; if they are all positive, μ belongs only to $Ind^G_{P^1}((\delta^1_3)_+ \otimes \nu^1)$; if some are positive and some are negative, μ belongs to both summands.

The normalized operator $\frac{R_\mu(\omega, \delta_3, a\epsilon_1)}{-C_0 C_1^2}$ acts by $+1$ on the fine K-types ϵ_1 and $-\epsilon_2$, and it acts by $\frac{1-a}{1+a}$ on $2\epsilon_1 + \epsilon_2$ and $-\epsilon_1 - 2\epsilon_2$.
Therefore, the two Langlands quotients are unitary if and only if $0 < a < 1$.