Weyl Group Representations and Signatures of Intertwining Operators

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An overview

- **MOTIVATION:**
  Study of the "non-unitarity" of a spherical principal series for a real split semi-simple Lie group.
- **STRATEGY:**
  1. Use Weyl group calculations to compute the intertwining operator on *petite* K-types.
  2. Use *petite* K-types to define a non-unitarity test.
- **MAIN RESULT:**
  A method to construct *petite* K-types.

Plan of the talk...

- **INTRODUCTION**
  1. The classical problem of studying the Unitary Dual.
  2. A non-unitarity test for a spherical principal series [Vogan-Barbasch].
     For this test we need to know "a lot" of petite K-types.
- **ORIGINAL WORK**
  An inductive argument to extend a certain class of Weyl group representations to petite K-types

Studying the Unitary Dual

By a theorem of Harish-Chandra, this is equivalent to:

1. Describing the **irreducible admissible repr.s of G**, up to infinitesimal equivalence.

2. Understanding which irreducible admissible repr.s of G admit a **non-degenerate invariant Hermitian form**.

3. Deciding whether the non-degenerate invariant Hermitian form on an admissible irreducible repr. of G is **positive definite**.

The irreducible admissible repr.s of G

Langlands, early 1970s:

- Every irreducible admissible representation of G is infinitesimally equivalent to a **Langlands quotient** $J_P(\delta \otimes \nu)$
- Two Langlands quotients $J_P(\delta \otimes \nu)$ and $J_{P'}(\delta' \otimes \nu')$ are infinitesimally equivalent if and only if there exists an element ω of K such that

$$\omega P \omega^{-1} = P' \quad \omega \cdot \delta = \delta' \quad \omega \cdot \nu = \nu'.$$

A Langlands Quotient

- $P = MAN$ a parabolic subgroup of G
- (δ, V^δ) an irreducible tempered unitary representation of M
- $\nu \in (\mathfrak{a}_0')^{\mathbb{C}}$, with real part in the open positive Weyl chamber
- $I_P(\delta \otimes \nu)$ the corresponding **principal series**

G acts by left translation on $\{F \colon G \to V^\delta \text{ s.t. } F\mid_K \in L^2(K, V^\delta);$
$F(xman) = e^{-(\nu+\rho) log(a)} \delta(m^{-1}) F(x), \forall\, man \in P, \forall\, x \in G\}$

- $J_P(\delta \otimes \nu)$: **the unique irreducible quotient** of $I_P(\delta \otimes \nu)$

$J_P(\delta \otimes \nu)$ is the quotient of $I_P(\delta \otimes \nu)$ modulo the kernel of
$A(\bar{P} : P : \delta : \nu) \colon I_P(\delta \otimes \nu) \to I_{\bar{P}}(\delta \otimes \nu),\ F \mapsto \int_{\Theta(N)} F(x\bar{n})\, d\bar{n}$

Hermitian Langlands Quotients

Knapp and Zuckerman, 1976:

$J_P(\delta \otimes \nu)$ admits **a non-degenerate invariant Hermitian form** if and only if there exists an element ω of K satisfying the following "**formal symmetry condition**":

$$\omega P \omega^{-1} = \bar{P} \qquad \omega \cdot \delta \simeq \delta \qquad \omega \cdot \nu = -\bar{\nu}.$$

(because the Hermitian dual of $J_P(\delta \otimes \nu)$ is $J_{\bar{P}}(\delta \otimes -\bar{\nu})$).
Any non-degenerate invariant Hermitian form on $J_P(\delta \otimes \nu)$ is a real multiple of the form induced by the Hermitian operator

$$\boxed{B = \delta(\omega) \circ R(\omega) \circ A(\bar{P} : P : \delta : \nu)}$$

from $I_P(\delta \otimes \nu)$ to $I_P(\delta \otimes -\bar{\nu})$.

Unitary Langlands Quotients

$J_P(\delta \otimes \nu)$ **is unitary**

$$\Updownarrow$$

B **is semi-definite.**

Next task: **Computing the signature of B**

Computing the signature of B

the first reduction: **A K-type by K-type calculation...**

- For every K-type (μ, E_μ), we have a Hermitian operator

$$R_\mu(\omega, \nu)\colon \operatorname{Hom}_K(E_\mu, I_P(\delta \otimes \nu)) \to \operatorname{Hom}_K(E_\mu, I_P(\delta \otimes -\bar{\nu}))$$

- By Frobenius reciprocity:

$$R_\mu(\omega, \nu)\colon \operatorname{Hom}_M(E_\mu\,|_{M\cap K}, V^\delta) \to \operatorname{Hom}_M(E_\mu\,|_{M\cap K}, V^\delta)$$

Constructing the operator

$$R_\mu(\omega,\,\nu)\colon\ \mathrm{Hom}_M(\mathrm{Res}^K_{M\cap K} E_\mu,\, V^\delta) \to \mathrm{Hom}_M(\mathrm{Res}^K_{M\cap K} E_\mu,\, V^\delta)$$

additional assumptions...

- G is **split**, e.g. $SL(n,\mathbb{R})$, $Sp(2n,\mathbb{R})$, $SO(n,n)$, E_6, E_7, E_8

  Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition of Lie(G), and let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$. G split if $Z_{\mathfrak{k}_0}(\mathfrak{a}_0) = \{0\}$.

- $P = MAN$ is a **minimal** parabolic subgroup of G
- δ is the **trivial** representation of M
- ν is a **real** character of A

Constructing the operator $R_\mu(\omega, \nu)$

... a reduction to rank-one computations

- When P is a minimal parabolic, the element ω is a Weyl group element, and it admits a minimal decomposition as a product of simple reflections. We can decompose $R_\mu(\omega, \nu)$ accordingly. (**Gindikin-Karpalevic**)

- When G is split, the operator corresponding to a simple reflection can be computed using the results known for $SL(2, \mathbb{R})$.

Constructing the operator $R_\mu(\omega, \nu)$

Some Considerations:

pros We obtain a decomposition of $R_\mu(\omega, \nu)$ as a product of operators corresponding to simple reflections, for which an *explicit formula* exists.

cons This formula depends on the decomposition of μ in $K^{(\beta)}$ types, and this decomposition changes when β varies. It is very hard to keep track of these different decompositions when you multiply the various rank-one operators to obtain $R_\mu(\omega, \nu)$.

A new strategy [Vogan-Barbasch]

When μ is "petite", compute $R_\mu(\omega, \nu)$
by means of Weyl group calculations

- A K-type μ is called **petite** if the $SO(2)$ subgroup attached to every simple root only acts with characters 0, ±1,±2.

- When μ is petite, we can compute $R_\mu(\omega, \nu)$ in a purely algebraic manner. Indeed, $R_\mu(\omega, \nu)$ **depends *only* on the representation of the Weyl group on the space of M-fixed vectors of E_μ.**

Constructing $R_\mu(\omega, \nu)$, for μ petite

- $R_\mu(\omega, \nu)$ is an endomorphism of $\mathcal{H} \equiv \mathrm{Hom}_M(\mathrm{Res}^K_M E_\mu, V^\delta)$
- $\mathcal{H} \simeq (E^M_\mu)^*$, and it carries a Weyl group representation ψ_μ
- Decompose $R_\mu(\omega, \nu)$ as a product of rank-one operators.
  For every simple root β, we can write

$$\mathcal{H} = \underbrace{\mathrm{Hom}_M(\mathrm{Res}^{MK^{(\beta)}}_M \varphi_0, \mathbb{C})}_{(+1)\text{-eigenspace of } \psi_\mu(s_\beta)} \oplus \underbrace{\mathrm{Hom}_M(\mathrm{Res}^{MK^{(\beta)}}_M (\varphi_2 \oplus \varphi_{-2}), \mathbb{C})}_{(-1)\text{-eigenspace of } \psi_\mu(s_\beta)}$$

$$\Rightarrow \; R_\mu(s_\beta, \gamma) = \begin{cases} +1 & \textbf{on the } (+1)\textbf{-eigenspace of } \psi_\mu(s_\beta) \\ \frac{1-\langle\gamma, \check\beta\rangle}{1+\langle\gamma, \check\beta\rangle} & \textbf{on the } (-1)\textbf{-eigenspace of } \psi_\mu(s_\beta) \end{cases}$$

A non-unitarity test

- **For each petite K-type (μ, E_μ), compute the representation ψ_μ of the Weyl group on the space of M-invariants in E_μ**
- **Use ψ_μ to construct the algebraic operator $R_\mu(\omega, \nu)$, and evaluate its signature**
- **If $R_\mu(\omega, \nu)$ fails to be (positive) semi-definite, then $J_P(\delta \otimes \nu)$ is not unitary.**

[**Barbasch**] **This non-unitarity test also detects unitarity when G is a classical group.**

A few more comments on the non-unitarity test

The non-unitarity test consists of computing the signature of the intertwining operator on petite K-types (by means of Weyl group calculations). For this test to be efficient, "**we need to know a large number of petite K-types**".

Next task: **Constructing petite K-types**

Plan of the talk. . .

- **INTRODUCTION** ✓
  1. The classical problem of studying the Unitary Dual.
  2. A non-unitarity test for a spherical principal series [Vogan-Barbasch].
     For this test we need to know “a lot” of petite K-types.
- **ORIGINAL WORK**
  An inductive argument to extend a certain class of Weyl group representations to petite K-types

trick: look at the example of $SL(3)$ to get insight!

The example of $SL(3, \mathbb{R})$

- $G = SL(3, \mathbb{R})$
- $K = SO(3, \mathbb{R})$
- $M = \{3 \times 3$ diag. matrices with det. 1 and diag. entries $= \pm 1\}$
- $A = \{3 \times 3$ diag. matrices with det. 1 and non-negative entries$\}$
- $W \simeq \mathcal{S}_3$ (symmetric group on 3 letters)
- $\widehat{M} = \{\delta_0, \delta_1, \delta_2, \delta_3\}$ with δ_0 the trivial representation of M, and

$$\delta_j : M \to \mathbb{R}, \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \mapsto m_j, \forall\, j = 1, 2, 3.$$

The petite K-types for $SL(3,\,\mathbb{R})\ldots$

- $\widehat{K}=\{\mathcal{H}_N : N\geqslant 0\}$

For each $N\geqslant 0$, $\mathcal{H}_N$ is the complex v.s. (of dimension $2N+1$) of **harmonic homogeneous polynomials of degree N** in three variables. $SO(3)$ acts by:

$$(g\cdot F)(x,\,y,\,z)=F((x,\,y,\,z)g)\quad \forall\, g\in SO(3),\,\forall\, F\in\mathcal{H}_N.$$

- $\widehat{K}_{petite}=\{\mathcal{H}_0,\,\mathcal{H}_1,\,\mathcal{H}_2\}$

For each simple root α, $\mathcal{H}_N\mid_{K^\alpha\simeq SO(2)}=\bigoplus_{l=-N}^{N}\xi_l$.

... the corresponding Weyl group representations

- $\underbrace{\mathcal{H}_0}_{dim\, 1} |_M = \delta_0 \Rightarrow (\mathcal{H}_0)^M$ is the **trivial repr. of** $W = \mathcal{S}_3$

- $\underbrace{\mathcal{H}_1}_{dim 3} |_M = \delta_1 \oplus \delta_2 \oplus \delta_3 \Rightarrow (\mathcal{H}_1)^M = \{0\}$

- $\underbrace{\mathcal{H}_2}_{dim\, 5} |_M = (\delta_0)^2 \oplus \delta_1 \oplus \delta_2 \oplus \delta_3 \Rightarrow (\mathcal{H}_2)^M = \mathbb{C}^2$

  $(\mathcal{H}_2)^M$ is the **standard repr. of** $W = \mathcal{S}_3$

The restriction to M' of the petite K-types

Repr.s of M'	trivial	sign	standard	ν_1	ν_2
dimension	1	1	2	3	3
eigenvalues of σ_α	1	-1	± 1	$1, \pm i$	$-1, \pm 1$

- $\mathcal{H}_0|_{M'} =$ trivial
- $\mathcal{H}_1|_{M'} = \nu_1$
- $\mathcal{H}_2|_{M'} =$ standard $\oplus\, \nu_2$

There is no sign repr. (of $W = \mathcal{S}_3$)!!

A closer look at $\mathcal{H}_2$

$$E_{\mathcal{H}_2} = \underbrace{\{ay^2 + bz^2 - (a+b)x^2 : a,\, b \in \mathbb{C}\}}_{F \equiv \{M\text{-fixed vectors}\}} \oplus \underbrace{\mathbb{C}\, xy \oplus \mathbb{C}\, xz \oplus \mathbb{C}\, yz}_{\text{“}Z_\beta \cdot v\text{”, with } v \in F,\, \sigma_\beta \cdot v = -v}$$

- $xy = Z_{\varepsilon_1 - \varepsilon_2} \cdot \underline{v}$, with $\underline{v}$ the “unique” (-1) eigenv. of $\sigma_{\varepsilon_1 - \varepsilon_2}$ in F
- $xz = Z_{\varepsilon_1 - \varepsilon_3} \cdot \underline{u}$, with $\underline{u}$ the “unique” (-1) eigenv. of $\sigma_{\varepsilon_1 - \varepsilon_3}$ in F
- $yz = Z_{\varepsilon_2 - \varepsilon_3} \cdot \underline{w}$, with $\underline{w}$ the “unique” (-1) eigenv. of $\sigma_{\varepsilon_2 - \varepsilon_3}$ in F

An algorithm to construct petite K-types

- **Input:** a representation ρ of $M' \subseteq SL(n)$ not containing the sign representation of $W(SL(3))$

  (when ρ is a Weyl group representation, this is equivalent to requiring that the partition ρ have at most two parts)

- **Output:** a petite representation μ_ρ of $SO(n)$ that extends ρ

- **Value:** When ρ is a Weyl group representation, the signature w.r.t. μ_ρ of the intertwining operator for a spherical principal series can be computed using *only* the Weyl group repr. ρ.

The main ideas...

- As a Lie algebra representation, the differential of a petite representation of K is generated by its restriction to M' through an iterated application of the Z_αs.
- Because the representation is petite, the eigenvalues of Z_α must lie in the set $\{0, \pm i, \pm 2i\}$. More precisely, Z_α must act by:

$$\begin{aligned} 0 &\quad \text{on the } (+1)\text{-eigenspace of } \sigma_\alpha \\ +i &\quad \text{on the } (+i)\text{-eigenspace of } \sigma_\alpha \\ -i &\quad \text{on the } (-i)\text{-eigenspace of } \sigma_\alpha \\ \pm 2i &\quad \text{on the } (-1)\text{-eigenspace of } \sigma_\alpha . \end{aligned}$$

The element $\sigma_\alpha = exp\left(\frac{\pi}{2} Z_\alpha\right)$ is a representative in M' for s_α.

We only need to know the action of Z_α on the (-1)-eigensp. of σ_α

A sketch of the construction...

It is an inductive argument. This is the mth step:

- By construction (ρ_m, F_{ρ_m}) is a representation of M' that does not include the sign of $\mathcal{S}_3$.
- We add some **generators** $\mathcal{G}_m$ (to specify the action of each Z_α on the (-1)-eigenspace of σ_α).
- We impose some **relations** $\mathcal{R}_m$ (to control the eigenvalues of Z_α and make sure not to introduce any copy of the sign representation of $\mathcal{S}_3$).
- The result is a new vector space

$$\frac{F_{\rho_m} + Span(\mathcal{G}_m)}{\mathcal{R}_m}$$

on which we define a representation ρ_{m+1} of M'.

A sketch of the construction... (continued)

- Because $(\rho_{m+1}, F_{\rho_{m+1}})$ does not include the sign of $\mathcal{S}_3$, we can iterate the construction.
- The number of steps is finite ($\mathcal{G}_m = \varnothing$ for m big).
- The final result is a representation of M' that extends the original representation. It is possible to define an action of Lie(K) on this space, that lifts to a petite representation of K.

Some details

I will describe:

- The set of generators $\mathcal{G}_m$ to be added at the mth step of the construction.
- The set of relations $\mathcal{R}_m$ to be added at the mth step of the construction.
- The vector space that results from this inductive construction, and the actions of M' and Lie(K) on this space.

The "new generators" $\mathcal{G}_m \ldots$

The set $\mathcal{G}_m$ consists of all the "formal strings" $Z_\nu v$, with v a (-1)-eigenvector of σ_ν in F_{ρ_m}, and ν a positive root.

...keep in mind the example of $SL(3)$!!

The "new relations" $\mathcal{R}_m \ldots$

The set $\mathcal{R}_m$ consists of the following four kinds of relations:

1. **"linearity relations"**

$$\boxed{Z_\nu(a_1 v_1 + a_2 v_2) = a_1 Z_\nu v_1 + a_2 Z_\nu v_2}$$

for all a_1, a_2 in $\mathbb{C}$, and for all (-1)-eigenv.s v_1, v_2 of σ_ν in F_{ρ_m}.

2. **"commutativity relations"**

$$\boxed{Z_{\nu_1} Z_{\nu_2} v = Z_{\nu_2} Z_{\nu_1} v}$$

for all *mutually orthogonal* positive roots ν_1, ν_2 and all *simultaneous* (-1)-eigenvectors of σ_{ν_1}, σ_{ν_2} in $F_{\rho_{m-1}}$.

The "new relations" $\mathcal{R}_m$ ... (continued)

3. **"no repetitions!"** $\boxed{Z_\nu Z_\nu v = -4v}$

   for all positive roots ν and all (-1)-eigenvectors of σ_ν in $F_{\rho_{m-1}}$.

4. **"$(\star)$-relations"** $\boxed{Z_\nu v = \sigma_\beta \cdot (Z_\nu v) + \sigma_\gamma \cdot (Z_\nu v)}$

   for all positive roots ν, all (-1)-eigenvectors v of σ_ν in F_{ρ_m}, and all triples of positive roots α, β, γ forming an $\mathcal{A}_2$, such that

   - $\alpha = \nu$ or $\alpha \perp \nu$; $\beta \not\perp \nu$ (so automatically $\gamma \not\perp \nu$), and
   - $\sigma_\alpha \cdot v = -v$; $\sigma_\beta^2 \cdot v = -v$ (so automatically $\sigma_\gamma^2 \cdot v = -v$).

The final result...

- **the vector space:** we have added to F_ρ the equivalence classes of all the strings of the form:

$$S = Z_{\alpha_1} \dots Z_{\alpha_r} v$$

with $\alpha_1, \dots, \alpha_r$ *mutually orthogonal* positive roots, and v a *simultaneous* (-1)-eigenvector for $\sigma_{\alpha_1}, \dots, \sigma_{\alpha_r}$ in F_ρ.

- **the action of M':** $\sigma \cdot [Z_\nu v] = [(\mathrm{Ad}(\sigma)(Z_\nu))(\sigma \cdot v)]$
- **the action of $\mathrm{Lie}(K)$:**

$$\begin{aligned} Z_\alpha \cdot [Z_\nu v] &= [0] && \text{if } \sigma_\alpha \cdot (Z_\nu v) = +(Z_\nu v) \\ Z_\alpha \cdot [Z_\nu v] &= [Z_\alpha Z_\nu v] && \text{if } \sigma_\alpha \cdot (Z_\nu v) = -(Z_\nu v) \\ Z_\alpha \cdot [Z_\nu v] &= \sigma_\alpha \cdot [Z_\nu v] && \text{if } \sigma_\alpha^2 \cdot (Z_\nu v) = -(Z_\nu v). \end{aligned}$$

Construction for $SL(2n, \mathbb{R})$

$$(2n) \Rightarrow L(0\psi_1 + 0\psi_2 + \cdots + 0\psi_n)$$

$$(2n-k, k) \Rightarrow L(2\psi_1 + 2\psi_2 + \cdots + 2\psi_k), \text{ for all } 0 < k < n$$

$$(n, n) \Rightarrow L(2\psi_1 + \cdots + 2\psi_{n-1} - 2\psi_n) \oplus L(2\psi_1 + \cdots + 2\psi_{n-1} + 2\psi_n)$$

Construction for $SL(2n+1, \mathbb{R})$

$$(2n+1) \Rightarrow L(0\psi_1 + 0\psi_2 + \cdots + 0\psi_n)$$

$$(2n+1-k, k) \Rightarrow L(2\psi_1 + 2\psi_2 + \cdots + 2\psi_k), \text{ for all } 0 < k \leq n$$

Possible generalizations...

- Our method for constructing petite K-types appears to be generalizable to other split semi-simple Lie groups, other than $SL(n)$, whose root system admits one root length.

- As ρ varies in the set of Weyl group repr.s that do not contain the sign of $\mathcal{S}_3$, the output will be a list of petite K-types on which the intertwining operator for a spherical principal series can be constructed by means of Weyl group computations.

- The final result will be a non-unitarity test for a spherical principal series for split groups of type $\mathcal{A}$, $\mathcal{D}$, E_6, E_7 and E_8.

Construction for the representation $S^{2,2}$ of $\mathcal{S}_4$

We can choose a basis $\{v, u\}$ of $F_\rho = \mathbb{C}^2$ that consists of a $(+1)$ and a (-1) eigenvector of σ_{12}. W.r.t. this basis, we have:

$$\sigma_{12},\, \sigma_{34} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

u: (-1) eigenvector of σ_{12}, σ_{34}

$$\sigma_{13},\, \sigma_{24} \rightsquigarrow \begin{pmatrix} -1/2 & -1/2 \\ -3/2 & 1/2 \end{pmatrix}$$

$u + v$: (-1) eigenvector of σ_{13}, σ_{24}

$$\sigma_{23},\, \sigma_{14} \rightsquigarrow \begin{pmatrix} -1/2 & 1/2 \\ 3/2 & 1/2 \end{pmatrix}$$

$v - u$: (-1) eigenvector of σ_{23}, σ_{14}.

Complete list of generators:

$$\begin{array}{cccc} u & v & Z_{12}u & Z_{34}u \\ Z_{13}(u+v) & Z_{24}(u+v) & Z_{23}(v-u) & Z_{14}(v-u) \\ Z_{12}Z_{34}u & Z_{13}Z_{24}(v+u) & Z_{23}Z_{14}(v-u). & \end{array}$$

There are no relations among strings of length one, and there is only one relation among strings of length two:

$$Z_{12}Z_{34}u = -\frac{1}{2}Z_{23}Z_{14}(v-u) - \frac{1}{2}Z_{13}Z_{24}(u+v).$$

The extension has dimension 10, and the corresponding representation of $Lie(K)$ is $\rho_{2\varepsilon_1+2\varepsilon_2} \oplus \rho_{2\varepsilon_1-2\varepsilon_2}$.

For all α, β in Δ, we have:

$$\mathrm{Ad}(\sigma_\beta^2)(Z_\alpha) = \begin{cases} +Z_\alpha & \text{if } \alpha = \beta \text{ or } \alpha \perp \beta \\ -Z_\alpha & \text{otherwise.} \end{cases}$$

For every string $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ and for every positive root β

$$\sigma_\beta^2 \cdot (Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v) = (-1)^{\#\{j:\ [Z_\beta, Z_{\alpha_j}] \neq 0\}} (Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v).$$

- Let Φ be a root system with one root-length. For all α, β in Φ, $s_\beta(\alpha)$ cannot be orthogonal to α.
- Let $\alpha_1, \ldots, \alpha_r$ be mutually orthogonal positive roots and let v be an element of F_ρ satisfying

$$\sigma_{\alpha_1} \cdot v = \cdots = \sigma_{\alpha_r} \cdot v = -v.$$

Let ν be any positive root. Then

(i) $S = Z_{\alpha_1} \ldots Z_{\alpha_r} v$ is a $(+1)$-eigenvector of σ_ν if and only if the following conditions are satisfied:

- $\sigma_\nu \cdot v = +v$
- $\nu \perp \{\alpha_1, \ldots, \alpha_r\}$.

(ii) $S = Z_{\alpha_1} \ldots Z_{\alpha_r} v$ is a (-1)-eigenvector of σ_ν if and only of the following conditions are satisfied:

- $\sigma_\nu \cdot v = -v$
- ν belongs to the set $\{\alpha_1, \ldots, \alpha_r\}$, or it is orthogonal to it.