# Parameters for Representations of Real Groups
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The basic references are [7] and [6]. The parameters given in these notes only exist in the unpublished preprint [4]. The case of regular integral infinitesimal character is discussed in [1]. Everything appears, sometimes in somewhat different form, in [2].

## 1 Algebraic Groups and Root Data

A root datum is a quadruple

$$(X, \Delta, X^\vee, \Delta^\vee)$$

where $X, X^\vee$ are free abelian groups of finite rank, and $\Delta, \Delta^\vee$ are finite subsets of $X, X^\vee$, respectively. In addition there is a perfect pairing $\langle , \rangle : X \times X^\vee \to \mathbb{Z}$ so $X^\vee \simeq \mathrm{Hom}(X, \mathbb{Z})$. There must exist a bijection $\alpha \to \alpha^\vee : \Delta \to \Delta^\vee$ such that for all $\alpha \in \Delta$,

$$\langle \alpha, \alpha^\vee \rangle = 2,\ s_\alpha(\Delta) = \Delta,\ s_{\alpha^\vee}(\Delta^\vee) = \Delta^\vee.$$

Heer $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ and $s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee$ $(x \in X, y \in X^\vee)$.

By [3] (applied to $\mathbb{Z}\langle\Delta\rangle$ and $\mathbb{Z}\langle\Delta^\vee\rangle$) the conditions determine the bijection uniquely. In particular $(X, \Delta, X^\vee, \Delta^\vee)$ is determined by $(X, \Delta)$ if $Z\langle\Delta\rangle = X$ (the semi–simple case).

Suppose $\Delta^+$ is a set of positive roots of $\Delta$. Then $\Delta^{\vee +} = \{\alpha^\vee \mid \alpha \in \Delta^+\}$ is a set of positive roots of $\Delta^\vee$, and

$$(X, \Delta^+, X^\vee, \Delta^{\vee +})$$

is a based root datum.

Two root systems are isomorphic if there exists an isomorphism $\phi : X \to X'$ such that $\phi(\Delta) = \Delta'$ and $\phi^t(\Delta'^\vee) = \Delta^\vee$. Here $\phi^t : X'^\vee \to X$ is given by

$$\langle \phi(x), y' \rangle = \langle x, \phi^t(y') \rangle \quad (x \in X, y' \in X'^\vee). \tag{1.1}$$

Let $\mathbb{G}$ be a connected reductive algebraic group and choose a Cartan subgroup $\mathbb{T}$. The corresponding root data is

$$D = (X^*(\mathbb{T}), \Delta, X_*(\mathbb{T}), \Delta^\vee)$$

where $X^*(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m), X_*(\mathbb{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$, $\Delta = \Delta(\mathbb{G}, \mathbb{T})$ is the set of roots of $\mathbb{T}$ in $\mathbb{G}$, and $\Delta^\vee = \Delta^\vee(\mathbb{G}, \mathbb{T})$ is the set of co-roots.

If $\mathbb{T}'$ is another Cartan subgroup the associated root data is isomorphic to the given one. This isomorphism is canonical up to the Weyl group.

There is an exact sequence

$$1 \to \mathrm{Int}(\mathbb{G}) \to \mathrm{Aut}(\mathbb{G}) \to \mathrm{Out}(\mathbb{G}) \to 1 \tag{1.2}$$

where $\mathrm{Int}(\mathbb{G}) \simeq \mathbb{G}$ is the group of inner automorphism of $\mathbb{G}$, $\mathrm{Aut}(\mathbb{G})$ is the automorphism group of $\mathbb{G}$, and and $\mathrm{Out}(\mathbb{G})$ is the group of outer automorphisms.

Fix a Borel subgroup $\mathbb{B}$, a Cartan subgroup $\mathbb{T}$ contained in $\mathbb{B}$, and a set of root vectors for the simple roots of $\mathbb{T}$ in $\mathbb{B}$. Associated to this *splitting data* is a splitting of (1.2); for $\gamma \in \mathrm{Out}(\mathbb{G})$ take $s(\gamma) \in \mathrm{Aut}(\mathbb{G})$ to be the unique pre–image fixing this data (as sets). This gives a canonical isomorphism

$$\mathrm{Out}(\mathbb{G}) \simeq \mathrm{Aut}(D). \tag{1.3}$$

Fix $\gamma \in \mathrm{Out}(\mathbb{G})$. Define

$$Z(\mathbb{G})^\gamma = \{z \in Z(\mathbb{G}) \mid s(\gamma) z s(\gamma)^{-1} = z\} \tag{1.4}$$

This is independent of the choice of a splitting $s$ of (1.2).

If $\mathbb{G}$ is semisimple then $\mathrm{Out}(\mathbb{G})$ is isomorphic to the automorphisms of the Dynkin diagram of $\mathbb{G}$.

If $\mathbb{G} = \mathbb{T}$ is a torus an automorphism $\theta$ is determined by an automorphism of $X_*(\mathbb{T})$. If $\theta$ has order 2 then there is a basis

$$x_1, \dots, x_r, y_1, \dots y_s, z_1, z_1', \dots, z_t, z_t'$$

of $X_*(\mathbb{T})$ so that $\theta(x_i) = x_i$, $\theta(y_i) = -y_i$, and $\theta(z_i) = z_i', \theta(z_i') = z_i$.

In general $\mathbb{G} = \mathbb{T}\mathbb{G}_d$ where $\mathbb{T} = Z(\mathbb{G})^0$ is a central torus, and an automorphism is given by automorphisms of $\mathbb{T}$ and $\mathbb{G}_d$, which agree on $\mathbb{T} \cap \mathbb{G}_d$.

# 2 The Dual Group and the Dual Automorphism

Suppose we are given $\mathbb{G}$ with corresponding root data $D = (X, \Delta, X^\vee, \Delta^\vee)$. The *dual root data* is $D^\vee = (X^\vee, \Delta^\vee, X, \Delta)$, and the *dual group* is the group ${}^\vee\mathbb{G}$ defined by $D^\vee$.

If $\tau \in \mathrm{Aut}(D)$ let ${}^\vee\tau = -\tau^t \in \mathrm{Aut}(D^\vee)$ (cf. Section 1). We obtain isomorphisms

$$\mathrm{Out}(\mathbb{G}) \simeq \mathrm{Aut}(D) \simeq \mathrm{Aut}(D^\vee) \simeq \mathrm{Out}({}^\vee\mathbb{G}). \tag{2.1}$$

where the middle term is $\tau \to {}^\vee\tau$.

**Definition 2.2** *For $\gamma \in Out(\mathbb{G})$ define ${}^\vee\gamma \in Out({}^\vee\mathbb{G})$ by* (2.1).

# 3 Real Forms of $\mathbb{G}$

To say that $\mathbb{G}$ is defined over $\mathbb{R}$ means that there is an anti–holomorphic involution $\sigma$ of $\mathbb{G}(\mathbb{C})$. Then $G(\mathbb{R}) = \mathbb{G}(\mathbb{C})^\sigma$, and we will write $G = \mathbb{G}(\mathbb{R})$. We say $\sigma$ is equivalent to $\sigma'$ if $\sigma' = int(g) \circ \sigma \circ \mathrm{int}(g^{-1})$ for some $g \in \mathbb{G}$, i.e.

$$\sigma(x) = g\sigma(g^{-1}xg)g^{-1} \quad (x \in \mathbb{G}(\mathbb{C})).$$

An involution of $\mathbb{G}$, i.e. an algebraic automorphism of $\mathbb{G}$ of order 2, may be considered a holomorphic involution of $\mathbb{G}(\mathbb{C})$. We say involutions $\theta, \theta'$ are equivalent if $\theta = int(g) \circ \theta' \circ \mathrm{int}(g^{-1})$ for some $g \in \mathbb{G}$.

Suppose $\mathbb{G}$ is defined over $\mathbb{R}$, with corresponding anti–holomorphic involution $\sigma$. We may choose an involution $\theta$ of $\mathbb{G}$, a "Cartan involution", such that $K = G^\theta$ is a maximal compact subgroup of $G$. Then $\mathbb{K} = \mathbb{G}^\theta$ is the algebraic group corresponding to $K$, and $\mathbb{K}(\mathbb{C}) = \mathbb{G}(\mathbb{C})^\theta$.

**Lemma 3.1** *The map taking an anti–holomorphic involution $\sigma$ to a corresponding Cartan involution $\theta$ is a bijection between equivalence classes of real forms and equivalence classes of involutions.*

We work entirely with Cartan involutions.

**Definition 3.2** *We say two involutions $\theta, \theta'$ are inner if they have the same image in $Out(\mathbb{G})$, i.e. there exists $g \in G$ such that $\theta' = int(g) \circ \theta$, or*

$$\theta'(x) = g\theta(x)g^{-1}. \quad (x \in \mathbb{G}).$$

*This is an equivalence relation, and an equivalence class is called an inner class. Such a class is determined by an involution $\gamma \in Out(\mathbb{G})$, and we refer to $\gamma$ as an inner class.*

**Definition 3.3** *We say two real forms of $\mathbb{G}$ are inner if their Cartan involutions $\theta, \theta'$ are inner.*

# 4 Basic Data

Fix $\mathbb{G}$. By Definition 3.2 an inner class of real forms is given by an involution $\gamma \in \mathrm{Out}(\mathbb{G})$.

Thus our basic data will be a pair $(\mathbb{G}, \gamma)$ where $\gamma$ is an involution in $\mathrm{Out}(\mathbb{G})$. By Section 2 we obtain $({}^\vee\mathbb{G}, {}^\vee\gamma)$.

# 5 Principal and Distinguished Involutions

**Definition 5.1** *An involution $\theta$ of $\mathbb{G}$ is principal if the corresponding real group $G$ is quasisplit, i.e. contains a Borel subgroup.*

**Lemma 5.2** *The following conditions are equivalent*

1. *$\theta$ is a principal involution*

2. *There is a $\theta$–stable Cartan subgroup $\mathbb{T}$ with no imaginary roots,*

3. *There are a $\theta$–stable Cartan subgroup $\mathbb{T}$ and a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$, such that every simple root of $\mathbb{T}$ is complex or non–compact imaginary.*

Every real form is inner to a quasiplit group:

**Lemma 5.3** *Any inner class of involutions contains a principal involution, which is unique up to conjugation by $\mathbb{G}$.*

That is given $\theta_0 \in \mathrm{Out}(\mathbb{G})$ there exists a principal involution $\theta \in \mathrm{Aut}(\mathbb{G})$ with image $\theta_0$, and if $\theta, \theta'$ are two such, then $\theta' = \mathrm{int}(g)\theta\mathrm{int}(g)^{-1}$ for some $g \in \mathbb{G}$.

**Definition 5.4** *An involution is said to be distinguished if there are $\theta$–stable Cartan and Borel subgroups $\mathbb{T} \subset \mathbb{B}$ so that every simple imaginary root is compact (equivalently: every simple root is compact imaginary or complex). A real form is said to be distinguished if its Cartan involution is distinguished.*

Every real group has a distinguished inner form:

**Lemma 5.5** *Any inner class of involutions contains a distinguished involution, and any two such are conjugate by $\mathbb{G}$.*

# 6 Encoding real forms

Fix $(\mathbb{G}, \gamma)$ as in Section 4. Let $\Gamma = \{1, \sigma\} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Choose a involution $\theta_0$ in the inner class of $\gamma$. Consider the group $\mathbb{G} \rtimes \Gamma$ where the action of $\sigma$ on $\mathbb{G}$ is by $\theta_0$. That is $\mathrm{int}(\sigma) = \theta_0$.

Suppose $\theta$ is a Cartan involution of a real form in the same inner class. Then $\theta = \mathrm{int}(g) \circ \theta_0$. Let $\delta = g\sigma \in \mathbb{G} \rtimes \Gamma - \mathbb{G}$. Then

$$\theta = \mathrm{int}(\delta).$$

That is every Cartan involution in this inner class is given by conjugation by an element of $\mathbb{G} \rtimes \Gamma$.

It is natural to take $\theta_0$ to be either a principal involution or a distinguished involution in the inner class (cf. Section 5).

Note that

$$\delta^2 = g\sigma(g) \in Z(\mathbb{G})^\gamma$$

(cf. 1.4).

# 7 L-Groups: Version 1

Fix $(\mathbb{G},\gamma)$ as in Section 4.

Roughly speaking the L–group of $\mathbb{G}$ is the semidirect product ${}^\vee\mathbb{G}\rtimes\Gamma$ where $\sigma$ acts on $\mathbb{G}$ by a distinguished involution in the inner class of ${}^\vee\gamma\in\mathrm{Aut}({}^\vee\mathbb{G})$ (Definition 2.2).

More precisely we need to incorporate a conjugacy class of such splittings into the data:

**Definition 7.1** *An L–group for $\mathbb{G}$ is a pair $({}^\vee\mathbb{G}^\Gamma,\mathcal{S})$, where ${}^\vee\mathbb{G}^\Gamma$ fits in an exact sequence*

$$1\to{}^\vee\mathbb{G}\to{}^\vee\mathbb{G}^\Gamma\to\Gamma\to 1$$

*and $\mathcal{S}$ is a ${}^\vee\mathbb{G}$–conjugacy class of splittings of this exact sequence, such that for $s\in\mathcal{S}$, int$(s(\sigma))$ is a distinguished involution in the inner class of ${}^\vee\gamma$.*

**Remark 7.2** There is a unique quasisplit group $G$ in the given inner class (in fact a unique strong inner form, cf. Section 9). This has a distinguished representation $\pi_0$: the spherical principal series with infinitesimal character 0.

The Weil group (cf. Section 15) maps to $\Gamma$, and therefore a homomorphsim $\phi:\Gamma\to{}^\vee\mathbb{G}^\Gamma$ defines an irreducible representation of $G$ (in fact an L–packat which is a singleton in this case). There is not necessarily a distinguished homomorphism $\phi:\Gamma\to{}^\vee\mathbb{G}^\Gamma$. The choice of L–group structure is such a homomorphism $\phi$, and the choice of L–group struture amounts to declaring that $\phi$ corresponds to $\pi_0$.

# 8 Basic Data Revisited

Fix $(\mathbb{G},\gamma)$ as in Section 4. We obtain ${}^\vee\mathbb{G}$ and ${}^\vee\gamma\in\mathrm{Out}(\mathbb{G})$ as in Section 2. We may therefore think of this as a quadruple

$$(\mathbb{G},\gamma,{}^\vee\mathbb{G},{}^\vee\gamma).$$

We may define $({}^\vee\mathbb{G}^\Gamma,{}^\vee\mathcal{S})$, as in Section 7. The same definition applied to $({}^\vee\mathbb{G},{}^\vee\gamma)$ gives us a group $(\mathbb{G}^\Gamma,\mathcal{S})$.

# 9 Strong Real Forms

Fix $(\mathbb{G},\gamma)$ as in Section 4, and $\mathbb{G}^\Gamma, {}^\vee\mathbb{G}^\Gamma$ as in Section 8. We apply the discussion of Section 6 to $\mathbb{G}^\Gamma$.

**Definition 9.1** *A strong real form of* $\mathbb{G}$ *is an element* $x \in \mathbb{G}^\Gamma - \mathbb{G}$ *satisfying* $x^2 \in Z(\mathbb{G})$. *We say two strong real forms* $x, x'$ *are equivalent if* $x$ *is conjugate to* $x'$.

**Lemma 9.2** *If* $x$ *is a strong real form of* $\mathbb{G}$ *let* $\theta_x = int(x)$. *This is the Cartan involution of a real form in the inner class* $\gamma$. *This map is surjective onto the real forms in this inner class. If* $\mathbb{G}$ *is adjoint it is a bijection.*

# 10 Representations

Fix $(\mathbb{G},\gamma)$ as in Section 4, Fix $\mathbb{G}$, an inner class $\gamma \in \mathrm{Out}(\mathbb{G})$, and $(\mathbb{G}^\Gamma, {}^\vee\mathbb{G}^\Gamma)$ as in Section 8.

**Definition 10.1** *A representation of a strong real form of* $\mathbb{G}$ *is a pair* $(x,\pi)$ *where* $x$ *is a strong real form of* $\mathbb{G}$ *and* $\pi$ *is a* $(\mathfrak{g},\mathbb{K}_x)$*–module.*

*We say* $(x,\pi)$ *is equivalent to* $(x',\pi')$ *if there exists* $g \in \mathbb{G}$ *such that* $gxg^{-1} = x'$ *and* $g\cdot\pi \simeq \pi'$. *Here* $g\cdot\pi(h) = \pi(g^{-1}hg)$ *for* $h \in \mathbb{K}_{x'}$, *and* $g\cdot\pi(X) = \pi(Ad(g^{-1})X)$ *for* $X \in \mathfrak{g}$.

Suppose $\zeta$ is a distinguished isomorphism. Then $\zeta$ induces bijections:

$$\Delta^\vee(\mathbb{G},\mathbb{T}) \simeq \Delta({}^\vee\mathbb{G},{}^d\mathbb{T}) \tag{10.2}$$

$$\Delta(\mathbb{G},\mathbb{T}) \simeq \Delta^\vee({}^\vee\mathbb{G},{}^d\mathbb{T}) \tag{10.3}$$

# 11 Distnguished Isomorphisms

Fix $(\mathbb{G},\gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma,\mathcal{S}),({}^\vee\mathbb{G}^\Gamma,{}^\vee\mathcal{S}))$ as in Section 8.

Suppose $\mathbb{T}$ is a Cartan subgroup of $\mathbb{G}$, and ${}^d\mathbb{T}$ is a Cartan subgroup of ${}^\vee\mathbb{G}$. By the construction of ${}^\vee\mathbb{G}$ there are isomorphisms

$$X_*({}^\vee\mathbb{T}) \simeq X_*({}^d\mathbb{T})$$

and

$$ {}^\vee\mathbb{T} \simeq {}^d\mathbb{T}, \quad {}^\vee\mathfrak{t} \simeq {}^d\mathfrak{t}. $$

Given Borel subgroups $\mathbb{B}, {}^d\mathbb{B}$ containing $\mathbb{T}, {}^d\mathbb{T}$ respectively, we obtain isomorphisms

$$ \zeta(\mathbb{B}, {}^d\mathbb{B}) : {}^\vee\mathbb{T} \simeq {}^d\mathbb{T}, \quad {}^\vee\mathfrak{t} \simeq {}^d\mathfrak{t}. $$

Also recall $X^*(\mathbb{T}) = X_*({}^\vee\mathbb{T})$ and $\mathfrak{t}^* = {}^\vee\mathfrak{t}$ (canonically). So $\zeta$ may be interpreted as an isomorphism

$$ \zeta : \mathfrak{t}^* \simeq {}^d\mathfrak{t} \tag{11.1} $$

**Definition 11.2** *We say an isomorphism* $\zeta : {}^\vee\mathbb{T} \simeq {}^d\mathbb{T}$ *is distinguished if it is equal to* $\zeta(\mathbb{B}, {}^d\mathbb{B})$ *for some* $\mathbb{B}, {}^d\mathbb{B}$.

Now suppose $\theta$ is an involution of $\mathbb{T}$, and ${}^d\theta$ is an involution of ${}^d\mathbb{T}$. Then (cf. Section 2) ${}^\vee\theta$ is an involution of ${}^\vee\mathbb{T}$. Suppose $\zeta : {}^\vee\mathbb{T} \simeq {}^d\mathbb{T}$ is a distinguished isomorphism. We define an involution $\zeta^*(\theta)$ by carrying the involution ${}^\vee\theta$ of ${}^\vee\mathbb{T}$ to ${}^d\mathbb{T}$ via $\zeta$, i.e.

$$ \zeta^*(\theta)(t) = \zeta({}^\vee\theta(\zeta^{-1}(t))) \quad (t \in {}^d\mathbb{T}). $$

# 12 Integral L–data

Fix $(\mathbb{G}, \gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma, \mathcal{S}), ({}^\vee\mathbb{G}^\Gamma, {}^\vee\mathcal{S}))$ as in Section 8.

Here is the data which will parametrize representations with integral infinitesimal character.

**Definition 12.1** *Fix* $(\mathbb{G}, \gamma)$ *as in Section 4, and* $((\mathbb{G}^\Gamma, \mathcal{S}), ({}^\vee\mathbb{G}^\Gamma, {}^\vee\mathcal{S}))$ *as in Section 8.*

*A set of weak integral L–data is a* 6*-tuple* $(x, \mathbb{T}, \mathbb{B}, y, {}^\vee\mathbb{T}, {}^\vee\mathbb{B})$ *where*

1. $\mathbb{T} \subset \mathbb{B} \subset \mathbb{G}$ *are a Cartan and Borel subgroup, respectively,*
2. $x^2 \in Z(\mathbb{G})$,
3. $\mathbb{T}$ *is* $\theta_x$*–stable where* $\theta_x = int(x)$,

4. ${}^\vee\mathbb{T} \subset {}^\vee\mathbb{B} \subset {}^\vee\mathbb{G}$ *are a Cartan and Borel subgroup, respectively,*

5. $y^2 \in Z({}^\vee\mathbb{G})$,

6. ${}^\vee\mathbb{T}$ *is* ${}^\vee\theta_y$*–stable where* ${}^\vee\theta_y = int(y)$,

7. *The isomorphism* $\zeta = \zeta(\mathbb{B}, {}^\vee\mathbb{B})$ *satisfies* $\zeta^*(\theta_x) = {}^\vee\theta_y$,

*A set of (integral) L–data is a pair* $(S, \lambda)$ *where* $S = (x, \mathbb{T}, \mathbb{B}, y, {}^\vee\mathbb{T}, {}^\vee\mathbb{B})$ *is a set of weak L–data,* $\lambda \in {}^\vee\mathfrak{t}$*, and* $exp(2\pi i\lambda) = y^2$.

If $(S, \lambda)$ is a set of strong integral L–data let $\zeta = \zeta(\mathbb{B}, {}^d\mathbb{B})$, and identify $\lambda$ with an element of $\mathfrak{t}^*$ via (11.1).

# 13 L–data

Fix $(\mathbb{G}, \gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma, \mathcal{S}), ({}^\vee\mathbb{G}^\Gamma, {}^\vee\mathcal{S}))$ as in Section 8.

We generalize the construction of the previous section to include representations with non–integral infinitesimal character.

**Definition 13.1** *Fix* $(\mathbb{G}, \gamma)$ *as in Section 4, and* $((\mathbb{G}^\Gamma, \mathcal{S}), ({}^\vee\mathbb{G}^\Gamma, {}^\vee\mathcal{S}))$ *as in Section 8.*

*A set of weak L–data is a* 6*-tuple* $(x, \mathbb{T}, P, y, {}^\vee\mathbb{T}, {}^\vee P)$ *where*

1. $\mathbb{T} \subset \mathbb{G}$ *is a Cartan subgroup,*

2. $x^2 \in Z(\mathbb{G})$,

3. $\mathbb{T}$ *is* $\theta_x$*–stable where* $\theta_x = int(x)$,

4. $P$ *is contained in a set of positive roots of* $\Delta(\mathbb{T}, \mathbb{G})$,

5. ${}^\vee\mathbb{T} \subset {}^\vee\mathbb{G}$ *is a Cartan subgroup,*

6. $y^2 \in {}^\vee\mathbb{T}$

7. ${}^\vee\mathbb{T}$ *is* ${}^\vee\theta_y$*–stable where* ${}^\vee\theta_y = int(y)$*, an involution of* ${}^\vee\mathbb{G}_{y^2} = Cent_{{}^\vee\mathbb{G}}(y^2)$,

8. ${}^\vee\mathbb{B}$ *is a Borel subgroup of* ${}^\vee\mathbb{G}_{y^2}$ *containing* $\mathbb{T}$,

9. *There is a distinguished isomorphism* $\zeta$ *satisfying:* $\zeta^*(\theta_x) = {}^\vee\theta_y$ *and* $\Delta({}^\vee\mathbb{B}, {}^\vee\mathbb{T}) = \{\zeta(\alpha^\vee) \,|\, \alpha \in P\}$.

*A set of L–data is a pair* $(S, \lambda)$ *where* $S = (x, \mathbb{T}, \mathbb{B}, y, {}^\vee\mathbb{T}, {}^\vee\mathbb{B})$ *is a set of weak L–data,* $\lambda \in {}^\vee\mathfrak{t}$*, and* $exp(2\pi i\lambda) = y^2$.

If $(S, \lambda)$ is a set of L–data let $\zeta$ be any distinguished isomorphism as in (9). Then we identify $\lambda$ with an element of $\mathfrak{t}^*$ via (11.1).

**Remark 13.2**

# 14 Parametrization of Representations

Fix $(\mathbb{G}, \gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma, \mathcal{S}), ({}^\vee\mathbb{G}^\Gamma, {}^\vee\mathcal{S}))$ as in Section 8.

Suppose $(S, \lambda)$ is a set of strong L–data. Associated to $(S, \lambda)$ is a standard $(\mathfrak{g}, \mathbb{K}_x)$–module $I(S, \lambda)$ with infinitesimal character (the $\mathbb{G}$–orbit of) $\lambda$.

We will idenfity $\lambda$ with an element of $\mathfrak{t}^*$ via $\zeta$ as at the end of Section 13.

## 14.1 Regular Infinitesimal Character

Assume $\lambda$ is regular, i.e. $\langle\alpha, \lambda\rangle \neq 0$ for all $\alpha \in \Delta({}^\vee\mathbb{G}, {}^\vee\mathbb{T})$. Then $I(S, \lambda)$ contains a unique irreducible (Langlands) submodule $J(S, \lambda)$.

**Theorem 14.1** *Suppose* $(x, \pi)$ *is an irreducible representation of a strong real form of* $\mathbb{G}$ *(Section 10) with regular infinitesimal character. Then* $\pi \simeq J(S, \lambda)$ *for some* $S = (x, \dots)$ *and* $\lambda$*. Two non–zero representations* $(x, J(S, \lambda))$ *and* $(x, J(S', \lambda'))$ *are isomorphic if and only if* $(S, \lambda)$ *is* $\mathbb{G} \times {}^\vee\mathbb{G}$ *conjugate to* $(S', \lambda')$.

## 14.2 General Infinitesimal Character

Let $J(S, \lambda)$ be the socle of $I(S, \lambda)$, i.e. the direct sum of the irreducible subrepresentations of $I(s, \lambda)$. [Q: we need to define this using the translation principle?]

We say $(S, \lambda)$ is $M$–regular if $\langle\lambda, \alpha^\vee\rangle \neq 0$ for all imaginary roots (with respect to $\theta_x$) of $\Delta(\mathbb{G}, \mathbb{T})$ [there may be a $\rho$–shift missing here].

**Theorem 14.2** *Suppose $(x,\pi)$ is an irreducible representation of a strong real form of $\mathbb{G}$ (Section 10). Then there exists strong, M–regular, L–data $(S,\lambda)$ so that $\pi$ is a subrepresentation of $J(S,\lambda)$. If $(S',\lambda')$ also satisfies these conditions then $(S',\lambda')$ is $\mathbb{G}\times {}^\vee\mathbb{G}$–conjugate to $(S,\lambda)$.*

This gives a finite to one map from equivalence classes of strong, M–regular, L–data $(S,\lambda)$ to equivalence classes $(x,\pi)$ of representations of strong real forms of $\mathbb{G}$. This map is a bijection in the case of regular infinitesimal character. In Section 16 we will describe how to compute the fiber of this map.

# 15 Sketch of the Construction of $I(S,\lambda)$

Fix $(\mathbb{G},\gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma,\mathcal{S}),({}^\vee\mathbb{G}^\Gamma,{}^\vee\mathcal{S}))$ as in Section 8. Let $(S,\lambda)$ be a set of L–data, where $S=(x,\mathbb{T},P,y,{}^\vee\mathbb{T},{}^\vee\mathbb{B})$.

Recall the Weil group is $W_\mathbb{R}=\langle\mathbb{C}^*,j\rangle$ where $j^2=-1$ and $jzj^{-1}=\overline{z}$.

The data $(y,{}^\vee\mathbb{T},{}^\vee\mathbb{B},\lambda)$ defines an L–homomorphism $\phi: W_\mathbb{R}\to {}^\vee\mathbb{G}^\Gamma$ as follows:

$$\phi(z)=z^\lambda\overline{z}^{Ad(y)\lambda}$$
$$\phi(j)=exp(-\pi i\lambda) \tag{15.3}$$

where $z^\lambda=exp(\lambda\log(z))$ ($z\in\mathbb{C}^*\subset W_\mathbb{R}$) (it requires a short argument that $\phi(z)$ is well defined).

Then $\phi: W_\mathbb{R}\to\langle\mathbb{T}^\vee,y\rangle$. This is not necessarily isomorphic to the L–group of $T$. It is isomorphic to an E–group ${}^\vee\mathbb{T}^\Gamma$ of $\mathbb{T}$, and maps into ${}^\vee\mathbb{T}^\gamma$ parametrize characters of the $\rho$–cover $T(\mathbb{R})_\rho$ of $T(\mathbb{R})$.

The extra data in $S$ gives us an isomorphism of $\langle\mathbb{T},y\rangle\simeq{}^\vee\mathbb{T}^\Gamma$, and hence a character $\Lambda$ of $T(\mathbb{R})_\rho$.

For example in the case of a discrete series representation $\Lambda$ is a character with differential the Harish–Chandra parameter $\lambda$; recall that $\lambda-\rho$ (and not necessarily $\lambda$) exponentiates to the compact Cartan.

If $\lambda$ is regular $\Lambda$ is all that is needed to define a standard module $I(\Psi,\Lambda)$ as in [5, Definition 8.27]. If $\Lambda$ is singular an extra choice of positive real roots is necessary. This is included in the data of $S$.

The module $I(\Psi, \Lambda)$ may be written as cohomological induction from a principal series representation of a quasisplit group $L$. The reducibility of $J(S, \lambda)$ (for singular $\lambda$) comes from the reducibility of the corresponding standard module for $L$.

Therefore the fiber of the map described in Theorem 14.2 is obtained from the discussion in the next section applied to $L$.

# 16 R–Groups

We need some definitions and results from [7, Chapter 4].

Suppose $G$ is quasiplit and $H = TA$ is the maximally split Cartan subgroup. Let $M = Cent_G(A)$; this is an abelian group. We say a character $\delta$ of $M$ is fine if its restriction to $(G^d \cap M)^0$ is trivial [7, Definition 4.3.8]. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$, the (non–zero) real roots, and $\overline{\Delta} \subset \Delta$ the reduced root system of $\Delta$. We say a root $\alpha$ of $\Delta$ is real if it is the restriction of a real root of $\Delta(\mathfrak{g}, \mathfrak{h})$, and complex otherwise.

$$W = Norm_K(A)/M \tag{16.1}$$
$$= W(\overline{\Delta}). \tag{16.2}$$

Let $\overline{\Delta}_\delta$ be the good roots ([7, Definition 4.3.11]). That is

$$\overline{\Delta}_\delta = \{\alpha \in \overline{\Delta} \,|\, \alpha \text{ is complex or } \alpha \text{ is real and } \delta(m_\alpha) = 1\}$$

Fix $\nu \in \hat{A}$. As in [7, Definitions 4.3.13 and 4.4.9] define

$$\begin{aligned}
W_\delta &= Stab_W(\delta)\\
W_\delta^0 &= W(\overline{\Delta}_\delta)\\
R_\delta &= W_\delta/W_\delta^0\\
W(\nu) &= Stab_W(\nu)\\
W_\delta(\nu) &= Stab_W(\delta \otimes \nu)\\
W_\delta^0(\nu) &= Stab_{W_\delta^0}(\delta \otimes \nu)\\
R_\delta(\nu) &= W_\delta(\nu)/W_\delta^0(\nu) \subset R_\delta\\
R_\delta^\perp(\nu) &= \text{annihlator of } R_\delta(\nu) \text{ in } \widehat{R_\delta}
\end{aligned}$$

Note that $\widehat{R_\delta}/R_\delta^\perp(\nu) \simeq \widehat{R_\delta(\nu)}$.

**Definition 16.3** *Suppose* $(S.\lambda)$ *is a set of strong L–data. The R–group for S, denoted* $R(S,\lambda)$ *is* $\widehat{R_\delta(\nu)}$ *computed on L ... [Assignment part 1: make this precise! It comes down to the real roots - a computation involving the principal series of the quasisplit group L].*

If $\lambda$ is regular then $R(S,\lambda) = 1$.

**Lemma 16.4** *The fiber of the map of Theorem 14.2 is naturally parametrized by* $R(S,\lambda)$ *[Assignment part 2: so that this lemma holds].*

# 17 L–packets and Blocks

Fix $(\mathbb{G},\gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma,\mathcal{S}),({}^\vee\mathbb{G}^\Gamma,{}^\vee\mathcal{S}))$ as in Section 8.

Fix $y, {}^\vee\mathbb{T}, {}^\vee\mathbb{B}$ as in the definition of L–data, and $\lambda$ satisfying $exp(2\pi i\lambda) = y^2$. Recall (Section 15) this data defines an L–homomorphism $\phi : W_\mathbb{R} \to {}^\vee\mathbb{G}^\Gamma$.

We assume $\lambda$ is regular.

**Definition 17.1** *An L–packet is the set of representation* $J(S,\lambda)$ *where* $(S = (x,\mathbb{T},P,y,{}^\vee\mathbb{T},{}^\vee\mathbb{B}),\lambda)$ *is a set of L–data.*

This is sometimes called a "super" L–packet; it includes representations on various strong real forms. Its restriction to a single strong real form is a conventional L–packet.

[Question: singular infinitesimal character?]

**Definition 17.2** *Fix* $x, y$ *satisfying* $x^2 \in Z(\mathbb{G})$. *The* $\mathbb{Z}$*–spane of the representation* $J(S,\lambda)$ *where* $(S = (x,\mathbb{T},P,y,{}^\vee\mathbb{T},{}^\vee\mathbb{B}),\lambda)$ *is a set of L–data is a block.*

Again this is sometimes called a super–block. The restriction to a strong real form is a block. This is a minimal subspace of the Grothendieck group which is spanned by both irreducible and standard modules. Thus the Kazhdan–Lusztig polynomials are defined on blocks.

# 18 Example: $SL(2)$

Let $\mathbb{G} = SL(2)$. Then $\mathrm{Out}(\mathbb{G}) = 1$ so $\gamma = 1$. We have $(\mathbb{G},\gamma,{}^\vee\mathbb{G},\gamma^\vee) = (SL(2),1,PSL(2),1)$.

We fix some notation. Let $\mathbb{B}^{\pm}$ be the upper and lower triangular matrices in $SL(2)$ respectively. Let $\mathbb{T}$ be the diagonal Cartan subgoup. Write $\mathbb{B}^{\pm}$ and $\mathbb{T}$ for $PSL(2)$ as well. (We abuse notation slightly and write $PSL(2)$ as $2\times 2$ matrices.)

Let $t(z) = diag(z, 1/z), m_\rho = t(i)$. Note that in $PGL(2)$ $t(z) = t(-z)$. Let $\lambda(z) = diag(z, -z) \in {}^\vee\mathfrak{t}$.

The group ${}^\vee\mathbb{G}^\Gamma$ is generated by ${}^\vee\mathbb{G}$ and an element ${}^\vee\delta$ satisfying ${}^\vee\delta^2 = 1$ and ${}^\vee\delta g {}^\vee\delta^{-1} = m_\rho g m_\rho^{-1}$.

The group $\mathbb{G}^\Gamma$ is generated by $\mathbb{G}$ and $\delta$, where $\delta^2 = -I$ and $\delta g \delta^{-1} = m_\rho g m_\rho^{-1}$.

There is a unique L–group structure $({}^\vee\mathbb{G}^\Gamma, \{{}^\vee\delta, \mathbb{B}^+\})$. Here $\{{}^\vee\delta, \mathbb{B}^+\}$ denotes the ${}^\vee\mathbb{G}$ conjugacy class of $({}^\vee\delta, \mathbb{B}^+)$.

There are two L–group structures $(\mathbb{G}^\Gamma, \{\pm\delta, \mathbb{B}^+)\}$. Note that $(\delta, \mathbb{B}^+)$ is conjugate to $(-\delta, \mathbb{B}^-)$. This corresponds to the fact that $PGL(2, \mathbb{R})$ has two one–dimensional representations, and this choice amounts to choosing one of these. Dually this corresponds to choosing a discrete series representation of $SL(2, \mathbb{R})$ with infinitesimal character $\rho$.

There are three inequivalent strong real forms of $\mathbb{G}$, given by $x = \delta, \pm m_\rho\delta$. The corresponding real groups are $SL(2, \mathbb{R})$ and $SU(2)$, respectively. These may be though of as $SU(2, 0), SU(1, 1)$ and $SU(0, 2)$.

There are two inequivalent strong real forms of ${}^\vee\mathbb{G}$, since it is adjoint, corresponding to $PGL(2, \mathbb{R})$ and $SO(3)$, respectively.

# 19 Other Parametrizations

There are several other ways to parametrize the standard and irreducible representations of real groups. The problem is how to conveniently write down characters of Cartan subgroups; disconnectedness is the main issue.
**Assignment:** Carefully write down how to go back and forth between these classifications.
**1** $\theta$–stable data $(\mathfrak{q}, H, \delta, \nu)$ ([7, Definition 6.5.1] This realizes the standard modules as derived functor modules from a minimal principal series of a quasipslit group $L$.
**2** Character data $(H, \gamma)$ with $\gamma = (\Gamma, \overline{\gamma})$, [7, Definition 6.6.1]. Here $\Gamma$ is a character of $H$, not of a two–fold cover as in (43). The infinitesimal character is $\overline{\gamma}$, which is $d\Gamma +$ a $\rho$–shift.

**3** Cuspidal data $(M, \delta, \nu)$ [7, Definition 6.6.11]. Here $M$ is a real Levi factor and $\delta$ is a (relative) discrete series representation of $M$. This is the original Langlands version of the classification.
**4** $I(\Psi, \Lambda)$ ([5, Definition 8.27 and Theorem 8.29]) Here $\Lambda$ is a character of the $\rho$–cover of $H$, and the infinitesimal character is $d\Lambda$.

# 20 Vogan Duality

The irreducible representations of strong real forms of $\mathbb{G}$ are parametrized by integral $L$–data $(x, \mathbb{T}, \mathbb{B}, y, {}^\vee\mathbb{T}, {}^\vee\mathbb{B})$ with $x^2 \in Z(\mathbb{G}), y^2 \in Z({}^\vee\mathbb{G})$. This data is symmetric: $(y, {}^\vee\mathbb{T}, {}^\vee\mathbb{B}, x, \mathbb{T}, \mathbb{B})$ is L–data with the roles of $\mathbb{G}$, ${}^\vee\mathbb{G}$ reversed, and this defines a representation of a strong real form of $\mathbb{G}$ with integral infinitesimal character. This realizes Vogan duality [8], analogous to duality for Verma modules given by multiplication by the long element of the Weyl group.

Now suppose $\lambda$ is regular but not integral. Then L–data satisfies $x^2$ is central in $\mathbb{G}$, but $y^2$ is not necessarily central in ${}^\vee\mathbb{G}$. To recover Vogan duality we have to allow $x^2$ not central in $\mathbb{G}$. This can be done, but requires some extra work. See [4].

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