Non-Cellularity of Signatures?

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For a reflection s through the hyperplane orthogonal to α, let $A_s(\lambda) = 1 + \langle \lambda, \alpha^\vee \rangle s$. Let $w_0 = s_1 \cdots s_n$ be a reduced expression and $A(\nu) = A_{s_n}(s_n \cdots s_1 \nu) \cdots A_{s_2}(s_1 \nu) A_{s_1}(\nu)$ the usual self-adjoint operator in the group algebra of W.

We would like to claim that the question of whether $A(\nu)$ is positive semidefinite depends only on the cell containing ν, where "cell" is defined by the hyperplanes (and sides thereof) where the various factors are singular; i.e., $\langle \nu, \beta^\vee \rangle = 1$.

Certainly for the big cells (of co-dimension 0), it is easy to show that all points in a given cell yield positive definite operators, or none do.

But in general, this claim looks questionable. For example, consider

$$A(t_1, t_2, t_3) := (1 + t_1 A_1)(1 + t_2 A_2)(1 + t_3 A_3),$$

where the t_i are parameters and the $A_i \in M_n(\mathbb{R})$ are fixed. The "interesting" values for the t_i's are the hyperplanes $t_i = -1/x$, where x is an eigenvalue of A_i.

In particular, consider the case

$$A_1 = A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Note that the A_i's, as well as $A(t_1, t_2, t_3)$, are real symmetric, and the hyperplanes defining the cells where the factors are singular are

$$t_1 = \pm 1, \ t_3 = \pm 1, \ t_2 = 1, \ t_2 = -1/2.$$

If we now specialize $t_1 = t_3 \to 1$, we obtain

$$A(t_1, t_2, t_3) \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & t_2 & t_2 \\ t_2 & 1 & t_2 \\ t_2 & t_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 4 \begin{bmatrix} 1 & t_2 & 0 \\ t_2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, the interesting values that control the rank of A are $t_2 = 1$ and $t_2 = -1$; the latter is not one of the original hyperplanes.