Notes on the step algebra

Siddhartha Sahi*

July 13, 2004

0 Introduction

These are informal notes of my talk during the Atlas of Lie groups workshop at AIM in Palo Alto, July 2003. All the ideas below are due to others, the primary references are Mickelsson [M] and Zhelebenko [Z1], [Z2]. Zhelebenko's papers contain statements without proofs, therefore one should probably verify the results independently. Assuming the results, one gets a fairly explicit algorithm for computing signatures on the K-types of a Harish-Chandra module. (See notes on David Vogan's Montreal talk.)

Let $\mathfrak{g}$ be a reductive Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and positive root system $\Delta_+(\mathfrak{t}, \mathfrak{k})$, and let $\mathfrak{k}_+ = \oplus_{\alpha \in \Delta_+} \mathfrak{k}_\alpha$.

Let $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let $\mathcal{I} = \mathcal{U}\mathfrak{k}_+$ be the left ideal generated by $\mathfrak{k}_+$. We define

$$\tilde{\mathcal{S}} = \{u \in \mathcal{U} \mid u\mathfrak{k}_+ \subset \mathcal{I}\}$$

Thus $\tilde{\mathcal{S}}$ is the *idealizer* of $\mathcal{I}$, *i.e.* the largest subalgebra of $\mathcal{U}$ which contains $\mathcal{I}$ as a two-sided ideal.

Following Mickelsson, we define the "step-algebra"

$$\mathcal{S} = \tilde{\mathcal{S}}/\mathcal{I}.$$

Let $\mathcal{M} = \mathcal{U}/\mathcal{I}$, then by "abstract nonsense" , we have

$$\mathcal{S} \approx \mathcal{M}^{\mathfrak{k}_+} \approx \mathrm{End}_{\mathcal{U}}(\mathcal{M})^{Op}.$$

If X is a $\mathfrak{g}$-module, then the subspace $X^{\mathfrak{k}_+}$ is an $\mathcal{S}$-module. The principal result regarding $\mathcal{S}$ (conjectured by Mickelsson, proved by van den Hombergh) is the following

Theorem 0.1 *If X is an irreducible admissible Harish-Chandra module then $X^{\mathfrak{k}_+}$ is an irreducible $\mathcal{S}$-module.*

*sahi@math.rutgers.edu, Math Department, Rutgers University, New Brunswick, NJ08903

1 Localization

Let $S(\mathfrak{t})$ be the symmetric algebra on $\mathfrak{t}$ and consider the imbedding

$$S(\mathfrak{t}) \approx \mathcal{U}(\mathfrak{t}) \subset \mathcal{U}(\mathfrak{g}).$$

This makes $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ a (free) left and right module over $S(\mathfrak{t})$. However $\mathcal{U}$ is *not* an $S(\mathfrak{t})$-algebra since

$$t \cdot u - u \cdot t = ad(t) \cdot u \text{ for } u \in \mathcal{U},\ t \in \mathfrak{t}.$$

In fact, we have a direct sum decomposition $\mathcal{U} = \oplus_{\lambda \in \mathfrak{t}^*} \mathcal{U}_\lambda$ into weight spaces under the adjoint action of $\mathfrak{t}$.

Let $\mathcal{R}$ be the field of fractions of $S(\mathfrak{t})$, and consider the localization $\mathcal{U}' = \mathcal{U} \otimes_{S(\mathfrak{t})} \mathcal{R}$. Then $\mathcal{U}'$ is an algebra (over $\mathbb{C}$) and a right and left $\mathcal{R}$-vector space, although it is not an $\mathcal{R}$-algebra. In fact, identifying $\mathcal{R}$ with the field of rational functions on $\mathfrak{t}^*$, we have

$$f \cdot u = u \cdot f_\lambda, \text{ for } u \in \mathcal{U}'_\lambda \text{ and } f \in \mathcal{R}$$

where $f_\lambda \in \mathcal{R}$ denotes the rational function

$$f_\lambda(\mu) = f(\mu + \lambda).$$

It is easy to see that $S(\mathfrak{t}) \subset \tilde{\mathcal{S}}$, and that the corresponding map from $S(\mathfrak{t}) \to \mathcal{S}$ is injective. Thus we can localize $\mathcal{S}$ to obtain

$$\mathcal{S}' = \mathcal{S} \otimes_{S(\mathfrak{t})} \mathcal{R}.$$

Alternatively one can consider the idealizer $\tilde{\mathcal{S}}'$ of $\mathcal{I}' = \mathcal{U}'\mathfrak{k}_+$ in $\mathcal{U}'$. Then for $\mathcal{M}' = \mathcal{U}'/\mathcal{I}'$one also has

$$\mathcal{S}' \approx \tilde{\mathcal{S}}'/\mathcal{I}' \approx (\mathcal{M}')^{\mathfrak{k}_+} \approx \mathrm{End}_{\mathcal{U}'}(\mathcal{M}')^{Op}.$$

2 Completion

We choose basis vectors $e_{\pm\alpha}$ in $\mathfrak{k}_{\pm\alpha}$ such that $e_\alpha, e_{-\alpha}$ and $h_\alpha = [e_\alpha, e_{-\alpha}]$ form an S-triple. We also fix an ordering $\alpha_1, \ldots, \alpha_d$ of the positive roots (satisfying some condition). Then by the PBW theorem, elements of the form

$$(e_{-\alpha_1})^{k_1} \ldots (e_{-\alpha_d})^{k_d} (e_{\alpha_d})^{l_d} \ldots (e_{\alpha_1})^{l_1}$$

give a basis for $\mathcal{U}'(\mathfrak{k}) = \mathcal{U}(\mathfrak{k}) \otimes_{S(\mathfrak{t})} \mathcal{R}$ as a left (and right) $\mathcal{R}$-module. The given element belongs to the weight space $\mathcal{U}'_\mu$ where

$$\mu = (l_1 - k_1)\alpha_1 + \ldots + (l_d - k_d)\alpha_d.$$

We define the formal completion F_μ of $\mathcal{U}'_\mu$ to consist of *infinite* linear combinations of PBW basis elements of weight μ; and we consider their direct sum

$$F = \oplus_\mu F_\mu.$$

Clearly $\mathcal{U}'(\mathfrak{k})$ imbeds in F, and it is easy to see that formal multiplication endows F with a natural algebra structure extending that of $\mathcal{U}'(\mathfrak{k})$.

Moreover if $\mathcal{M}'$ is the $\mathcal{U}'(\mathfrak{g})$-module defined in the previous section, then the (left) action of $\mathfrak{k}_+$ coincides with the adjoint action and thus is locally finite. This implies that the action of $\mathcal{U}'(\mathfrak{k})$ on $\mathcal{M}'$ extends to an action of F.

3 Projection

Let $\rho = 1/2\left(\sum_{\alpha\in\Delta_+}\alpha\right)$, and for each α in Δ_+ put

$$f_{\alpha,k} = h_\alpha + \rho(h_\alpha) + 1,\ f_{\alpha,k} = f_\alpha(f_\alpha+1)\cdots(f_\alpha+k-1).$$

We define the following elements of weight 0 in F

$$P_\alpha = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!f_{\alpha,k}}(e_{-\alpha})^k(e_\alpha)^k \text{ and } P = P_{\alpha_1}\cdots P_{\alpha_d}.$$

The main properties of these elements are the following

Theorem 3.1 *P is the unique element of F which satisfies*

1. $e_\alpha P = Pe_{-\alpha} = 0$

Thus P is independent of the choice of the (normal) ordering of Δ_+. Moreover we have

2. $P^2 = P$ *and* $P^* = P$

   (where $*$ is the hermitian involution on F, defined on "generators" by $e_{\pm\alpha} = e_{\mp\alpha}$)

(This stuff needs to be checked, especially to clarify the role of the ordering of Δ_+)

4 Presentation

Recall the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}.$$

We now fix a basis $e_1,\ldots,e_p$ of $\mathfrak{p}$ with weights λ_i, so that $\lambda_i\le\lambda_j$ implies $i\le j$. We regard the e_i as elements of $\mathcal{M}' = \mathcal{U}'/\mathcal{I}'$. Combining the previous two sections, we see that $P\cdot e_1,\ldots,P\cdot e_p$ are well-defined elements of $\mathcal{M}'$. Moreover by the previous theorem, $e_\alpha(P\cdot e_i) = 0$, and thus $P\cdot e_i\in(\mathcal{M}')^{\mathfrak{k}_+}\approx\mathcal{S}'$.

We write z_i for $P\cdot e_i$ regarded as element of $\mathcal{S}'$. The main structural result proved by Zhelebenko is the following:

Theorem 4.1 *The monomials* $(z_1)^{k_1} \dots (z_p)^{k_p}$ *form a basis for* $\mathcal{S}'$ *as a left (and right) vector space over* $\mathcal{R}$*. Moreover for* $i < j$*, the relations are of the form*

$$z_j z_i = \sum_{a<b} f_{ij}^{ab} z_a z_b + g_{ij}$$

where the various coefficients f_{ij}*,* g_{ij} *belong to* $\mathcal{R}$*.*

References

[M] Mickelsson, J., *Step algebras of semisimple subalgebras of Lie algebras,* Rep. Math. Phys. **4** (1973), 307–318

[Z1] Zhelobenko, D., *S-algebras and Verma modules over reductive Lie algebras,* Sov. Math. Dokl. **28** (1983), 696–700

[Z2] Zhelobenko, D., *Z-algebras over reductive Lie algebras,* Sov. Math. Dokl. **28** (1983), 777–781