PARAMETRIZING REAL EVEN NILPOTENT COADJOINT ORBITS USING `atlas`

PETER E. TRAPA

Suppose $G_{\mathbb{R}}$ is the real points of a complex connected reductive algebraic group G defined over $\mathbb{R}$. (The class of such groups is exactly the class treated by the software package `atlas`.) Write $\mathfrak{g}_{\mathbb{R}}^*$ for the dual of the Lie algebra of $G_{\mathbb{R}}$ and $\mathcal{N}^{\mathbb{R}}$ for the nilpotent elements in $\mathfrak{g}_{\mathbb{R}}$. Then $G_{\mathbb{R}}$ acts with finitely many orbits on $\mathcal{N}^{\mathbb{R}}$ via the coadjoint action, and it is clearly very desirable to parametrize these orbits in a way that `atlas` can manipulate. For instance, it is definitely not sufficient to produce a list of tables of such orbits for, say, G simple.

The purpose of these notes is to describe such a parametrization of *even* nilpotent orbits. This can be extracted from [ABV, Chapter 20], but we offer a more easily accessible treatment here and make the `atlas`-based algorithms explicit and effective. The parametrization is in terms of certain closed orbits of a symmetric group K on partial flag varieties $\mathcal{P}$ for G. Since the geometry of K orbits on partial flag varieties is "dual" to the study of translation families of Harish-Chandra modules with singular infinitesimal character, the description of $K\backslash\mathcal{P}$ is of independent interest. We give complete details in Section 2 before turning to the parametrization of nilpotent orbits in Section 3.

1. The Sekiguchi correspondence

Instead of dealing with real nilpotent coadjoint orbits, we will instead work on the other side of the Sekiguchi correspondence (e.g. [CM, Chapter 9]). We begin with some notation.

Fix $G_{\mathbb{R}}$ as above. Let $K_{\mathbb{R}}$ denote a maximal compact subgroup of $G_{\mathbb{R}}$, let θ denote the corresponding Cartan involution, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ for the complexified Cartan decomposition. Let $\mathcal{P}$ denote the variety of parabolic subalgebras in $\mathfrak{g}$ and of a fixed type. Let $\mathcal{N}^\theta$ denote the cone of nilpotent elements in $(\mathfrak{g}/\mathfrak{k})^*$. (Using an invariant form on $\mathfrak{g}$, we may identify $\mathcal{N}^\theta$ with the nilpotent elements of $\mathfrak{s}$, but we will generally avoid doing so.) Finally let $\mathcal{N}^\theta_{\mathcal{P}}$ denote the cone of nilpotent elements in

$$(G \cdot (\mathfrak{g}/\mathfrak{p}_\circ)^*) \cap (\mathfrak{g}/\mathfrak{k})^*$$

where $\mathfrak{p}_\circ$ denotes a fixed base-point in $\mathcal{P}$; and, similarly, let $\mathcal{N}^{\mathbb{R}}_{\mathcal{P}}$ denote the cone of nilpotent elements in

$$(G \cdot (\mathfrak{g}/\mathfrak{p}_\circ)^*) \cap (\mathfrak{g}_{\mathbb{R}})^*$$

(Here and elsewhere we implicitly invoke the inclusion of $(\mathfrak{g}/\mathfrak{p}_\circ)^*$, $\mathfrak{g}_{\mathbb{R}}^*$, and $(\mathfrak{g}/\mathfrak{k})^*$ into $\mathfrak{g}^*$ and take the intersection there.) The complexification K of $K_{\mathbb{R}}$ of course acts with finitely many orbits on $\mathcal{P}$ and $\mathcal{N}^\theta_{\mathcal{P}}$. Note that $\mathcal{N}^\theta = \mathcal{N}^\theta_{\mathcal{P}}$ and $\mathcal{N}^{\mathbb{R}} = \mathcal{N}^{\mathbb{R}}_{\mathcal{P}}$ if $\mathcal{P} = \mathcal{B}$ is the full flag variety.

The following well-known result shows that we may consider nilpotent K orbits on $(\mathfrak{g}/\mathfrak{k})^*$ instead of real nilpotent orbits.

Proposition 1.1 (Sekiguchi)**.** *There is a bijection from the set of K orbits on $\mathcal{N}^\theta_{\mathcal{P}}$ and the set of $G_{\mathbb{R}}$ orbits on $\mathcal{N}^{\mathbb{R}}_{\mathcal{P}}$.*

Remark 1.2. The bijection preserves the closures on each set by a result of Barbasch and Sepanski.

These notes are based on a a lecture I gave at an Atlas workshop at the University of Maryland in March, 2008. I was supported by NSF grant DMS-0554118.

We conclude by introducing the relationship between $K\backslash\mathcal{P}$ and nilpotent K orbits on $(\mathfrak{g}/\mathfrak{k})^*$. Let $\mu_{\mathcal{P}}$ denote the moment map from $T^*\mathcal{P}$ to $\mathfrak{g}^*$. Concretely $\mu_{\mathcal{P}}$ maps a point $(\mathfrak{p}, \xi)$ in $T^*\mathcal{P}$, with

$$\xi \in T^*_{\mathfrak{p}}\mathcal{P} \simeq (\mathfrak{g}/\mathfrak{p})^* \tag{1.3}$$

simply to ξ. Consider now the conormal variety for K orbits on $\mathcal{P}$,

$$T^*_K\mathcal{P} = \bigcup_{Q \in K\backslash\mathcal{P}} T^*_Q\mathcal{P},$$

where $T^*_Q\mathcal{P}$ denotes the conormal bundle to the K orbit Q. In general we may identify

$$T^*_Q\mathcal{P} = \{(\mathfrak{p}, \xi) \mid \mathfrak{p} \in Q, \xi \in (\mathfrak{g}/\mathfrak{k} + \mathfrak{p})^*\}, \tag{1.4}$$

and hence we see that the image of $T^*_K\mathcal{P}$ under $\mu_{\mathcal{P}}$ is simply $\mathcal{N}^\theta_{\mathcal{P}}$.

Since $\mu_{\mathcal{P}}$ is G-equivariant, a short argument shows that the image of a particular conormal bundle $T^*_Q\mathcal{P}$ contains a unique dense orbit of K on $\mathcal{N}^\theta_{\mathcal{P}}$. Hence we obtain a map

$$\Phi = \Phi_{\mathcal{P}} \ : \ K\backslash\mathcal{P} \longrightarrow K\backslash\mathcal{N}^\theta_{\mathcal{P}}. \tag{1.5}$$

Concretely, Φ maps $Q \in K\backslash\mathcal{P}$ to the dense K orbit in $K \cdot (\mathfrak{g}/\mathfrak{p})^*$ where $\mathfrak{p}$ is an element of Q.

2. PARAMETRIZING $K\backslash\mathcal{P}$

In this section, we give a parametrization of $K\backslash\mathcal{P}$. We begin with a discussion of the set $K\backslash\mathcal{B}$ of K orbits on the full flag variety $\mathcal{B}$. Basic references for this material are [M] or [RS]. The point of this whole discussion is that everything we say below has already been (or can easily be) implemented in `atlas`. We make this explicit below.

The set $K\backslash\mathcal{B}$ is partially ordered by the inclusion of orbit closures. It is generated by closure relations in codimension one. We will need to distinguish two kinds of such relations. To do so, we fix a base-point $\mathfrak{b}_\circ \in \mathcal{B}$, a decomposition $\mathfrak{b}_\circ = \mathfrak{h}_\circ \oplus \mathfrak{n}_\circ$, and let Δ^+ denote the corresponding set of positive roots. For a simple root α, let $\mathcal{P}_\alpha$ denote the set of parabolic subalgebras of type α, and write π_α for the projection $\mathcal{B} \to \mathcal{P}_\alpha$.

Fix K orbits Q and Q' on $\mathcal{B}$. If K is connected, then Q is irreducible, and hence so is $\pi_\alpha^{-1}(\pi_\alpha(Q))$. Thus $\pi_\alpha^{-1}(\pi_\alpha(Q))$ contains a unique dense K orbit. In general, K need not be connected and Q need not be irreducible. But it is easy to see that the similar reasoning applies to conclude $\pi_\alpha^{-1}(\pi_\alpha(Q))$ always contains a dense K orbit. We write $Q \xrightarrow{\alpha} Q'$ if

$$\dim(Q') = \dim(Q) + 1$$

and

$$Q' \text{ is dense in } \pi_\alpha^{-1}(\pi_\alpha(Q)).$$

This implies that $\pi_\alpha^{-1}(\pi_\alpha(Q'))$ fibers as a $\mathbb{P}^1$ bundle over $\pi_\alpha(Q)$, and hence that Q is codimension one in the closure of Q'. The relations $Q \xrightarrow{\alpha} Q'$ do not generate the closure order however. Instead we must also consider a kind of saturation condition. More precisely, whenever a codimension one subdiagram of the form

$$\begin{tikzcd} & Q_1 & \\ Q_2 \arrow[ur, "\alpha"] & & Q_3 \\ & Q_4 \arrow[ul] \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{2.1}$$

is encountered, we complete it to

(2.2)
$$\begin{tikzcd}
 & Q_1 & \\
Q_2 \arrow[ur, "\alpha"] & & Q_3 \arrow[ul, dotted] \\
 & Q_4 \arrow[ul] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

New edges added in this way are dashed in the diagrams below. Note that this operation must be applied recursively, and thus some of the edges in the original diagram (2.1) may be dashed as the recursion unfolds. Following the terminology of [RS, 5.1], we call the partially ordered set determined by the solid edges the weak closure order.

Now fix a variety of parabolic subalgebras $\mathcal{P}$ of an arbitrary fixed type and write $\pi_{\mathcal{P}}$ for the projection from $\mathcal{B}$ to $\mathcal{P}$. For definiteness fix $\mathfrak{p}_\circ = \mathfrak{l}_\circ \oplus \mathfrak{u}_\circ \in \mathcal{P}$ containing $\mathfrak{b}_\circ$. Then $K\backslash\mathcal{P}$ may be parametrized from a knowledge of the weak closure on $K\backslash\mathcal{B}$ as follows. Consider the relation $Q \sim_{\mathcal{P}} Q'$ if $\pi_{\mathcal{P}}(Q) = \pi_{\mathcal{P}}(Q')$; this is generated by the relations $Q \sim Q'$ if $Q \xrightarrow{\alpha} Q'$ for α simple in $\Delta(\mathfrak{h}_\circ, \mathfrak{l}_\circ)$. Equivalence classes in $K\backslash\mathcal{B}$ clearly are in bijection with $K\backslash\mathcal{P}$. Fix an equivalence class C and fix a representative $Q \in C$. The same reasoning that shows that $\pi_\alpha^{-1}(\pi_\alpha(Q))$ contains a unique dense K orbit also shows that

$$\pi_{\mathcal{P}}^{-1}(\pi_{\mathcal{P}}(Q))$$

contains a unique dense K orbit $Q_C \in K\backslash\mathcal{B}$. In other words, Q_C is the unique largest dimensional orbit among the elements in C. In fact Q_C is characterized among the elements of C by the condition

$$\dim \pi_\alpha^{-1}(\pi_\alpha(Q_C)) = \dim(Q_C) \tag{2.3}$$

for all α simple in $\Delta(\mathfrak{h}_\circ, \mathfrak{l}_\circ)$. Moreover, the full closure order on $K\backslash\mathcal{P}$ is simply the restriction of the full closure on $K\backslash\mathcal{B}$ to the subset of all maximal-dimensional representatives of the form Q_C. By restricting only the weak closure order, we may speak of the weak closure order on $K\backslash\mathcal{P}$.

We now give and *explicit* description of how to read off the partially ordered set of K orbits on $\mathcal{P}$ from the output of `atlas`. (This algorithm could easily be implemented in future versions.) First recall that the `atlas` command `kgb`. It outputs a table, and each row corresponds to a K orbit on $\mathcal{B}$. The command `kgborder` then provides a list of the covering relations in the closure partial order. An example of version 0.3 output for $\mathrm{Sp}(4,\mathbb{R})$ is given in Figure 2.1.

In general, the first column of the `kgb` output is a label representing an orbit of K on $\mathcal{B}$. For the row labeled `i`, write Q_i for the corresponding orbit. (The labeling is more or less random but is consonant with the entry in the second column which denotes the dimension of the orbit minus the dimension of any closed orbit.) The remaining columns are broken into three chunks, the first two of which consist of $n = \mathrm{rank}(G)$ columns, and a third (which shall not concern us) consisting of a possibly empty list of numbers separated by commas. Fix an index i corresponding to a simple root α_i with Bourbaki label i. The weak order is generated by the following rules:

(1) In row `k`, if the character `C` appears in the ith entry between [ ], let `l` denote the entry in the ith position in the *first* chunk of columns in row `k`. Then
$$Q_k \xrightarrow{\alpha_i} Q_l$$
if the relative dimension of Q_l is greater than that of Q_k (which happens if and only if $k < l$) and $Q_l \xrightarrow{\alpha_i} Q_k$ otherwise.

(2) In row `k`, if the character `n` appears in the ith entry between [ ], let `l` denote the entry in the ith position in the *second* chunk of columns in row `k`. In the notation introduced above $Q_k \xrightarrow{\alpha_i} Q_l$, and it turns out that the relative dimension of Q_l is always greater than that of Q_k.

```
empty: type
Lie type: C2
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
sc
enter inner class(es): s
main: realform
(weak) real forms are:
0: sp(2)
1: sp(1,1)
2: sp(4,R)
enter your choice: 2
real: kgb
kgbsize: 11
Name an output file (return for stdout, ? to abandon):
 0:  0  0  [n,n]    1   2     6   4
 1:  0  0  [n,n]    0   3     6   5
 2:  0  0  [c,n]    2   0     *   4
 3:  0  0  [c,n]    3   1     *   5
 4:  1  2  [C,r]    8   4     *   *  2
 5:  1  2  [C,r]    9   5     *   *  2
 6:  1  1  [r,C]    6   7     *   *  1
 7:  2  1  [n,C]    7   6    10   *  2,1,2
 8:  2  2  [C,n]    4   9     *  10  1,2,1
 9:  2  2  [C,n]    5   8     *  10  1,2,1
10:  3  3  [r,r]   10  10     *   *  1,2,1,2
real: kgborder
kgbsize: 11
Name an output file (return for stdout, ? to abandon):
0:
1:
2:
3:
4: 0,2
5: 1,3
6: 0,1
7: 4,5,6
8: 4,6
9: 5,6
10: 7,8,9
Number of comparable pairs = 44
```

FIGURE 2.1. `atlas` output for $\mathrm{Sp}(4,\mathbb{R})$.

This explains how to read off the weak order on $K\backslash\mathcal{B}$ from the output of `kgb`.

Now fix a subset of simple roots defining the type of $\mathcal{P}$. This corresponds to a list of indices $i_1, i_2, \ldots$ in the Bourbaki labeling. (Instead of writing something like α_{i_j}, we will instead identify the indices with the corresponding simple roots.) The equivalence relation $\sim_{\mathcal{P}}$ is generated by all arrows $\xrightarrow{i_j}$ in the weak order on $K\backslash\mathcal{B}$ which we just specified. As explained above, each equivalence class contains an element of maximal dimension, which (once the equivalence classes are tabulated) is easy to read off: it is the element of the class corresponding to the largest label.

We have thus explained how to read off a list of rows of the `kgb` command which parametrize the orbits of K on $\mathcal{P}$. The closure order on $K\backslash\mathcal{P}$ is then simply the restriction of closure order on $K\backslash\mathcal{B}$ which is outputted, as mentioned above, using the command `kgborder`.

Example 2.4. Consider the case of $G_{\mathbb{R}} = \mathrm{Sp}(4,\mathbb{R})$ and $\mathcal{P} = \mathcal{B}$. Then 1 is the Bourbaki label for the short simple root and 2 is the label for the long root. We may read off the closure order from Figure 2.1 as follows. Orbits on the same row of the diagram below all have the same dimension. Dashed lines represent relations in the full closure order which are not in the weak order.

(2.5)

$$\begin{tikzcd}
 & & & Q_{10} & & & \\
 & Q_8 \arrow[urr, "2"] & & Q_7 \arrow[u, "1"] & & Q_9 \arrow[ull, "2"'] & \\
 & Q_4 \arrow[u, "1"] \arrow[urr, dotted] & & Q_6 \arrow[u, "2"'] \arrow[ull, dotted] \arrow[urr, dotted] & & Q_5 \arrow[u, "1"'] \arrow[ull, dotted] & \\
Q_2 \arrow[ur, "2"] & & Q_0 \arrow[ul, "2"'] \arrow[ur, "1"'] & & Q_1 \arrow[ul, "1"] \arrow[ur, "2"] & & Q_3 \arrow[ul, "2"']
\end{tikzcd}$$

Adopt the parametrization of $K\backslash\mathcal{N}^\theta$ given in [CM, Theorem 9.3.5] in terms of signed tableau. Let $(i_1)^{j_1}_{\epsilon_1}(i_2)^{j_2}_{\epsilon_2}\cdots$ denote the tableau with j_k rows of length i_k beginning with sign ϵ_k for each k. Then the closure order on $K\backslash\mathcal{N}^\theta$ is given by

(2.6)

$$\begin{tikzcd}
 & 4^1_+ & & 4^1_- & \\
2^2_+ \arrow[ur] & & 2^1_+2^1_- \arrow[ul] \arrow[ur] & & 2^2_- \arrow[ul] \\
 & 2^1_+1^1_+1^1_- \arrow[ul] \arrow[ur] & & 2^1_-1^1_+1^1_- \arrow[ul] \arrow[ur] & \\
 & & 1^2_+1^2_- \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Then $\Phi_{\mathcal{B}}$ maps Q_{10} to $1^2_+1^2_-$; Q_8 to $2^1_+1^1_+1^1_-$; Q_9 to $2^1_-1^1_+1^1_-$; Q_6 and Q_7 to $2^1_+2^1_-$; Q_2 and Q_4 to 2^2_+; Q_3 and Q_5 to 2^2_-; Q_0 to 4^1_+. and Q_1 to 4^1_-. Note that $\Phi_{\mathcal{B}}$ reverses all closure relations *except* the two dashed edges indicating Q_4 and Q_5 are contained in $\overline{Q_7}$.

Now let $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) consist of parabolic subalgebras of type 1 (resp. 2) and write π_1 and π_2 in place of $\pi_{\mathcal{P}_1}$ and $\pi_{\mathcal{P}_2}$. Then the closure order on $K\backslash\mathcal{P}_1$ is obtained by the procedure described above. (Dashed edges are those covering relations present in the full order but not the weak one, and the terminal subscripts indicate actual dimensions.)

(2.7)
$$\begin{tikzcd}
 & & \pi_1(Q_{10})_3 & & \\
 & \pi_1(Q_8)_2 \arrow[ur] & & \pi_1(Q_9)_2 \arrow[ul] & \\
\pi_1(Q_2)_0 \arrow[ur] & & \pi_1(Q_6)_1 \arrow[ul, dotted] \arrow[ur, dotted] & & \pi_1(Q_3)_0 \arrow[ul]
\end{tikzcd}$$

The closure order on $K\backslash\mathcal{P}_2$ is again obtained by the algorithm described above. Terminal subscripts indicate actual dimensions.

(2.8)
$$\begin{tikzcd}
 & \pi_2(Q_{10})_3 & \\
 & \pi_2(Q_7)_2 \arrow[u] & \\
\pi_2(Q_4)_1 \arrow[ur, dotted] & & \pi_2(Q_5)_1 \arrow[ul, dotted]
\end{tikzcd}$$

In this case $\mathcal{N}_1^\theta = \mathcal{N}_2^\theta$, and the closure order on $K\backslash\mathcal{N}_{\mathcal{P}}^\theta$ is just the bottom three rows of (2.6),

(2.9)
$$\begin{tikzcd}
2_+^2 & & 2_+^1 2_-^1 & & 2_-^2 \\
 & 2_+^1 1_+^1 1_-^1 \arrow[ul] \arrow[ur] & & 2_-^1 1_+^1 1_-^1 \arrow[ul] \arrow[ur] & \\
 & & 1_+^2 1_-^2 \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

In fact, Φ_1 is the obvious order reversing bijection of (2.7) onto (2.9).

By contrast, Φ_2 does not invert the dashed edges in (2.8): Φ_2 maps $\pi_2(Q_{10})$ to the zero orbit, and the three remaining orbits to the three orbits of maximal dimension in $\mathcal{N}_{\mathcal{P}}^\theta$.

Example 2.10. Suppose now $G_{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$ and $\mathcal{P}$ consists of maximal parabolic of type corresponding to the subset of simple roots obtained by deleting the long one. (So if $n = 1$, $\mathcal{P} = \mathcal{P}_1$ in Example 2.4 above.) Then the analysis of the preceding example extends to show that $\Phi_{\mathcal{P}}$ is an order-reversing bijection. The closure order on $K\backslash\mathcal{N}_{\mathcal{P}}^\theta$ (and hence $K\backslash\mathcal{P}$) is as follows.

(2.11)
$$\begin{tikzcd}
2_+^n & & 2_+^{n-1} 2_-^1 & \cdots & 2_-^{n-1} 2_+^1 & & 2_-^n \\
 & \cdots \arrow[ul] \arrow[ur] & & \cdots \arrow[ul] \arrow[ur] & & \cdots \arrow[ul] \arrow[ur] & \\
 & & 2_+^1 1_+^{n-1} 1_-^{n-1} \arrow[ul, dash] \arrow[ur, dash] & & 2_-^1 1_+^{n-1} 1_-^{n-1} \arrow[ul, dash] \arrow[ur, dash] & & \\
 & & & 1_+^n 1_-^n \arrow[ul] \arrow[ur] & & &
\end{tikzcd}$$

Here, as before, we are using the parametrization of $K\backslash\mathcal{N}_{\mathcal{P}}^\theta$ given in [CM, Theorem 9.3.5].

Let $\mathcal{O}$ be a nilpotent coadjoint orbit of G on $\mathfrak{g}^*$. Suppose that $\mathcal{O}$ is *even*. This means that all of the nodes in the weighted Dynkin diagram are either labeled by 0 or 2. Consider the variety of parabolic subalgebras $\mathcal{P} = \mathcal{P}(\mathcal{O})$ corresponding to the nodes labeled 0. Then it is well-known that $\mathcal{O}$ is dense in the image of $T^*\mathcal{P}$ under $\mu_{\mathcal{P}}$; see [CM, Theorem 7.1.6]. Furthermore, according to a result usually attributed to Hesselink, $\mu_{\mathcal{P}}$ is birational onto its image. See [ABV, Lemma 27.8].

Here is the parametrization to which we have been alluding.

Proposition 3.1 (`atlas`**-based parametrization of even real nilpotent orbits).** *Fix an even complex nilpotent coadjoint orbit $\mathcal{O}$ and define $\mathcal{P} = \mathcal{P}(\mathcal{O})$ as above. Then there is a subset $S = S(\mathcal{O})$ of closed K orbits on $\mathcal{P}$ such that the restriction of $\Phi_{\mathcal{P}}$ is a bijection from S to the set of K orbits on $\mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^*$ and hence (according to Proposition 1.1) the $G_{\mathbb{R}}$ orbits on $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}^*$. The set S is explicitly computable from the output of* `atlas`.

In a moment we will explain how to use `atlas` to find S, but first we give a more down-to-earth formulation of the proposition. The equivalence of the two statements of Proposition 3.1 and 3.2 follows from unwinding the definitions and from the easy fact that any K orbit of θ-stable parabolic subalgebra (of the type specified by $\mathcal{P}$) is closed in $\mathcal{P}$.

Proposition 3.2 (alternate formulation of Proposition 3.1 for nilpotent adjoint orbits). *Fix an even complex nilpotent adjoint orbit $\mathcal{O}$ and define $\mathcal{P} = \mathcal{P}(\mathcal{O})$ as above. Recall the Cartan involution θ and the complexified Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Let S denote the set of K-conjugacy classes of θ-stable parabolic subalgebras $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \in \mathcal{P}$ such that*

$$G \cdot (\mathfrak{u} \cap \mathfrak{s}) = G \cdot \mathfrak{u}. \tag{3.3}$$

(This is precisely the set S which appears in Proposition 3.1.) Then the map which assigns to each conjugacy class $K \cdot \mathfrak{p}$ in S the dense K orbit in $K \cdot (\mathfrak{u} \cap \mathfrak{p})$ is a bijection from S to the set of K orbits on $\mathcal{O} \cap \mathfrak{s}$ and hence (according to Proposition 1.1) the $G_{\mathbb{R}}$ orbits on $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$.

This formulation is essentially proved in [ABV, Theorem 27.10]. (The "essentially" means that some nontrivial extraction is required on the part of the reader.) We omit any details here, but focus instead of how to read off the set S of Propositions 3.1 and 3.2. The examples of Section 2 provide excellent orientation.

According to the discussion in Section 2, we can read off the set of closed orbits of K on $\mathcal{P}$ from the output of `atlas`. They are given as list of indices labeling rows in the `kgb` output representing maximal elements in $\sim_{\mathcal{P}}$ equivalence classes corresponding to closed orbits. Write S' for this set of indices. The question is to determine which closed K orbits comprise the set S of Propositions 3.1 and 3.2. For this we must use the full power of the computation of Kazhdan-Lusztig-Vogan polynomials in general.

We describe how to pare down the set of indices S' to the set S we want. Fix an index `i` $\in S'$. The first requirement is that the orbit on $\mathcal{P}$ parametrized by `i` must consist of θ-stable parabolics. For this we use the output of the command `blocku`. (When executing `blocku`, a choice of dual block is required; it suffices to choose the most split real form at the prompt.) Then `i` corresponds to an orbit of θ-stable parabolics only if there is an row in the output of `blocku` which has `i` appearing immediately after the first open parenthesis. If not, we discard `i` from consideration. If there is such a row, it has a potentially different label, say `j`, which now represents an element of the block of the trivial representation of $G_{\mathbb{R}}$. We next turn to the cell C containing the representation π_j parametrized by `j`. The condition in (3.3) is equivalent to requiring that the (special constituent of the) Weyl group representation of W afforded by C contain the sign representation of the parabolic subgroup $W(\mathcal{P})$ of type $\mathcal{P}$. Here is a simple effective procedure to decide that. For each entry `k` in the cell containing `j` (as specified by the output of `wcells`), look at the row labeled `k` in the output of the `block` command. (Again a choice of dual block must be specified when executing the `block` command. The same choice should be made as when executing the `blocku` command above.) In the entries between square braces, change each `C-`, `ic`, `r1`, and `r2` to a `-`, and all other entries to a `+`. The `-`'s represent the simple roots in the τ-invariant of π_k; i.e. those which act by minus one on the class of π_k in the coherent continuation representation. As `k` varies over all indices in the cell C, there is a unique maximal set $\tau(C)$ (with respect to the inclusion partial order) which contains the set of simple roots defining $\mathcal{P}$. If $\tau(C)$ corresponds to the simple roots defining $\mathcal{P}$, then `i` belongs to S (i.e. the sign representation of $W(\mathcal{P})$ appears in the representation afforded by C). Otherwise `i` does not belong to S.

This completes the `atlas` description of the set S appearing in Propositions 3.1 and 3.2, and hence completes an `atlas` based parametrization of real even nilpotent coadjoint orbits.

References

[ABV] Adams, J., D. Barbasch, and D. A. Vogan, Jr., *The Langlands Classification and Irreducible Characters for Real Reductive Groups*, Progress in Math, Birkhäuser (Boston), **104**(1992).

[CM] D. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold (New York), 1993.

[KR] B. Kostant and S. Rallis, Orbits and representations associated with symmetric spaces, *Amer. J. Math.*, **93** (1971), 753–809.

[M] T. Matsuki, Closure relations for orbits on affine symmetric spaces under the action of minimal parabolic subgroups, *Representations of Lie groups, Kyoto, Hiroshima, 1986*, Adv. Stud. Pure Math., **14** (1988), 541–559.

[RS] R. W. Richardson and T. A. Springer, The Bruhat order on symmetric varieties, *Geom. Dedicata*, **35** (1990), no. 1-3, 389–436.

Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA

E-mail address: `ptrapa@math.utah.edu`