ADMISSIBLE W-GRAPHS

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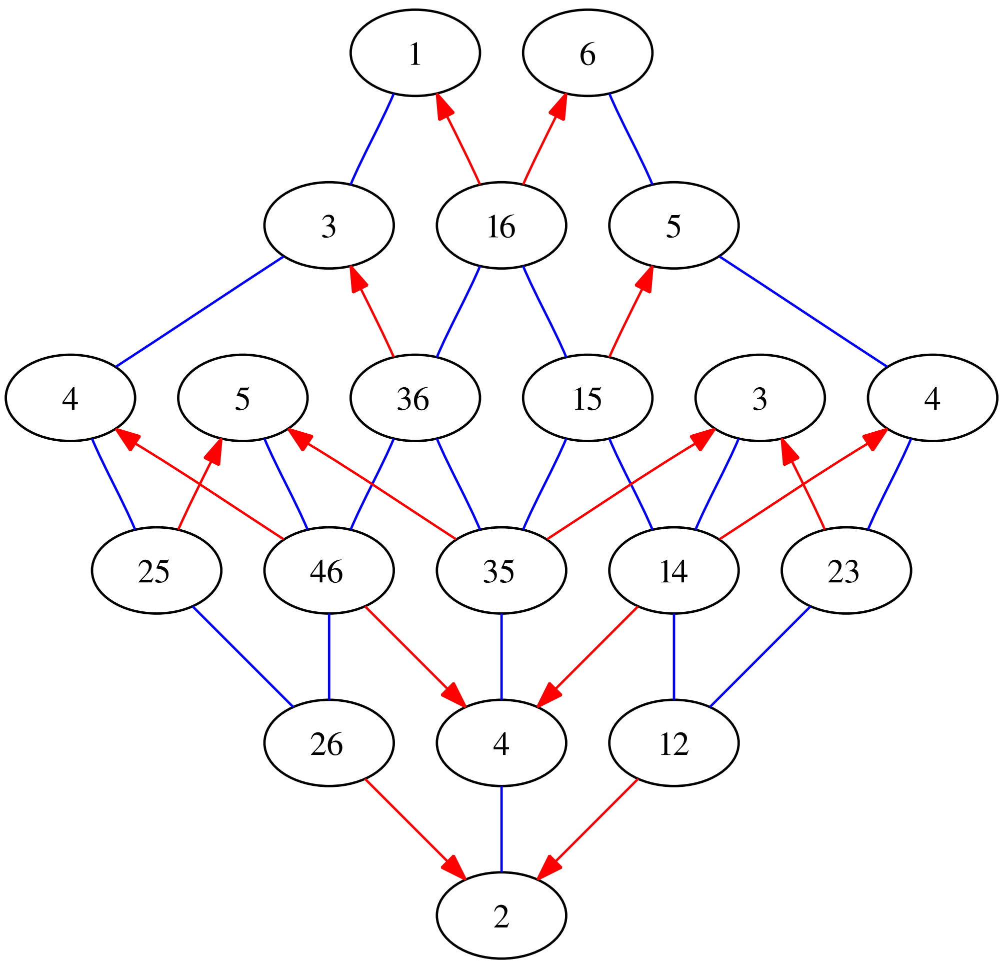


1. General W-Graphs

Let (W, S) be a Coxeter system, $S = \{s_1, \ldots, s_n\}$.

Primarily, $W =$ a finite Weyl group.

Let $\mathcal{H} = \mathcal{H}(W, S) =$ the associated Iwahori-Hecke algebra over $\mathbb{Z}[q^{\pm 1/2}]$.

$$= \langle T_1, \ldots, T_n \mid (T_i - q)(T_i + 1) = 0, \text{ braid relations} \rangle.$$

Definition. An S-**labeled graph** is a triple $\Gamma = (V, m, \tau)$, where

- V is a (finite) vertex set,
- $m : V \times V \to \mathbb{Z}[q^{\pm 1/2}]$ (i.e., a matrix of edge-weights),
- $\tau : V \to 2^S = 2^{[n]}$.

Conventions.

- $u \xrightarrow{-3} v$ means $m(u \to v) = -3$,
- u —— v means $m(u \to v) = m(v \to u) = 1$.

Let $M(\Gamma) =$ free $\mathbb{Z}[q^{\pm 1/2}]$-module with basis V.

Introduce operators T_i on $M(\Gamma)$:

$$T_i(v) = \begin{cases} qv & \text{if } i \notin \tau(v), \\ -v + q^{1/2} \sum_{u : i \notin \tau(u)} m(v \to u) u & \text{if } i \in \tau(v). \end{cases}$$

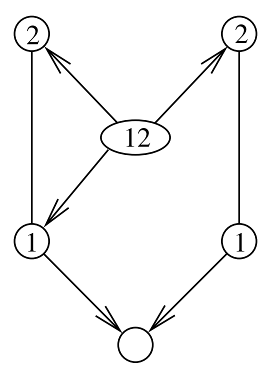
Definition (K-L). Γ is a W-**graph** if this yields an $\mathcal{H}$-module.

Note: $(T_i - q)(T_i + 1) = 0$ (always), so W-graph $\Leftrightarrow$ braid relations.

$$T_i(v) = \begin{cases} qv & \text{if } i \notin \tau(v), \\ -v+q^{1/2}\sum_{u:i\notin\tau(u)} m(v\to u)u & \text{if } i \in \tau(v). \end{cases} \tag{1}$$

Remarks.

- Kazhdan-Lusztig use T_i^t, not T_i.
- Restriction: for $J \subset S$, $\Gamma_J := (V, m, \tau|_J)$ is a W_J-graph.
- At $q = 1$, we get a W-representation.
- However, braid relations at $q = 1 \not\Rightarrow W$-graph:



- If $\tau(v) \subseteq \tau(u)$, then (1) does not depend on $m(v \to u)$.

Convention. $m(v \to u) := 0$ whenever $\tau(v) \subseteq \tau(u)$.

Definition. A W**-cell** is a strongly connected W-graph.

For every W-graph Γ, $M(\Gamma)$ has a filtration whose subquotients are cells.

Typically, cells are not irreducible as $\mathcal{H}$-reps or W-reps.

However (Gyoja, 1984):

if W is finite every irrep may be realized as a W-cell.

2. Admissible W-graphs

$\mathcal{H}$ has a distinguished basis $\{C'_w : w \in W\}$ (the Kazhdan-Lusztig basis).

The action of T_i on C'_w is encoded by a W-graph $\Gamma_W = (W, m, \tau)$, where

- $\tau(v) = \{s \in S : \ell(sv) < \ell(v)\}$ (left descent set),
- m is determined by the Kazhdan-Lusztig polynomials:

$$m(u \to v) = \begin{cases} \mu(u,v) + \mu(v,u) & \text{if } \tau(u) \not\subseteq \tau(v), \\ 0 & \text{if } \tau(u) \subseteq \tau(v), \end{cases}$$

where $\mu(u,v) =$ coeff. of $q^{(\ell(v)-\ell(u)-1)/2}$ in $P_{u,v}(q)$ ($= 0$ unless $u \leqslant v$).

Remarks.

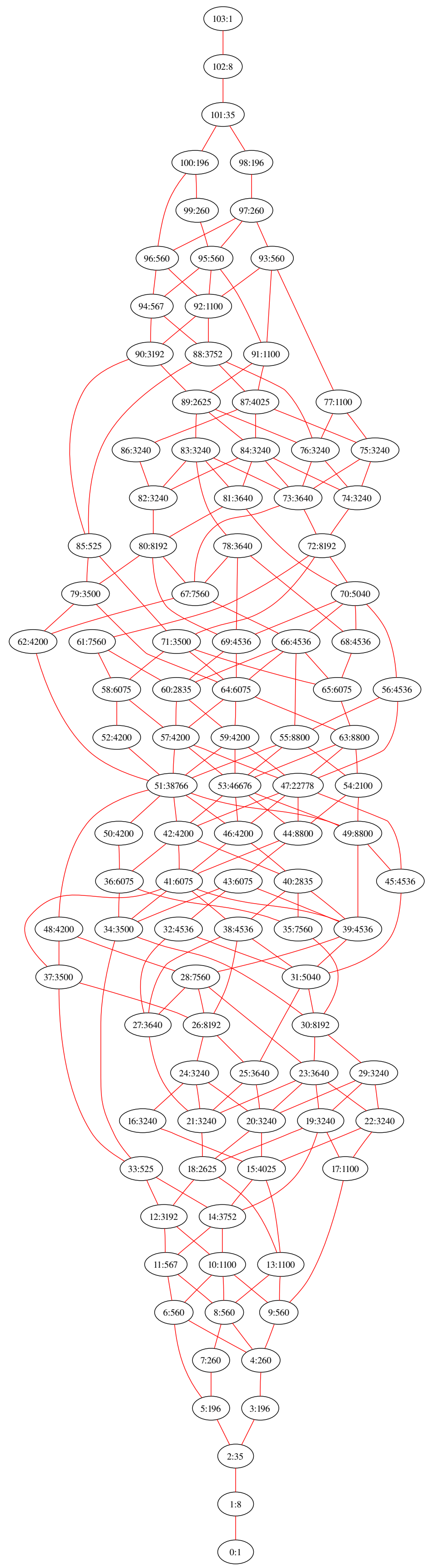
- This graph is generally very sparse, and has edge weights in $\mathbb{Z}$.
- The cells of Γ_W decompose the regular representation of $\mathcal{H}$.
- These cells are often not irreducible as $\mathcal{H}$-reps or W-reps.
- For all W of interest (finite or crystallographic), we know that $P_{u,v}(q)$

has *nonnegative* coefficients.

- These W-graphs are **edge-symmetric**; i.e.,

$$m(u \to v) = m(v \to u) \quad \text{if } \tau(u) \not\subseteq \tau(v) \text{ and } \tau(v) \not\subseteq \tau(u). \tag{2}$$

- If $\mu(u,v) \neq 0$, then $\ell(u) \neq \ell(v)$ mod 2, so these graphs are bipartite.
- (Vogan) Similar W-graphs, cells, and K-L polys exist for real Lie groups.


103:1
102:8
101:35
100:196
98:196
99:260
97:260
96:560
95:560
93:560
94:567
92:1100
90:3192
88:3752
91:1100
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86:3240
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15:4025
17:1100
12:3192
14:3752
11:567
10:1100
13:1100
6:560
8:560
9:560
7:260
4:260
5:196
3:196
2:35
1:8
0:1

DEFINITION. An S-labeled graph $\Gamma = (V, m, \tau)$ is **admissible** if

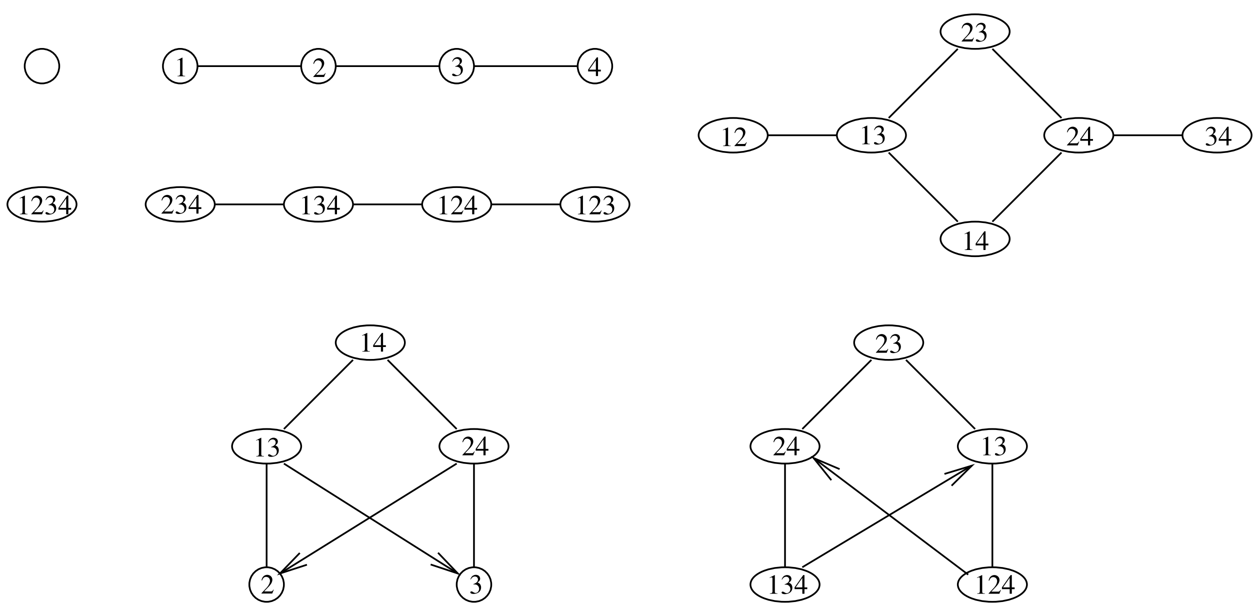
- it is edge-symmetric; i.e.,

$$m(u \to v) = m(v \to u) \quad \text{if } \tau(u) \not\subseteq \tau(v) \text{ and } \tau(v) \not\subseteq \tau(u),$$

- all edge weights $m(u \to v)$ are nonnegative integers, and
- it is bipartite.

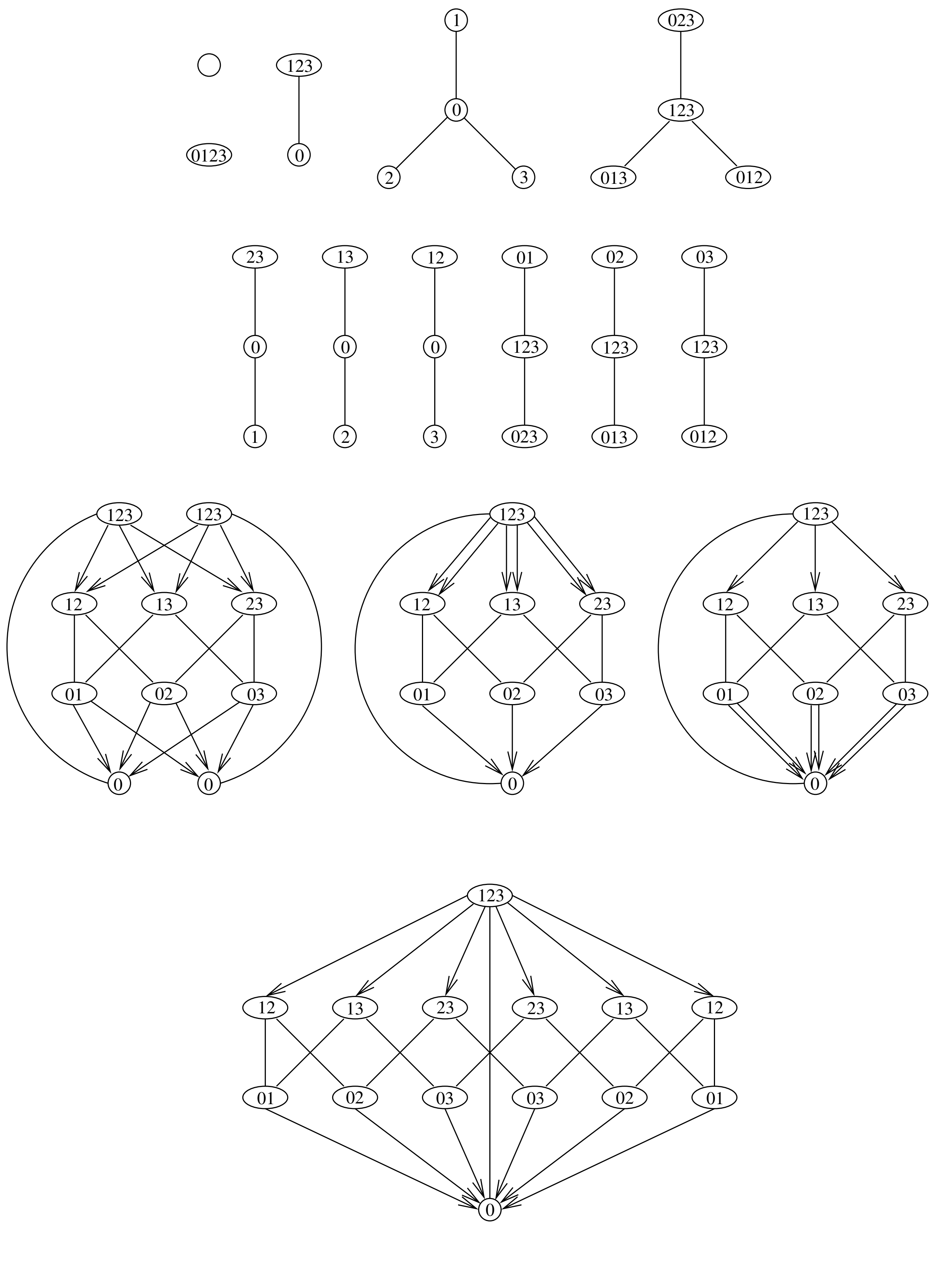
MAIN HYPOTHESIS. These axioms capture the W-graphs that we care about, and are sufficiently rigid that there should be few "synthetic" cells. Sufficient understanding of admissible W-cells could yield constructions of K-L cells without having to compute K-L polynomials.

EXAMPLE. The admissible A_4-cells:



All of these are K-L cells; none are synthetic.

The admissible D_4-cells (three are synthetic):



3. The Agenda

PROBLEM 1 (W finite). *Are there finitely many admissible W-cells?*

- Confirmed for $A_1, \ldots, A_9$, B_2, B_3, D_4, D_5, D_6, E_6, and rank 2.

PROBLEM 2. *Classify/generate all admissible W-cells.*

- Are the only admissible A_n-cells the K-L cells?
- Caution (McLarnan-Warrington): Interesting things happen in A_{15}.

PROBLEM 3. *Understand "combinatorial rigidity" for cells.*

- Rigidity means $M(\Gamma_1) \cong M(\Gamma_2)$ (as W-reps) $\Rightarrow \Gamma_1 \cong \Gamma_2$.
- Example: Are K-L cells rigid? True for A_n.
- Admissible W-cells are not rigid in general.

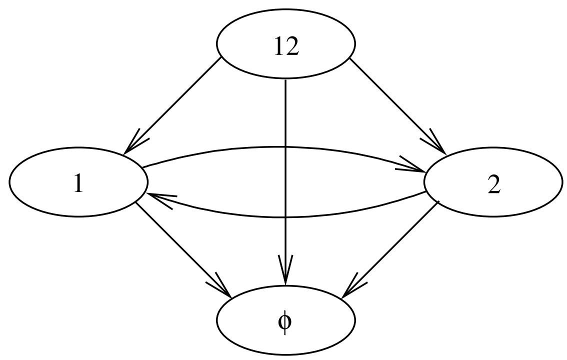
PROBLEM 4. *Understand "compressibility" of cells.*

- A given cell or W-graph should be recursively constructible from a small amount of data.

4. The Admissible Cells in Rank 2

Consider $W = I_2(m)$, $m < \infty$. (When $m = \infty$, anything goes.)

Given an $I_2(m)$-graph, partition the vertices according to τ:



Focus on non-trivial cells: $\tau(v) = \{1\}$ or $\{2\}$ for all $v \in V$.

Encode edge weights $\{1\} \to \{2\}$ (resp., $\{2\} \to \{1\}$) by a matrix A (resp. B).

The conditions on A and B are as follows:

- $m = 2$: $A = 0$, $B = 0$.
- $m = 3$: $AB = 1$, $BA = 1$.
- $m = 4$: $ABA = 2A$, $BAB = 2B$.
- $m = 5$: $ABAB - 3AB + 1 = 0$, $BABA - 3BA + 1 = 0$.

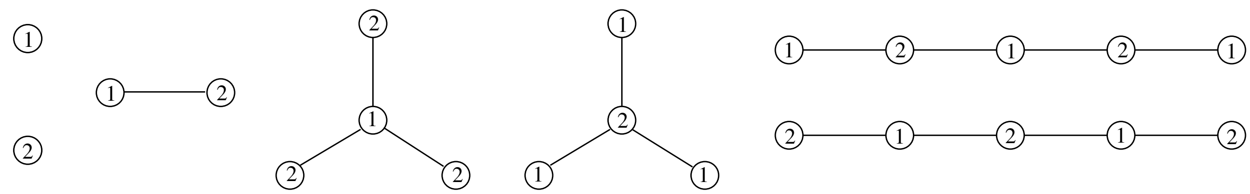
$$\vdots$$

REMARKS.

- If we assume only $\mathbb{Z}$-weights, no classification is possible (cf. $m = 3$).
- Edge symmetry $\Leftrightarrow A = B^t$.
- When $m = 3$, edge weights $\in \mathbb{Z}^{\geqslant 0} \Rightarrow$ edge symmetry, but not in general.

Theorem 1. *A 2-colored graph is an admissible $I_2(m)$-cell iff it is a properly 2-colored A-D-E Dynkin diagram whose Coxeter number divides m.*

Example. The Dynkin diagrams with Coxeter number dividing 6 are A_1, A_2, D_4, and A_5. Therefore, the (nontrivial) admissible G_2-cells are



Note: The nontrival K-L cells for $I_2(m)$ are paths of length $m-2$.

Proof Sketch. Let Γ be any properly 2-colored graph.

Let $M = \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}$ encode the edge weights of Γ.

Let $\phi_m(t)$ be the Chebyshev polynomial such that $\phi_m(2\cos\theta) = \dfrac{\sin m\theta}{\sin\theta}$.

Then Γ is an $I_2(m)$-cell $\Leftrightarrow \phi_m(M) = 0$

$\Leftrightarrow M$ is diagonalizable with eigenvalues $\subset \{2\cos(\pi j/m) : 1 \leqslant j < m\}$.

Now assume Γ is admissible ($M = M^t$, $\mathbb{Z}^{\geqslant 0}$-entries).

If Γ is an $I_2(m)$-cell, then $2 - M$ is positive definite.

Hence, $2 - M$ is a (symmetric) Cartan matrix of finite type.

Conversely, let A be any Cartan matrix of finite type (symmetric or not).

Then the eigenvalues of A are $2 - 2\cos(\pi e_j/h)$, where $e_1, e_2, \ldots$ are the exponents and h is the Coxeter number. $\square$

5. Combinatorial Characterization

For brevity, we restrict to the simply-laced case.

THEOREM 2. *(Assume (W, S) is simply-laced.) An admissible S-labeled graph is a W-graph if and only if it satisfies*

- *the Compatibility Rule,*
- *the Simplicity Rule,*
- *the Frontier Rule,*
- *the Diamond Rule, and*
- *the Hexagon Rule.*

THE COMPATIBILITY RULE (applies to all W-graphs for all W):
If $m(u \to v) \neq 0$, then

every $i \in \tau(u) - \tau(v)$ is bonded to every $j \in \tau(v) - \tau(u)$.

Necessity follows from analyzing commuting braid relations.

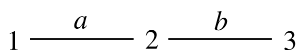
REFORMULATION: Define the **compatibility graph** $\mathrm{Comp}(W, S)$:

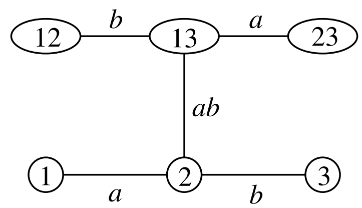
- vertex set $2^S = 2^{[n]}$,
- edges $I \to J$ when $I \supset J$,
- edges $I — J$ when $I \not\subset J$ and $J \not\subset I$ and

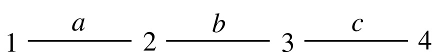
every $i \in I - J$ is bonded to every $j \in J - I$.

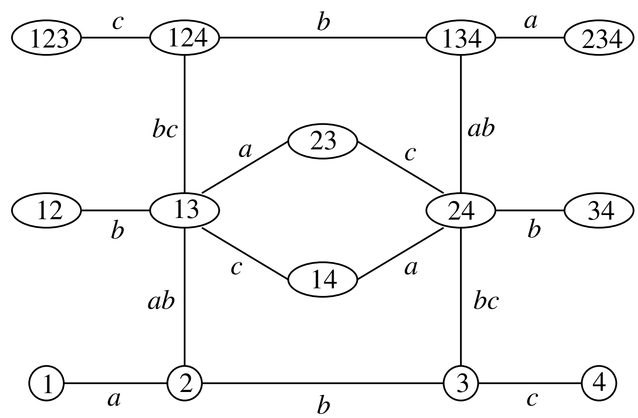
Compatibility means that $\tau : \Gamma \to \mathrm{Comp}(W, S)$ is a graph morphism.

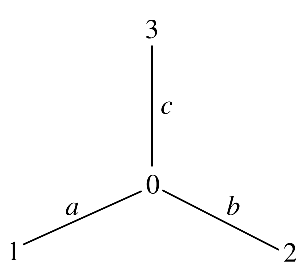
Compatibility graphs for A_3, A_4, and D_4:

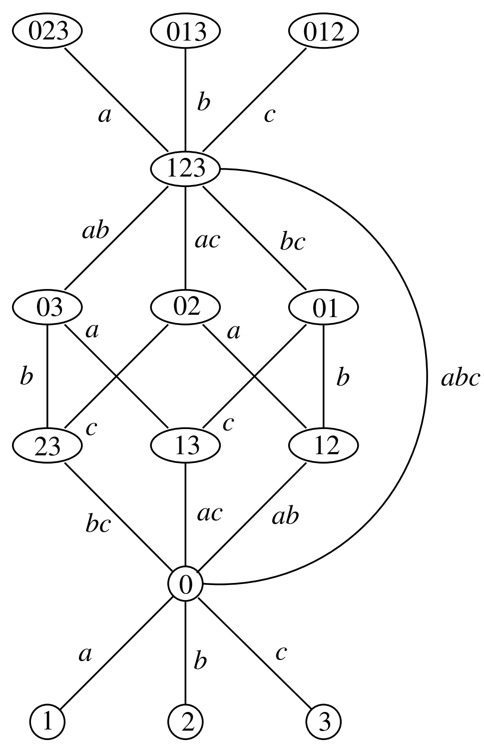












The Simplicity Rule (applies whenever $o(s_i s_j) < \infty$ for all i, j):

All edges are either simple or are inclusion arcs.

That is, $m(u \to v) \neq 0$ implies $m(u \to v) = m(v \to u) = 1$ or $\tau(u) \supset \tau(v)$. Necessity follows from Theorem 1.

The Frontier Rule (simply-laced only):

For each simple edge $u - v$, define

$$\mathrm{Bonds}(u, v) := \big\{\{i, j\} : i \in \tau(u) - \tau(v),\ j \in \tau(v) - \tau(u)\big\}.$$

Compatibility $\Rightarrow$ this is a set of bonds in the Dynkin diagram of (W, S).

Define the **frontier** of v:

$$\mathrm{Fr}(v) := \big\{\text{bonds } \{i, j\} : i \in \tau(v),\ j \notin \tau(v)\big\}.$$

The Frontier Rule requires that

$$\mathrm{Fr}(v) = \bigcup_{u : u - v} \mathrm{Bonds}(u, v) \qquad \text{(disjoint union)}.$$

Necessity follows from the $m = 3$ case of Theorem 1.

Example. Say $(W, S) = 1 \overset{a}{\text{------}} 2 \overset{b}{\text{------}} 3 \overset{c}{\text{------}} 4 \overset{d}{\text{------}} 5$ and $\tau(v) = \{1, 3, 4\}$. Then $\mathrm{Fr}(v) = \{a, b, d\}$ and

$$135 \overset{d}{\text{------}} 134 \overset{ab}{\text{------}} 24 \qquad \text{is legal at } v,$$

$$135 \overset{d}{\text{------}} 134 \overset{a}{\text{------}} 234 \qquad \text{is not.}$$

Remark. The Compatibility, Simplicity, and Frontier Rules suffice to determine all admissible A_3-cells.

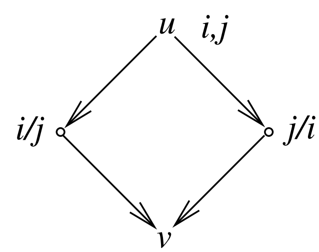
[Compare with G. Lusztig, *Represent. Theory* **1** (1997), Prop. A.4.]

$$\text{Define } V_{i/j} := \{v \in V : i \in \tau(v),\ j \notin \tau(v)\}.$$

The Diamond Rule:

For all $i \neq j$ and all vertices u, v such that $i, j \in \tau(u)$ and $i, j \notin \tau(v)$,

$$\sum_{w \in V_{i/j}} m(u \to w) m(w \to v) \ = \sum_{w \in V_{j/i}} m(u \to w) m(w \to v).$$



The Hexagon Rule:

For all bonded $\{i, j\}$ and all vertices u, v such that $i, j \in \tau(u)$ and $i, j \notin \tau(v)$,

$$\sum_{w \in V_{i/j}} m(u \to w) m(w' \to v) \ = \sum_{w \in V_{i/j}} m(u \to w') m(w \to v),$$

where w' is the unique vertex in $V_{j/i}$ such that $m(w \to w') = 1$.