Tau signatures, Cells and Orbits

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Atlas Workshop
University of Maryland
March 18, 2008

The Atlas Setting

Goal: parameterize $\widehat{G_{\mathbb{R}}}_{,unitary}$

- $G_{\mathbb{R}}$: a real reductive Lie group realizable as the set of real points of a reductive algebraic group defined over $\mathbb{R}$;
- $\widehat{G_{\mathbb{R}}}_{,unitary} \subset \widehat{G_{\mathbb{R}}}_{,adm}$: set of equivalence classes of irreducible admissible representations
- $\mathcal{L}_\lambda$: a set of Langlands parameters for irreducible admissible representations of **regular integral infinitesimal character** λ (a finite set).
- $\mathcal{HC}_\lambda = \{V_x = \pi_x|_{K\text{-finite}} \mid x \in \mathcal{L}_\lambda\}$: set of irreducible Harish-Chandra modules corresponding to irreducible admissible representations $\pi_x \in \widehat{G_{\mathbb{R}}}$, $x \in \mathcal{L}_\lambda$.

The Atlas software catalogs and analyzes $\mathcal{HC}_\lambda$.

Notation / Lie Algebraic Apparatus

- $\mathfrak{g} = Lie\left(G_{\mathbb{R}}\right)_{\mathbb{C}}$; $\quad \mathfrak{h}$, a CSA for $\mathfrak{g}$;
  $\Delta = \Delta\left(\mathfrak{h}, \mathfrak{g}\right)$, roots of $\mathfrak{h}$ in $\mathfrak{g}$;
  $\Pi \subset \Delta$, choice of simple roots in Δ;
- G : adjoint group of $\mathfrak{g}$
- $\mathcal{N}_{\mathfrak{g}}$: nilpotent cone in $\mathfrak{g}$ (identifying $\mathfrak{g}^*$ with $\mathfrak{g}$)
- $\mathcal{O}_x$: nilpotent orbit attached to $x \in \mathcal{L}_\lambda$
  $x \to Ann(V_x) \subset U(\mathfrak{g}) \xrightarrow{\ gr\ } \mathcal{I}_x \subset S(\mathfrak{g}) \to \mathcal{O}_x \subset \mathcal{N}_{\mathfrak{g}}$
- $x \in \mathcal{L}_\lambda$, with λ regular, integral inf. char.
  $\Longrightarrow \quad \mathcal{O}_x$ is **special** nilpotent orbit.
- Set $\mathcal{S} \equiv$ {special nilpotent orbits}
- $d : G\backslash\mathcal{N}_{\mathfrak{g}} \to \mathcal{S}$: the Spaltenstein-Barbasch-Vogan duality map that restricts to an involution on $\mathcal{S} = image(d)$.

Cells of Harish-Chandra modules

Definition: Let $x, y \in \mathcal{L}_\lambda$. Write $x \rightharpoonup y$ if there exists a f.d. rep F occurring in $T(\mathfrak{g})$ such that

$$V_y \text{ occurs as subquotient of } V_x \otimes F$$

A **cell** of H-C modules is a maximal collection of $x \in \mathcal{L}_\lambda$ such that

$$x, y \in C \quad \Longrightarrow \quad x \rightharpoonup y \text{ and } y \rightharpoonup x$$

Easy facts:

(i) $\mathcal{L}_\lambda = \coprod_{\text{cells } C} C$

(ii) If $x, y \in C$, then $\mathcal{AV}(V_x) = \mathcal{AV}(V_y)$
$\mathcal{AV}(V_x) \equiv$ associated variety of H-C module V_x, a union of $K_{\mathbb{C}}$-orbits in $(\mathfrak{g}/\mathfrak{k})^*$

(iii) (ii) implies

$$x, y \in C \Longrightarrow \mathcal{O}_x = \mathcal{O}_y$$

Problem: which cells correspond to which special nilpotent orbits?

The association

$$\text{cell} \quad \longrightarrow \quad \text{special orbit}$$

will be several to one.

(the associated variety of representation is a finer invariant than the associated variety of its annihilator)

Atlas Output

The Atlas software not only catalogs the KLV polynomials for the representations in $\mathcal{L}_\rho$, it computes the entire W-graph of $\mathcal{L}_\rho$: a weighted directed graph such that

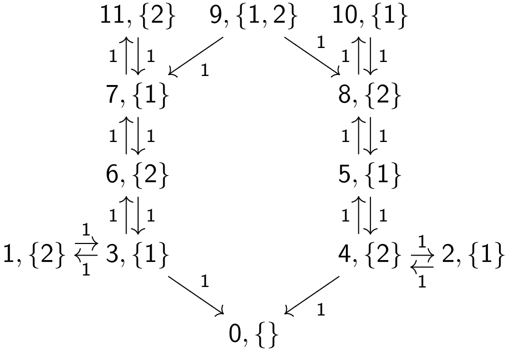
- vertices $\leftrightarrow$ $x \in \mathcal{L}_\rho$
- vertex weights $\leftrightarrow$ descent sets $\tau(x)$ of $x \in \mathcal{L}_\rho$
  For each $x \in \mathcal{L}_\lambda$, $\tau(x)$ is a certain subset of Π
  $\tau(x)$ is the tau invariant of $Ann(V_x)$.
- edges $\leftrightarrow$ relations $y \rightarrow x \quad \equiv V_y$ occurs in $V_x \otimes \mathfrak{g}$
- edge multiplicities: $mult(y \rightarrow x) =$ multiplicity of V_y in $V_x \otimes \mathfrak{g}$

H-C cells correspond to bidirectionally connected subgraphs

Example: the big block of the split real form of G_2.

block element	descent set	(edge vertex,multiplicity)
0	{}	{}
1	{2}	{(3,1)}
2	{1}	{(4,1)}
3	{1}	{(0,1),(1,1),(6,1)}
4	{2}	{(0,1),(2,1),(5,1)}
5	{1}	{(4,1),(8,1)}
6	{2}	{(3,1),(7,1)}
7	{1}	{(6,1),(11,1)}
8	{2}	{(5,1),(10,1)}
9	{1,2}	{(7,1),(8,1)}
10	{1}	{(8,1)}
11	{2}	{(7,1)}

The W-graph for this block thus looks like



Cell #	Members
0	0
1	1, 3, 6, 7, 11
2	2, 4, 5, 8, 10
3	9

The Spaltenstein-Vogan Criterion

Theorem. (Spaltenstein, Vogan) Suppose C is a cell of H-C modules with associated special nilpotent orbit $\mathcal{O}_C$ and let $\mathfrak{l}$ be a (standard) Levi subalgebra of $\mathfrak{g}$. Then

$$\mathcal{O}_C \subset \overline{ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}})} \quad \Longleftrightarrow \quad \exists\, x \in C \;\; s.t.\; \Pi_{\mathfrak{l}} \subset \tau(x)$$

where $\Pi_{\mathfrak{l}} =$ the simple roots of $\mathfrak{l}$. Here $\Pi_{\mathfrak{l}} \subset \Pi_{\mathfrak{g}}$ and

$$ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}}) \equiv \text{ unique dense orbit in } G \cdot \mathfrak{n}$$

where $\mathfrak{n}$ is nilradical of any parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$.

Orbits of the form $ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}})$ are called *Richardson orbits*.

Upshot: tau invariants of a cell constrain which Richardson orbit closures can contain $\mathcal{O}_C$

Problem: Every Richardson orbit is special, but not every special orbit is Richardson.

How do we separate configurations like

$$\mathcal{O}_C = \mathcal{O}_{Richardson}$$

$$\uparrow$$

$$\mathcal{O}_{C'} = \mathcal{O}_{special}$$

Levi subalgebras and Richardson orbits

- $\Gamma \subset \Pi$: a subset of the simple roots.
- $\mathfrak{l}_\Gamma$: standard Levi subalgebra attached to

$$\mathfrak{l}_\Gamma = \mathfrak{h} + \sum_{\alpha \in \langle \Gamma \rangle} \mathfrak{g}_\alpha$$

- $R_\Gamma = ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}} (\mathbf{0}_{\mathfrak{l}_\Gamma})$: the Richardson orbit induced from the trivial orbit of a Levi subalgebra $\mathfrak{l}_\Gamma$ of $\mathfrak{g}$

Fact: every special orbit $\mathcal{O}$ is determined by

(i) the Richardson orbits that contain $\mathcal{O}$

(ii) the Richardon orbits that contain $d(\mathcal{O})$

David Vogan's Idea: The tau invariants of a cell should tell us which Richardson orbits contain $\mathcal{O}_C$ and which Richardson orbits contain the SBV dual of $\mathcal{O}_C$.

Tau signatures for cells

Set

$$\tau(C) \equiv \{\tau(x) \mid x \in C\}$$

Facts

- $\#$ distinct $\tau(C) = \#$ special nilpotent orbits
- Let

$$\tau^{\vee}(C) = \{\Pi - \tau(x) \mid x \in C\}$$

  then $\tau(C) \mapsto \tau^{\vee}(C)$ is an involution on $\{\tau(C)\}$.
  $\Longrightarrow$ Spaltenstein-Barbasch-Vogan duality for tau sets.

Definition:

$\Psi = \{\Gamma \subset \Pi\}$: a set of *standard Gammas*: a collection of $\Gamma \in 2^{\Pi}$ such that

$$i : \Psi \leftrightarrow \{\text{conjugacy classes of Levi subalebras}\}$$

is a bijection.

Let $\Gamma, \Gamma' \in \Psi$ and let $\mathfrak{l}_\Gamma$ and $\mathfrak{l}_{\Gamma'}$ be the corresponding standard Levi subalgebras of $\mathfrak{g}$. We shall say

$$\Gamma \leq \Gamma' \Longleftrightarrow ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}) \subset \overline{ind_{\mathfrak{l}_{\Gamma'}}^{\mathfrak{g}}(\mathbf{0})}$$

Remark: this ordering tends to reverse the ordering by cardinality.

Definition: The **tau signature** of an H-C cell C is the pair

$$\tau_{sig}(C) \equiv \left(min \left(\tau(C) \cap \Psi \right) \ , \ min \left(\tau^{\vee}(C) \cap \Psi \right) \right)$$

Tau signatures for Special Orbits

Definition: Let $\mathcal{O}$ be a special orbit. The *tau signature* of $\mathcal{O}$ is the pair $\left(\tau\left(\mathcal{O}\right), \tau^{\vee}\left(\mathcal{O}\right)\right)$ where

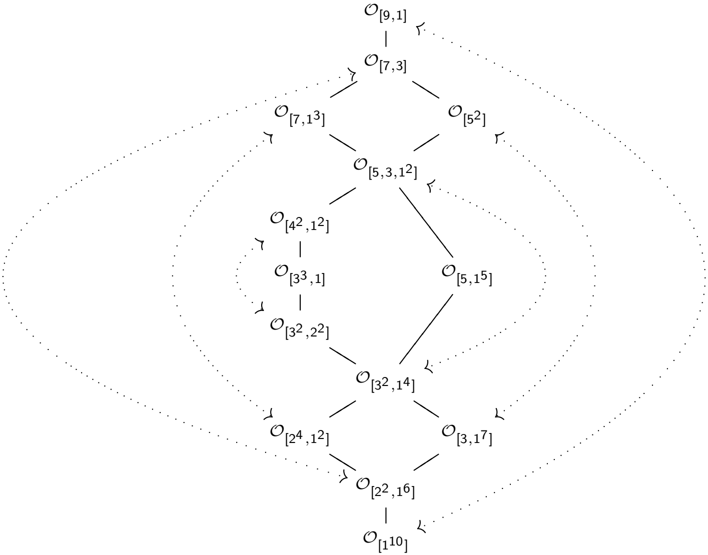
$$\tau\left(\mathcal{O}\right) = min\left\{\Gamma \in \Psi \mid \mathcal{O} \subset \overline{ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}_\Gamma}\right)}\right\}$$

$$\tau^{\vee}\left(\mathcal{O}\right) = min\left\{\Gamma \in \Psi \mid d\left(\mathcal{O}\right) \subset \overline{ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}_\Gamma}\right)}\right\}$$

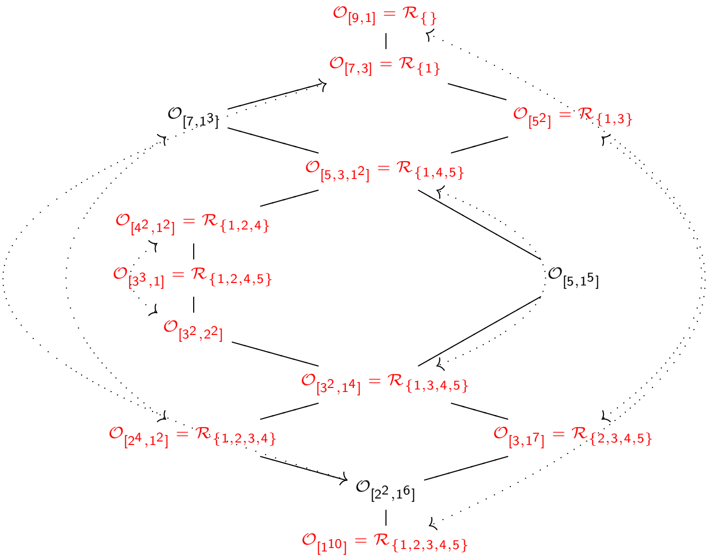
Corollary (to S-V criterion)

$$\mathcal{O}_C = \mathcal{O} \quad \Longleftrightarrow \quad \tau_{sig}(C) = \tau_{sig}(\mathcal{O})$$

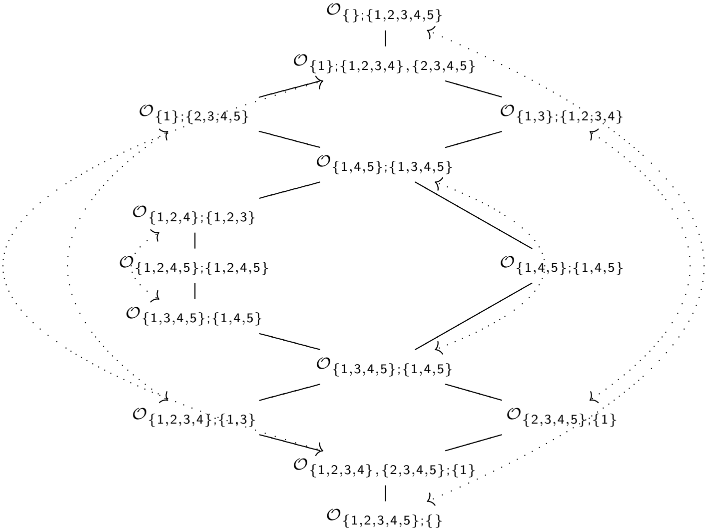
Example: Special Orbits of D_5



Richardson Orbits of D_5



Tau Signatures of Special Orbits of D_5



Tau signatures for cells in the big block of $SO(5,5)$

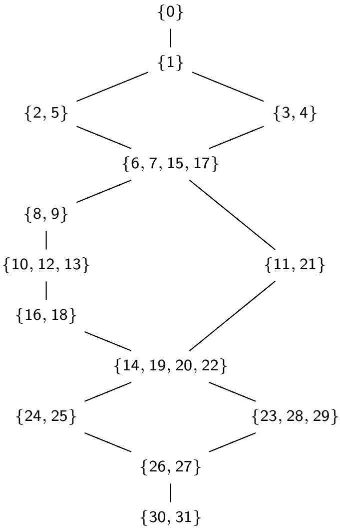
- 365 representations with inf. char. ρ in big block
- 32 cells in the big block

Output of `extract-cells`

```
// Individual cells.
// cell #0:
0[0]: {}

// cell #1:
0[1]: {2} --> 1,2
1[3]: {1} --> 0
2[5]: {3} --> 0,3,4
3[13]: {5} --> 2
4[14]: {4} --> 2
  *
  *
  *
// cell #29:
0[328]: {1,2,4,5} --> 2,3
1[340]: {2,3,4,5} --> 2
2[358]: {1,3,4,5} --> 0,1
3[364]: {1,2,3} --> 0

// cell #30:
0[353]: {1,2,3,4,5}

// cell #31:
0[357]: {1,2,3,4,5}
```

cell #	tau signature
0	{} , {1,2,3,4,5}
1	{1} , {1,2,3,4}
2	{1} , {2,3,4,5}
3	{1,3} , {1,3,4,5}
*	*
*	*
*	*
28	{2,3,4,5} , {1}
29	{2,3,4,5} , {1}
30	{1,2,3,4,5} , {}
31	{1,2,3,4,5} , {}

Cell-Orbit Correspondences for $SO(5,5)$



More Generally:

Exceptional Groups: tables by Spaltenstein list induced orbits, and Hasse diagrams.

Even E_8 can be done by hand.

Classical Groups:

Partition classification $\longrightarrow$ closure relations

Just need algorithms to determine

- which partitions correspond to special orbits;
- given $\Gamma \subset \Pi$, which partition corresponds to the Richardson orbit $\mathcal{R}_\Gamma \equiv ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}_\Gamma})$;

Cell-Orbit correspondences have now been computed for all exceptional and classical cases up to rank 8.

Conclusion:

- Atlas data $\Longrightarrow$ new algorithm mapping Langlands parameters to nilpotent orbits.
  Key is to first collect Langlands parameters into cells.
- Can one actually identify even finer invariants?
  - Can one tell when $Ann(V_x) = Ann(V_y)$? (yes!).
  - What about the associated variety of V_x (union of $K_{\mathbb{C}}$-orbits)?
- Representation theoretical intepretations of other combinatorial aspects of W-graphs?

Some References

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Acknowledgements

The Atlas for Lie Groups Team

Jeffrey Adams	Annegret Paul
Dan Barbasch	Patrick Polo
Birne Binegar	Siddhartha Sahi
Bill Casselman	Susana Salamanca
Dan Ciubotaru	John Stembridge
Fokko du Cloux	Peter Trapa
Scott Crofts	Marc van Leeuwen
Tatiana Howard	David Vogan
Monty McGovern	Wai Ling Yee
Alfred Noel	Gregg Zuckerman
Alessandra Pantano	

and the National Science Foundation for support via the *Atlas for Lie Groups and Representations FRG* (DMS 0554278).