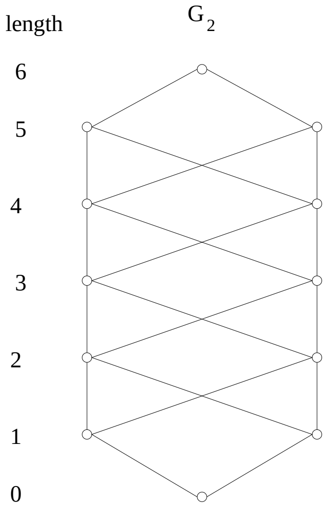
Can someone answer: what is maximal degree of a Kazhdan-Lusztig polynomial?

(Throughout this note, look for definitions in [KL79].)



eg. If y, x belong to the same block (here: $W(G_2)$) then the Kazhdan-Lusztig polynomials satisfy:

- $P_{x,y} \neq 0$ only if $x \leq y$
- $\deg P_{x,y} \leq \frac{\ell(y)-\ell(x)-1}{2}$ if $y \neq x$
- $P_{y,y} = 1$

In the G_2 example:
Length differences: $0, \ldots, 6$
Guess for degrees of $P_{x,y}$: $0, \ldots, \lfloor \frac{6-1}{2} \rfloor = 2$
Reality: all polynomials are 1 or 0, so degree 0. Bad guess.
Suggests: $\deg P_{x,y}$ typically much less than $\frac{\ell(y)-\ell(x)-1}{2}$

$\mu(x, y) =$ coefficient of $q^{(\ell(y)-\ell(x)-1)/2}$ in $P_{x,y}$
If x, y are in the same cell, $\mu(x, y) =$ multiplicity of edge joining $x < y$

W-graph sparse $\iff$ few edges
$\iff$ most edges have multiplicity 0
$\iff$ $\deg P$ usually $< \frac{\ell(y)-\ell(x)-1}{2}$

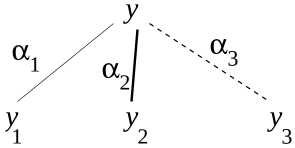
E_8: typical cell has $\sim 5,000$ block elements
Potentially: $\sim 1.25 \times 10^7$ edges
Actually: $\sim 10^5$ edges $\sim 1\%$ of potential edges

Real E_8:
Lengths: $0, 1, \ldots, 64$
Degree bound: $\lfloor \frac{64-1}{2} \rfloor = 31$ **achieved** (Atlas)
Average degree: ~ 12 (Atlas)
The distribution of length differences over pairs of elements in a block cannot be uniform:

$$\text{If so, then average length difference over pairs} = \frac{1}{64}\int_0^{64} \ell/2 \, d\ell = \frac{1}{64}\frac{\ell^2}{4}\bigg|_0^{64} = 16$$

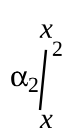
But then average degree $= 8 \neq 12$

Distribution actually Gaussian



$\leftarrow$ Can draw a diagram for a block indicating descents and ascents.

- some y_i's may be missing (eg. α_i compact imaginary)
- may have two $y_i's$ for one α_i (eg. α_i real type I)

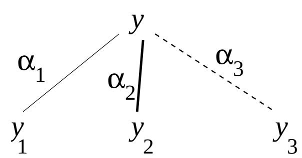


Non-primitive case: May happen that a descent for y is an ascent for x as in the diagram. Then in this case:

$$P_{x,y} = P_{x_2,y} \qquad x_2 \text{ may not exist (eg. } \alpha_2 \text{ "real non-parity"} \Rightarrow P_{x_2,y} = 0)$$

$$\text{may be two } x_2\text{'s (eg. } \alpha_2 \text{ type II nc imaginary)}$$

$\deg P_{x,y} = \deg P_{x_2,y} \leq \dfrac{\ell(y) - \ell(x_2) - 1}{2}$ and $\ell(x_2) = \ell(x) + 1$ so the degree bound for $P_{x,y}$ has been improved by $\frac{1}{2}$. In the non-primitive case, the maximal degree bound is therefore never achieved.



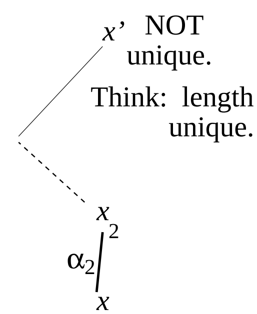
Given y, x: ascend x along descents of y as long as possible.
Arrive at: x' such that every descent for y is a descent for x'.

Definition: the pair (x, y) is **extremal** if

$$desc(x) \supset desc(y)$$

(*few* of these)
eg. x' as chosen above is extremal.

Observe: maximal degree is achieved for extremal pairs.



Definition: the pair (x, y) is **primitive** if

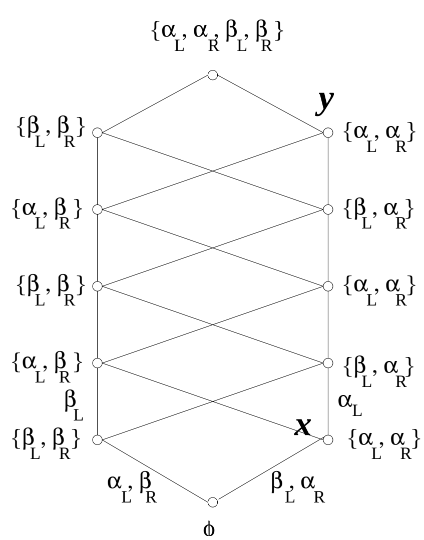
$$desc(x) \cup (\text{type II nc imaginary ascents of })\text{x} \supset desc(y)$$

(*many* of these)

Reformulated question:

What is the maximal length difference for an extremal pair (x, y)in a block?

eg. Complex G_2:



Four simple roots $\alpha_L, \beta_L, \alpha_R, \beta_R$
Outer edges: left action
Crosses: right action

Primitive pairs $x \neq y$:
Maximal length difference in extremal pair:

$$y = w_0 s_\alpha \qquad \text{length difference} = 4$$
$$x = s_\beta \qquad \Rightarrow \deg P_{x,y} \leq 1$$

What's biggest length difference in W, $x \le y$, $desc(x) \supset desc(y)$?

Guess:

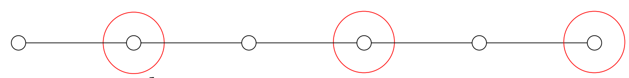
- partition simple roots as $\Pi = S \sqcup T$ where both S and T are type A_1^* (i.e. any two elements of S are orthogonal, any two elements of T are orthogonal)
- choose:

$$\begin{aligned} x &= \text{long element of } W(S) \\ y &= w_0 \text{long element of } W(T) \end{aligned}$$

(x, y) is primitive if w_0 commutes with long element of $W(T)$

$$\begin{aligned} \ell(y) &= \#\Delta^+ - \#T \\ \ell(x) &= \#S \\ \text{length difference:} &\quad \#\Delta^+ - \#\Pi \\ \text{degree bound:} &\quad \frac{\#\Delta^+ - \#\Pi - 1}{2} \qquad \leftarrow \text{Not sharp!} \sim \frac{n^2}{4} \qquad \text{Lower bound: } \frac{n^2}{16} \end{aligned}$$

eg. A_6:
Pick S and T as indicated in diagram.



- long element of $W(T)$ does not commute with w_0
- guess for largest degree from above parition: $\frac{21-6-1}{2} = 7$
- Atlas: 5

Take extra pair

$$\begin{array}{ccc} (x & , & y) \\ & \downarrow & \\ (x' & , & y') \end{array} \qquad \begin{array}{l} \text{recursion relation (descend } y) \\ \leftarrow \text{may not be extremal} \\ \text{therefore degree bound smaller than expected} \end{array}$$

Geometric picture:

$$\begin{array}{ccc} \mathscr{L}_y & \longleftrightarrow & \cdot\, y \\ \downarrow & & \\ \mathscr{O}_y & & \end{array}$$

$$\begin{array}{ccc} \mathscr{L}_x & \longleftrightarrow & \cdot\, x \\ \downarrow & & \\ \mathscr{O}_x & & \end{array}$$

What is constant term of $P_{y,x}$?

Guess: constant term $\longleftrightarrow$ extensions of local systems $\mathscr{L}_y$ to $\overline{\mathscr{O}}_y$ agreeing with $\mathscr{L}_x$ on $\mathscr{O}_x$

eg. E_8: degree 31 $\longleftrightarrow$ length difference 63 or 64, say 64

$$\begin{aligned} \mathscr{L}_y &= \text{local system on open orbit of } K \text{ on flag manifold} \\ \mathscr{O}_x &= \text{closed orbit} \\ \mathscr{L}_x &= \text{trivial} \end{aligned}$$

$$\begin{aligned} desc(x) &= \text{compact imaginary roots that are simple} \\ &\quad - \text{proper subset} \\ &\quad \text{Therefore } y \text{ has ``real non-parity roots''} \end{aligned}$$

Guess: $\mathscr{L}_y$ cannot extend all the way to $\mathscr{O}_x \Rightarrow$ constant term is 0?

References

[KL79] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53:165–184, 1979.