On the ω-Regular Unitary Representations of $Mp\,(2n,\mathbb{R})$
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2. Introduction.

Let G, K, T, $\mathfrak{g}_0$, $\mathfrak{g}$, $\mathfrak{t}_0$, θ, $\mathfrak{p}_0$ and $\Delta\,(\mathfrak{g},\mathfrak{t}) \subset i\,(\mathfrak{t}_0)$, $\Delta\,(\mathfrak{k},\mathfrak{t})$ $\Delta\,(\mathfrak{p},\mathfrak{t})$, etc. as ususal.

Let $\langle\ ,\ \rangle$ symmetric, G-invariant, θ-invariant non degenerate bilinear form on $\mathfrak{g}_0$, $\mathfrak{g}$, $\mathfrak{g}^*$.

Fix a positive root system $\Delta^+\,(\mathfrak{k},\mathfrak{t})$ and define

$$\rho_c = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})}\alpha \tag{2.1}$$

2.1. Strongly Regular Case. For a weight $\phi \in t^*$, choose a positive root system from the set of roots positive on ϕ.

$$\Delta^+\,(\phi) \subseteq \{\alpha \in \Delta\,(\mathfrak{g},\mathfrak{t}) \ \mid\ \langle\phi,\alpha\rangle \geq 0\} \tag{2.2}$$

Then define

$$\rho_\phi = \rho\left(\Delta^+\,(\phi)\right) \tag{2.3}$$

Definition 1. *Assume $\phi \in \mathfrak{t}^*$ is real. We say that ϕ is strongly regular if*

$$\langle \phi - \rho_\phi, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta^+(\phi)$$

Proposition 1. *Let X be an irreducible Hermitian $(\mathfrak{g}, K)$ module, infinitesimal character associated to a weight ϕ. Assume that ϕ is a strongly regular infinitesimal character.*

Then X is unitary if and only if there is

1. *a θ-stable parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$;*
2. *an admissible unitary character $(\lambda, \mathbb{C}_\lambda)$ of the Levi subgroup of q (zero on $\Delta(\mathfrak{l}, \mathfrak{t}^c)$ and positive on $\Delta(\mathfrak{u}, \mathfrak{t}^c)$), such that*

$$X \cong A_{\mathfrak{q}}(\lambda) = \mathfrak{R}_{\mathfrak{q}}(\mathbb{C}_\lambda) \tag{2.4}$$

Remark 1. *Admissible $A_{\mathfrak{q}}(\lambda)$ representations are always in the good range. So, nonzero, irreducible and unitary.*

2.2. ω-Regular case. Let $G = Mp(2n)$. Then $\mathfrak{g} = \mathfrak{sp}(2n)$. To extend the above result, consider the genuine representations of G.

If

$$\mathfrak{l} = \prod_{i=1}^{t} \mathfrak{u}(p_i, q_i)$$

then we can construct $A_{\mathfrak{q}}(\lambda)$'s if the infinitessimal character is also strongly Regular (SR).

Also, we can construct genuine $A_{\mathfrak{q}}(\lambda)$ representations which are not SR but in the good range.

But, if

$$\mathfrak{l} = \prod_{i=1}^{t} \mathfrak{u}(p_i, q_i) \oplus \mathfrak{sp}(2m)$$

Then there is a surjection

$$\prod_{i=1}^{r} \widetilde{U}(p_i, q_i) \times Mp(2m) \longrightarrow L, \tag{2.5}$$

So, irreducible admissible of $L \leftrightarrow \bigotimes_{i=1}^{r} \pi_i \otimes \sigma$,

To descend to L, either all representations in the product are genuine or all are non-genuine.

So, we have no genuine $A_{\mathfrak{q}}(\lambda)$ representations for L if $m > 0$ (since there are no genuine one-dimensional representations of $Mp(2m)$).

To extend these representations:

We use the metaplectic representation of $Mp(2m)$. Then construct a representation of L

1. λ_i genuine one dimensional of $\widetilde{U}(p_i, q_i)$ and
2. ω^Leither $\omega_o^{\pm}$, or $\omega_e^{\pm}$ for $Mp(2m)$

Definition 2. *An $A_{\mathfrak{q}}(\Omega)$ is a (genuine) representation X of G of the following form. Let*

(1) $q = l \oplus u$ *be a theta stable parabolic subalgebra of* g *with* $L = \prod_{i=1}^{r} \widetilde{U}(p_i, q_i) \times Mp(2m)$.
(2) *Let* $\mathbb{C}_\lambda$ *be a genuine one-dimensional representation of* $\prod_{i=1}^{r} \widetilde{U}(p_i, q_i)$ *and* ω^L *an oscillator representation of* $Mp(2m)$ *as above.*
(3) *Assume that* $\Omega = \mathbb{C}_\lambda \otimes \omega^L$ *is in the good range for* q.

Let $A_{\mathfrak{q}}(\Omega) = R_{\mathfrak{q}}(\Omega)$.

Definition 3. *A Meta-*$A_{\mathfrak{q}}(\lambda)$ *is a (non-genuine) representation* X *of* G *of the following form. Let*

(1) $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ *be a theta stable parabolic subalgebra of* $\mathfrak{g}$ *with* $L = \prod_{i=1}^{r} \widetilde{U}(p_i, q_i) \times Mp(2m)$.
(2) $\mathbb{C}_\lambda$ *be a non-genuine one-dimensional representation of* $\prod_{i=1}^{r} \widetilde{U}(p_i, q_i)$
(3) J_ν *the spherical constituent of the spherical principal series of* $Mp(2m)$ *with infinitesimal character* ν.
If $m \neq 1$ *then* $\nu = \rho$ *so that* $J_\nu = J_\rho$ *is the trivial representation;*
If $m = 1$ *then* $\frac{1}{2} \leq \nu \leq 1$, *so that* J_ν *is a complementary series of* $Mp(2)$.
(4) $C_\lambda \otimes J_\nu$ *is in the good range for* $\mathfrak{q}$.
(5) *Denote by* $A_{\mathfrak{q}}(\lambda, \nu) = R_{\mathfrak{q}}(\mathbb{C}_\lambda \otimes J_\nu)$.

Proposition 2. *With notation as above,*
(1) $A_{\mathfrak{q}}(\Omega)$*'s and Meta-*$A_{\mathfrak{q}}(\lambda)$*'s are nonzero, irreducible and unitary.*
(2) *Meta-*$A_{\mathfrak{q}}(\lambda)$*'s with* $\nu = \rho$ *are admissible* $A_{\mathfrak{q}}(\lambda)$*'s, and they have strongly regular infinitesimal character.*
(3) $A_{\mathfrak{q}}(\Omega)$*'s and Meta-*$A_{\mathfrak{q}}(\lambda)$*'s are* ω*-regular (definition below).*

Definition 4. *Let* X *be a genuine Hermitian* $(\mathfrak{g},K)$ *module of* $Mp(2n)$ *with infinitesimal character associated to* $\phi \in t^*$ *as above. Assume that* ϕ *is real. Let* γ^ω *be a weight representating the infinitesimal character of the oscillator representation of* $Mp(2n)$ *such that* ϕ *belongs to the Weyl chamber determined by* γ^ω. *We say that* ϕ *(as well as* X*) is* ω*-regular if*

$$\langle \phi - \gamma^\omega, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta^+(\phi) \quad (2.6)$$

Conjecture 1. *(Adams, Barbasch, Vogan) The* $A_{\mathfrak{q}}(\Omega)$ *and Meta-*$A_{\mathfrak{q}}(\lambda)$ *representations exhaust all the* ω*-regular unitary irreducible representations of* G.

3. Main Theorem

Theorem 1. *Assume* $G = Mp(2n)$ *for* $n \leq 3$. *Then the* $A_{\mathfrak{q}}(\Omega)$*'s exhaust all the genuine* ω*-regular unitary representations of* G.

Remark 2. *The non-genuine part of the conjecture is true for* $Mp(4)$*; we will restrict our attention to the genuine case for the remainder of this talk.*

4. Sketch of proof

Proposition 3. *Let $n \leq 3$, and let $\mu = \mu(\mathfrak{q},\Omega)$ be the LKT of an $A_{\mathfrak{q}}(\Omega)$ representation of G.*

Let $\phi = \phi(\mathfrak{q},\Omega)$ be its infinitesimal character. Then if X is an ω-regular, unitary representation of G with LKT μ and infinitesimal character γ, then

$$\gamma = \phi(\mathfrak{q},\Omega).$$

Proposition 4. *Let $n \leq 3$. If X is genuine ω-regular representation of G with LKT $\mu(\mathfrak{q},\Omega)$ and infinitesimal character $\phi(\mathfrak{q},\Omega)$ then*

$$X \cong A_{\mathfrak{q}}(\Omega).$$

Proof. For $n \leq 3$, these two propositions can be proved case by case, going through all the possible choices for $\mathfrak{q}$, listing the corresponding LKT's. For Proposition 3, we show that Parthasarathy's Dirac operator inequality ($PDOI$), together with the ω-regular condition force the infinitesimal character to be $\gamma = \phi(\mathfrak{q},\Omega)$. For Proposition 4 one shows that for representations with these special LKT's, the infinitesimal character uniquely determines the continuous parameter. □

The two propositions should be true for all n; we are working on a general argument.

It remains to show that all representations which do not have the LKT of an $A_{\mathfrak{q}}(\Omega)$ are nonunitary.

(1) $n = 1$. Let $\mu \leftrightarrow a \in \mathbb{Z}+\frac{1}{2}$ be the LKT of the representation. If $|a| \geq \frac{3}{2}$, then representation is a discrete series. The K-types $\mu = \pm\frac{1}{2}$ are LKT's of oscillator representations, hence of $A_{\mathfrak{q}}(\Omega)$ modules, so the propositions tell the whole story for $Mp(2)$.

(2) Now assume $n = 2$. We can separate all those K representations that are LKT's of $A_{\mathfrak{q}}(\Omega)$ modules from those that are not. The $A_{\mathfrak{q}}(\Omega)$ LKT's are

$$\left\{ \begin{array}{c} (a,b), a \geq b \geq \frac{5}{2}; a \geq \frac{5}{2}, b \leq -\frac{1}{2}; \\ a \geq \frac{1}{2}, b \leq -\frac{5}{2}; b \leq a \leq -\frac{5}{2} \\ -b = a \geq \frac{3}{2}; \\ a \geq \frac{5}{2}, b = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2} \\ a = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}; \\ \left(\frac{3}{2},\frac{1}{2}\right), \left(-\frac{1}{2},-\frac{3}{2}\right), \pm\left(\frac{1}{2},\frac{1}{2}\right) \end{array} \right\} \tag{4.1}$$

By the above propositions, there is a unique ω-regular unitary representation containing each of these LKT's. We are left with the following LKT's

$$\left\{ \begin{array}{c} (-b+1,b), b \leq -\frac{1}{2}; (a,-a-1), a \geq \frac{1}{2}; \\ \pm\left(\frac{3}{2},\frac{3}{2}\right); \left(\frac{1}{2},-\frac{1}{2}\right) \end{array} \right\} \tag{4.2}$$

Using $PDOI$, one shows that any ω-regular representation with one of these LKT's must be non-unitary...except for the K-type $\mu = \left(\frac{1}{2},-\frac{1}{2}\right)$ and infinitessimal character $\phi = \gamma^{\omega}$. Up to contragredients, there is a unique such representation, the LKT constituent of a (non-pseudospherical) principal series. We call this representation Mystery (this reflects the long time it took us to figure out how to determine that it is non-unitary). Here one needs to calculate the intertwining operator; then one can check that the form changes signs on the K-type $\left(-\frac{1}{2},-\frac{3}{2}\right)$.

(3) Let $n=3$. Then similar (considering many more cases than in the previous case) arguments take care of the reps with $A_{\mathfrak{q}}(\Omega)$ LKT's, and $PDOI$ rules out all representations except:
   (a) Three representations with LKT $\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$ and infinitesimal character γ^{ω}; we call these Mystery representations as well; here we use the same technique as for Mystery of $Mp(4)$, but we have to work harder; the signature of the form is negative on $\left(\frac{3}{2},\frac{1}{2},\frac{1}{2}\right)$ in two of them; for the third we have to go to $\left(\frac{1}{2},-\frac{1}{2}.-\frac{3}{2}\right)$;
   (b) One representation with LKT $\left(\frac{5}{2},\frac{1}{2},\frac{1}{2}\right)$ and infinitesimal character γ^{ω}; this is a representation of the form $\mathcal{R}_{\mathfrak{q}}\left(\lambda\otimes Y\right)$ and Y a pseudo-spherical principal series with inf char $\left(\frac{5}{2}.\frac{1}{2}\right)$; a "Pothole" representation. Here we show that in Y, the form is negative on the K-type $\left(\frac{1}{2},-\frac{3}{2}\right)$; this K-type survives in the Bottom Layer, so the our Pothole representation is non-unitary as well;
   (c) A family of representations with LKT $\left(a,\frac{3}{2},\frac{1}{2}\right)$, $a\geq\frac{7}{2}$, and infinitesimal character $\left(a-1,\frac{3}{2},\frac{1}{2}\right)$; these are representations of the form $\mathcal{R}_{\mathfrak{q}}\left(\lambda\otimes Mystery\right)$ in the good range, which we call "Pseudo-$A_{\mathfrak{q}}(\Omega)$'s". Since the K-type of Mystery for $Mp(4)$ which detects non-unitarity survives in the Bottom Layer, our "Pseudo-$A_{\mathfrak{q}}(\Omega)$'s" are proved to be non-unitary.

5. Outlook

Here is our strategy for proving the general case:

(1) Prove Propositions 3 and 4 in general (almost done).
(2) Identify all ω-regular representations which are not $A_{\mathfrak{q}}(\Omega)$'s and for which the PDOI, applied to the LKT, does not detect non-unitarity. This includes the following families of representations:
   (a) Non-pseudospherical principal series with LKT $\left(\underbrace{\frac{1}{2},...,\frac{1}{2}}_{p},\underbrace{-\frac{1}{2},...,-\frac{1}{2}}_{q}\right)$ and infinitesimal character γ^{ω} ("Mystery" representations);
   (b) Pseudo-$A_{\mathfrak{q}}(\Omega)'s:\mathcal{R}_{\mathfrak{q}}\left(\lambda\otimes Mystery\right)$ in the good range;
   (c) 'Pothole' $A_{\mathfrak{q}}(\Omega)$'s: $X=\mathcal{R}_{\mathfrak{q}}\left(\lambda\otimes Y\right)$, where Y is a pseudospherical principal representation of $Mp(2m)$ with infinitesimal character $\nu=(\frac{1}{2},\frac{3}{2},...\widehat{x_i},...\frac{m-1}{2},\frac{m+1}{2})$ and $\widehat{x_i}$ means the i-th entry is deleted, and the infinitesimal character of X is γ^{ω};
   (d) Others, yet to be identified?
(3) Prove that the representations in (2) are non-unitary, using techniques similar to the ones for the small cases.
(4) Describe the $LKT's$ of all remaining representations.
(5) Use $PDOI$ to show that any ω-regular representation with such a LKT is non-unitary.