Spherical Unitary Representations of Split Groups

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Abstract

This is an expository version of the first few sections of *Spherical Unitary Dual for Split Classical Groups* by Dan Barbasch. See www.math.cornell.edu/~barbasch.

1 Introduction

Let G be a split symplectic or orthogonal group over $\mathbb{R}$ or a p-adic field. We compute the irreducible unitary spherical representations of G.

Suppose $\lambda = (a_1, \ldots, a_n)$ where $n = rank(G)$ $a_i \in \mathbb{C}$ for all i. Then associated to λ is a principal series representation $X(\lambda)$. This representation has a unique spherical constituent which we denote $\overline{X}(\lambda)$. This is tempered and hence unitary if $a_i \in i\mathbb{R}$ for all i. Unitarity for general λ reduces to the case $a_i \in \mathbb{R}$ for all i [?]. From now on we assume this is the case. Then $X(\lambda)$ has an invariant Hermitian form if and only if $-\lambda$ is conjugate to λ by the Weyl group. This is automatic if the long element of the Weyl group is equal to -1, i.e. G is not of type D_n with n odd. In the latter case the condition holds if and only if $a_i = 0$ for some i.

Let $G^\vee$ be the complex dual group of G. Fix a unipotent orbit $\mathcal{O}^\vee$ of $G^\vee$. According to the Arthur conjectures [?] associated to $\mathcal{O}^\vee$ is (among other things) a spherical unitary representation π of G. By standard theory attached to $\mathcal{O}^\vee$ is semi-simple $Ad(G^\vee)$ orbit in the Lie algebra $\mathfrak{g}^\vee$ of $G^\vee$, which in turn gives rise to element $\lambda \in \mathfrak{h}^*$. We write $\lambda = \lambda(\mathcal{O}^\vee)$. Thie spherical representation associated to $\mathcal{O}^\vee$ by Arthur's conjecture is $\overline{X}(\lambda)$, i.e. we expect that $\overline{X}(\lambda)$ is unitary.

For example the principal nilpotent orbit gives $\lambda = \lambda(\mathcal{O}^\vee) = \rho$ and $\overline{X}(\lambda)$ is the trivial representation. On the other hand if $\mathcal{O}^\vee = 0$ then $\lambda = \lambda(\mathcal{O}_c) = 0$, and $X(\lambda) = \overline{X}(\lambda)$ is irreducible and unitary.

Associated to $\mathcal{O}^\vee$ is the Bala–Carter [?] Levi factor $M^\vee$ of $G^\vee$. If $M^\vee = G^\vee$ the orbit $\mathcal{O}^\vee$ is said to be distinguished. The Levi factor $M^\vee$ has the property that $\mathcal{O}^\vee \cap M^\vee$ is a distinguished nilpotent orbit $\mathcal{O}^\vee_M$ in $M^\vee$. Furthermore the split $\mathbb{F}$–form M of the dual of $M^\vee$ is then a Levi subgroup of G. Suppose M is a proper subgroup of G. By the preceding discussion we expect that the spherical representation $\overline{X}_M(\mathcal{O}_M^\vee)$ of M_c is unitary.

In a bit more detail, we have

$$M \simeq M_0 \times GL(m_1) \times \cdots \times GL(m_r)$$

where M_0 is of the same type as G. The only distinguished nilpotent orbit in $GL(m)$ is the principal nilpotent, so $\mathcal{O}^\vee$ is the product of a distinguished nilpotent orbit in M_0 with the principal nilpotent orbits in each GL factor.

Let us assume for the moment that for any distinguished nilpotent orbit of M_0 the corresponding spherical representation $\overline{X}_{M_0}$ is unitary.

Now suppose $\mathcal{O}^\vee$ is not distinguished, with corresonding Levi factor M and $\mathcal{O}_M^\vee = \mathcal{O}^\vee \cap M^\vee$. Let $\lambda = \lambda(\mathcal{O}^\vee) = \lambda(\mathcal{O}_M^\vee)$. By the preceding discussion we assume $\overline{X}_M(\lambda)$ is unitary. Then $\overline{X}(\lambda)$ is the spherical constituent of

$$Ind_P^G(X_M(\lambda) \otimes 1)$$

where Ind_P^G denotes unitary induction from $P = MN$ to G. In particular $\overline{X}(\lambda)$ is unitary. Henceforth we drop N from the notation and write $Ind_M^G(X_M(\lambda))$.

From this realization of $\overline{X}(\lambda)$ we see it may be possible to embed $\overline{X}(\lambda)$ in a continuous family of unitary representations. Let χ be a real-valued character χ of M (trivial on M_0), i.e. χ restricted to each GL factor is a real power of $|det|$. We may then consider $Ind_M^G(X_M(\lambda)\chi)$ Letting $\nu = d\chi$ we write this as

$$Ind_M^G(X_M(\lambda + \nu)) \tag{1.1}$$

and the spherical consitutent of this representation is $\overline{X}(\lambda + \nu)$. In fact the induced representation (1.1) is reasonably close to be irreducible. More precisely the multiplicities of certain K–types which determine unitarity are the same in (1.1) and $\overline{X}(\lambda + \nu)$.

Suppose $Ind_P^G(X_M(\lambda))$ is irreducible. It is well known that the signature of the invariant Hermitian form on $Ind_M^G(X_M(\lambda + \nu))$, as ν varies, can only change sign at a point where it is reducible. We conclude that $X(\lambda + \nu)$ is unitary for all ν in some open set. This is the *complementary series* attached to $\mathcal{O}^\vee$ and containing $\overline{X}(\lambda)$. This complementary series exists for the induced representation 1.1 (even it is not irreducible). We seek to describe this set.

If $\mathcal{O}^\vee$ is the 0–orbit then $M \simeq GL(1)^n$ is the split torus in G, and $\overline{X}(\nu)$ is the spherical consituent of the minimal principal series representation $Ind_T^G(\nu)$. The 0–complementary series may be considered as an open subset of $\mathbb{R}^n$. We are going to reduce to this case, so we assume that we have computed this set for all classical groups.

We return to the consideration of a general nilpotent orbit $\mathcal{O}^\vee$.

Definition 1.1 *Given $\mathcal{O}^\vee$ we let $H^\vee$ be the reductive part of the centralizer of $\mathcal{O}^\vee$ in $G^\vee$. Let H be the $\mathbb{F}$–points of the split $\mathbb{F}$–form of the dual group of $H^\vee$.*

Remark 1.2 *The identity component of H is a product of symplectic and orthogonal groups.*

Note that H is not necessarily a subgroup of G. By [?] $M^\vee$ is the centralizer in $G^\vee$ of a maximal torus $T^\vee$ in $H^\vee$ and $T^\vee$ is the center of $M^\vee$. (In particular $\mathcal{O}^\vee$ is distinguished if and only if $H^\vee$ is finite.) Taking duals we see that the maximal split torus T of H may identified with the center of M. Consequently the character $\nu = d\chi$ of M may be identified with a minimal principal series representation $Ind_T^H(\nu)$.

The key observation is that the complementary series containing $\overline{X}(\lambda)$ is determined by the 0–complementary series of H:

Proposition 1.3 *The representation $\overline{X}_G(\lambda+\nu)$ is unitary if and only if $\overline{X}_H(\nu)$ is unitary, i.e. $\overline{X}_H(\nu)$ is in the 0–complementary series for H.*

We now have a large family of unitary representations obtained by continuous deformation of the representations associated to a nilpotent orbit. The main theorem is that this gives the entire spherical unitary dual.

Theorem 1.4 *Let $G = Sp(n, \mathbb{F})$ or $SO(n, \mathbb{F})$ be a split group over a $\mathbb{F} = \mathbb{R}$ or a p–adic field.*

1. *Let $\mathcal{O}^\vee$ be a distinguished nilpotent orbit in $G^\vee$, and let $\lambda = \lambda(\mathcal{O}^\vee)$. Then $\overline{X}(\lambda)$ is unitary.*
2. *Fix a nilpotent orbit $\mathcal{O}^\vee$ and let $\lambda = \lambda(\mathcal{O}^\vee)$. Let $H = H(\mathcal{O}^\vee)$ (Definition 1.1). The complementary series $\overline{X}(\lambda+\nu)$ associated to $\mathcal{O}^\vee$ is in bijection with the 0–complementary series $\overline{X}_H(\nu)$ of H.*
3. *Suppose π is an irreducible unitary spherical representation of G. Then there is a unique nilpotent orbit $\mathcal{O}^\vee$ such that $\pi \simeq \overline{X}(\lambda+\nu)$ where $\lambda = \lambda(\mathcal{O}^\vee)$ and $\overline{X}(\lambda+\nu)$ is in the complementary series attached to $\mathcal{O}^\vee$.*

By Remark 1.2 the next result completes the classification of the spherical unitary dual.

Theorem 1.5 *Classification of 0-complementary series for types B,C,D.*

By the preceding discussion is an algorithm which associates to any λ a group H and a parameter ν for H such that $\overline{X}_G(\lambda)$ is unitary if and only if $\overline{X}_H(\nu)$ is unitary. We make this algorithm explicit in Section 3.

2 Data associated to a nilpotent orbit

We describe some data associated to a nilpotent orbit in a classical group. This will be applied to $G^\vee$.

Let $G = Sp(n, C)$ or $SO(n, C)$. The nilpotent orbits of G are parametrized by partitions $(a_1, \dots, a_r)$ with $a_1 \geq \dots a_n \geq 0$ and $\sum a_i = n$. The multiplicity of each even (respectively odd) part must be even in the case of $O(n)$ (resp. $Sp(n)$). We view the partition as a Young diagram with rows of length $a_1, \dots, a_r$.

The parameter h: We first give an algorithm to compute $h = h(O)$. For each row of length $a_i > 1$ we attach the set $\{1, 2, \dots, \frac{a_i-1}{2}\}$ if a_i is odd, or $\{\frac{1}{2}, \frac{3}{2}, \dots, \frac{a_i-1}{2}\}$ if a_i is even. Let S be the union of these sets and let h_0 be the elements of S arranged in decreasing order. Then h is obtained by appending $0's$ to h_0 so that the number of coordinates is the rank of G.

The group H: Fix a partition P. We write

$$P = (a_1^{m_1}, a_2^{m_2}, \dots, a_r^{m_r})$$

where $a^m = \overbrace{a, a, \dots a}^{m}$.

For each i we let

$$H_i = \begin{cases} O(m_i) & G = Sp(n), a_i \text{ even} \\ O(m_i) & G = O(n), a_i \text{ odd} \\ Sp(n) & G = Sp(n), a_i \text{ odd} \\ Sp(n) & G = O(n), a_i \text{ even} \end{cases}$$

Then

$$H = S[H_1 \times \dots H_r]$$

Note that H contains a non–trivial torus if and only if $m_i > 1$ for some i. Therefore

O is distinguished if and only if all rows have distinct length

A nilpotent is even [?] if all rows have the same parity. If a row of length a multiplicity one then a is even (resp. odd) if $G = Sp(n)$ (resp. $SO(n)$). Therefore all distinguished nilpotent orbits are even.

The group M:

Let P be a partition

$$P = (a_1^{m_1}, \dots, a_r^{m_r})$$

as above, corresponding to a nilpotent orbit O of G. We make new partitions (P_0, P_1) as follows. The partition P_1 is obtained from P by replacing each odd m_i with $m_i - 1$, and P_0 has a single row of length a_i for each odd m_i. That is $P_0 \cup P_1 = P$, the multiplicity of each row in P_1 is even, and the rows of P_0 each have multiplicity one.

Now suppose P corresponds to a nilpotent orbit for G. Write

$$P_0 = (a_1, \dots, a_r)$$
$$P_1 = (b_1^{m_1}, \dots, b_s^{m_s})$$

Each m_i is even. Let M_0 be a classical group of the same type as G and of rank $\sum a_i$. Then

$$M = M_0 \times GL(b_1)^{\frac{m_1}{2}} \times GL(b_s)^{\frac{m_s}{2}}.$$

Note that the orbit O_0 in M_0 corresponding to P_0 is distinguished.

3 Algorithm

In this section we given an explicit algorithm realizing Theorem 1.4. That is we show how to decide whether a given representation $\overline{X}(\lambda)$ is unitary.

Fix λ. To determine if $\overline{X}(\lambda)$ is unitary we need to know if we can write $\overline{X}(\lambda)$ as in Theorem 1.4 (3). We begin with some combinatorial considerations.

Define an equivalence relation $\sim$ on $\mathbb{R}$: $a \sim b$ if $a+b$ *or* $a-b$ is an integer. The equivalence classes are in bijection with $[0, 1/2]$. If S is a finite subset of $\mathbb{R}$ we write S as a disjoint union of equivalence classes $S_0 \cup S_1 \cup \dots S_r$. Here we will require S_0 is the set of elements of S in $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ depending on the situation.

By a *string* we mean a set of real numbers of the form $\{a, a-1, \dots, a-\ell\}$. By a *balanced string* we mean a string of the form $\{a, a-1, \dots, -a\}$. Note that this implies $2a \in \mathbb{Z}$.

Fix a set $T = \{b_1, \dots, b_r\}$ of non–negative real numbers which are all equivalent. We seek to write T as a disjoint union of strings, where we allow each b_i to be replaced by $-b_i$. That is we write

$$T = |T_1| \cup |T_2| \cup \dots \cup |T_s|$$

each T_i is a string and $|T_i| = \{|b| \mid b \in T_i\}$.

We construct these sets inductively. Assume $b_1 \geq b_2 \dots b_r \geq 0$.

Let T_1 be the maximal string containing b_1 made from $b_1, \pm b_2, \dots, \pm b_r$. That is $T_1 = \{b_1, b_1 - 1, \dots, b_1 - \ell\}$ where ℓ is maximal so that $b_1, \pm b_2, \dots \pm b_1 - \ell \in T$. Write $T = T_1 \cup (T - T_1)$. Apply the same procedure to $T - T_1$. Proceeding in this way we obtain sets T_i as stated.

We say T is the union of the strings T_i. (This is a slight abuse of notation: in fact $T = \cup |T_i|$.)

If each $b_i \in \frac{1}{2}\mathbb{Z}$ we may further require that each string T_i is balanced. We can not necessarily write T as a union of balanced strings. However there is a unique maximal subset which can be so written, and we have

$$T = T' \cup |T_1| \cup \dots \cup |T_r|$$

where $T_1, \dots, T_r$ are balanced and T' contains no balanced strings.

For example if $T = \{3, 2, 2, 2, 1, 1, 1, 0, 0\}$ then $T_1 = \{3, 2, 1, 0, -1, -2\}, T_2 = \{2, 1, 0, -1, -2\}$ and $T_3 = \{0\}$. If we require the strings to be balanced we have $T_0 = \{3, 2, 1, 0\}$ and $T_1 = \{2, 1, 0, -1, -2\}$.

We return to our set S, and first consider the set S_0. We write $S_0 = S_0' \cup S_{0,1} \dots \cup S_{0,s}$ as a union of a set of maximal balanced strings as above, where S_0' contains no balanced strings. Let $X = \{\#(S_{0,1}), \#(S_{0,1}), \dots, \#(S_{0,s}), \#(S_{0,s})\}$ (each term counted twice).

Now write each set $S_0', S_1, \dots, S_r$ as a disjoint union of strings. For each string T which arises append $\#(T)$ to X.

Then X is a set of positive integers. We write these in decreasing order and consider X as a partition.

Now fix G and let $\lambda = (a_1, \ldots, a_n)$ with $a_i \in \mathbb{R}$. If $G = SO(2n)$ with n odd assume $a_i = 0$ for some i, i.e. λ is W–conjugate to $-\lambda$. After conjugating by the Weyl group we may assume $a_1 \geq \ldots a_n \geq 0$. If $G = SO(2n)$ and $a_i \neq 0$ for all i we may also need to apply an outer automorphism of G to make $a_n > 0$; this is allowed since outer automorphisms preserve unitarity.

Let $S = \{a_1, \ldots, a_n\}$. Write $S = S_0 \cup S_1 \cup \cdots \cup S_r$ as above, where

$$S_0 = \begin{cases} \{a_i \in S \,|\, a_i \in \mathbb{Z}\} & G = Sp(n) \text{ or } SO(2n) \\ \{a_i \in S \,|\, a_i \in \mathbb{Z} + \frac{1}{2}\} & G = SO(2n+1) \end{cases}$$

Apply the above procedure to S. We obtain a partion X which we denote $X(\lambda)$.

Proposition 3.1 *Fix λ.*

1. *The partion X corresponds to a nilpotent orbit, denoted $\mathcal{O}^\vee(\lambda)$, of $G^\vee$.*

2. *The map $\lambda \to \mathcal{O}^\vee(\lambda)$ is a left inverse to the map $\mathcal{O}^\vee \to \lambda(\mathcal{O}^\vee)$: $\mathcal{O}^\vee(\lambda(\mathcal{O}^\vee)) = \mathcal{O}^\vee$.*

3. *Let $h = \lambda(\mathcal{O}^\vee(\lambda))$ and $M = M(\mathcal{O}^\vee(\lambda))$. Then $\lambda = h + \nu$ where ν is the differential of the character of the center of M.*

4. *Let $H = H(\mathcal{O}^\vee(\lambda))$. Then $\overline{X}(\lambda)$ is unitary if and only if ν is in the 0–complementary series of H.*