

Hermitian forms for $Sp(4, \mathbb{R})$

Jeffrey Adams and Annegret Paul

April 19, 2012

1 Notation for $Sp(4, \mathbb{R})$

Basic references are Vogan's notes on $Sp(4, \mathbb{R})$: [3] (branching) and [4] (Hermitian forms). In fact these notes are largely a rewriting of [4] in more explicit atlas terms.

$G = Sp(4, \mathbb{R})$ with all the usual notation, including `atlas` stuff.

Write (x, y) for the usual coordinates, in which $\rho = (2, 1)$. The *software* coordinates (assuming we define G as `sc`) are fundamental weight coordinates. Write $[a, b]$ for these coordinates. The changes of coordinates are

$$(1.1) \quad \begin{aligned} (x, y) &\rightarrow [x - y, y] \\ (a + b, b) &\leftarrow [a, b] \end{aligned}$$

For example $\rho = (2, 1) = [1, 1]$.

2 Standard Modules

We always write H for the once-and-for-all fixed Cartan of G , and $X^* = X^*(H)$, $X_* = X_*(H)$. We may also have use for H^\vee , $X^{\vee,*} = X^*(H^\vee)$ and $X_*^\vee = X_*(H^\vee)$. We also fixed once and for all a set of positive roots.

In `atlas` language a *parameter* is a triple (x, λ, ν) where:

- (1) x is a `kgb` element, set $\theta = \theta_x$;
- (2) $\lambda \in (X^* + \rho)/(1 - \theta)X^*$;
- (3) $\nu \in (X^* \otimes_{\mathbb{Z}} \mathbb{Q})^{-\theta} = \mathfrak{a}(\mathbb{Q})^*$.

This defines a standard module $I(x, \lambda, \nu)$. The infinitesimal character is

$$(2.1) \quad \gamma = \frac{1}{2}(1 + \theta)\lambda + \frac{1}{2}(1 - \theta)\nu \in X^* \otimes \mathbb{Q}.$$

For example $\lambda = \nu$ gives $\gamma = \lambda$ (note that γ is integral).

Recall θ defines positive imaginary and real roots, and ρ_r, ρ_i and ρ_{cx} .

Some conditions are:

- (a) The parameter is *standard* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all positive imaginary roots.
- (b) The parameter is *final* if for all positive real roots, $\langle \nu, \alpha^\vee \rangle = 0$ implies $\langle \lambda + \rho_r, \alpha^\vee \rangle$ is even. Equivalently: $\langle \lambda, \alpha^\vee \rangle$ is *odd* for all real-simple roots (simple roots in the subsystem of real roots).
- (c) We say $(x, \lambda, \nu) \equiv (x, w(\lambda + \rho_r) - \rho_r, w\nu)$ for $w \in W_r$. Using this we can assume $\langle \nu, \alpha \rangle \geq 0$ for all positive real roots.
- (d) If α is simple and θ_x -complex we say $(x, \lambda, \nu) \equiv (s_\alpha \times x, s_\alpha \lambda, s_\alpha \nu)$. This allows us to move x to any fiber on the same Cartan.
- (e) For a standard parameter, $I(x, \lambda, \nu)$ is zero if and only if $\langle \lambda, \alpha^\vee \rangle = 0$ for some imaginary-simple root which is compact.

A nonzero final standard module $I(x, \lambda, \nu)$ has a unique irreducible quotient $J(x, \lambda, \nu)$.

We generally write the subscript K to indicate restriction to K ; so here we have $I_K(x, \lambda, \nu)$ and $J_K(x, \lambda, \nu)$.

Recall each fiber has a distinguished basepoint, and each conjugacy class of fibers has a canonical fiber. In the output of `KGB`, the canonical fiber is labelled $\#$, and the basepoints have entry $(0, \dots, 0)$ preceding the $\#$.

Using (c) we can write every final standard parameter in the form (x, λ, ν) where x is in a canonical fiber.

2.1 Standard Modules for $Sp(4, \mathbb{R})$

There are 11 `kgb` elements numbered 0 – 10. I'll call them x_0, \dots, x_{10} . Here is the output of `KGB`.

0:	0	[n,n]	1	2	4	5	(0,0)#0	e
1:	0	[n,n]	0	3	4	6	(1,0)#0	e
2:	0	[c,n]	2	0	*	5	(0,1)#0	e
3:	0	[c,n]	3	1	*	6	(1,1)#0	e
4:	1	[r,C]	4	9	*	*	(0,0)	1 1
5:	1	[C,r]	7	5	*	*	(0,0)	2 2

6:	1	[C,r]	8	6	*	*	(1,0)	2	2
7:	2	[C,n]	5	8	*	10	(0,0)#2	1,2,1	
8:	2	[C,n]	6	7	*	10	(0,1)#2	1,2,1	
9:	2	[n,C]	9	4	10	*	(0,0)#1	2,1,2	
10:	3	[r,r]	10	10	*	*	(0,0)#3	2,1,2,1	

The basepoints in the canonical fibers are x_0, x_7, x_9, x_{10} .

Cartan 0 This is the compact Cartan subgroup, so $\nu = 0$, and $\lambda \in X^* \simeq \mathbb{Z}^2$. The standard modules are ($a, b \in \mathbb{Z}, a \geq b \geq 0$):

$$\begin{aligned}
I(x_0, (a, b)) & \quad (\text{large DS}) \\
I(x_1, (a, b)) & \quad (\text{large DS}) \\
I(x_2, (a, b)) & \quad (\text{holomorphic DS}) \\
I(x_3, (a, b)) & \quad (\text{antiholomorphic DS})
\end{aligned}$$

These are always final. The standard modules are always nonzero in the first two cases, and if and only if $a > b$ in the last two.

Cartan 1 This is the \mathbb{C}^* Cartan subgroup. The canonical fiber has $\theta = w = 2, 1, 2$, so $\theta(x, y) = (-y, -x)$. Then $(1 - \theta)X^* = \{(c, c) \mid c \in \mathbb{Z}\}$ so $\lambda = (a, b) \bmod (c, c)$ with $a, b, c \in \mathbb{Z}$. (This is isomorphic to \mathbb{Z} by the map $(a, b) \rightarrow a - b$). On the other hand $(X^* \otimes \mathbb{Q})^{-\theta} = \{(x, x) \mid x \in \mathbb{Q}\}$.

The root $(1, -1)$ is imaginary, and $(1, 1)$ is real. The standard limit modules are

$$(2.1.2)(a) \quad I(x_9, (a, b) \bmod (c, c), (x, x)) \quad (a \geq b, x \geq 0)$$

with infinitesimal character

$$\gamma = \left(x + \frac{a-b}{2}, x - \frac{a-b}{2}\right)$$

These standard modules are always nonzero.

In this case $M \simeq GL(2, \mathbb{R})$. The standard module given by a, b, x is the one associated to the representation on M with restriction to $SL(2, \mathbb{R})^\pm$ the discrete series with infinitesimal character $a - b = 0, 1, 2, \dots$, and such that $\text{diag}(e^t, e^t)$ acts by the scalar e^{2tx} . (These may be not quite the inducing data since we are working with the “ Λ -parameters”; see [2] or [1].)

The final condition is

$$x = 0 \Rightarrow a - b \in 2\mathbb{Z} + 1.$$

Suppose $r \geq s \geq 0$. Provided $r, s \in \mathbb{Z}$ there are two standard modules with infinitesimal character (r, s) :

$$(2.1.2)(b) \quad I(x_9, (r-s, 0), \frac{1}{2}(r+s, r+s)) \rightarrow \gamma = (r, s)$$

$$(2.1.2)(c) \quad I(x_9, (r+s, 0), \frac{1}{2}(r-s, r-s)) \rightarrow \gamma = (r, -s)$$

In these coordinates, the final condition is:

$$\text{If } x = 0 \text{ then } I(x_9, (c, 0), (0, 0)) \text{ is final} \Leftrightarrow c \in 2\mathbb{Z} + 1.$$

In (b) γ is dominant, whereas in (c) it is not if $s > 0$. Using the cross action of the complex root $(0, 2)$ (note that $\theta_{x_4}(x, y) = (y, x)$):

$$(2.1.3) \quad I(x_9, (r+s, 0), \frac{1}{2}(r-s, r-s)) = I(x_4, (r+s, 0), \frac{1}{2}(r-s, -s+r))$$

It is convenient to change variables, and renormalize by $\frac{1}{2}$ as follows. Define

$$(2.1.4)(a) \quad I(x_9, c, x) = I(x_9, (c, 0), \frac{1}{2}(x, x)) \quad (c \in \mathbb{Z}_{\geq 0}, x \in \mathbb{R}_{\geq 0}).$$

This has infinitesimal character

$$(2.1.4)(b) \quad \gamma = \frac{1}{2}(x+c, x-c).$$

Cartan 2 This is the $\mathbb{R}^* \times S^1$ Cartan subgroup. The canonical involution is $1, 2, 1$, i.e. $\theta(a, b) = (-a, b)$. Therefore $(1-\theta)X^* = (2\mathbb{Z}, 0)$, and $\lambda = (\bar{a}, b) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. On the other hand $\nu = (x, 0)$ with $x \in \mathbb{Q}$. The root $(2, 0)$ is real, and $(0, 2)$ is imaginary. So the standard modules are

$$I(x_7, (\bar{a}, b), (x, 0))$$

$$I(x_8, (\bar{a}, b), (x, 0))$$

with $b \in \mathbb{Z}_{\geq 0}, x \in \mathbb{Q}_{\geq 0}$ and $\bar{a} \in \mathbb{Z}/2\mathbb{Z}$. The infinitesimal character is

$$\gamma = (x, b)$$

The final condition is

$$x = 0 \Rightarrow \bar{a} = 1.$$

These standard modules are always nonzero.

The standard modules are those given by the holomorphic discrete series of $SL(2, \mathbb{R})$ with infinitesimal character $b = 0, 1, 2, \dots$, and the character of $\mathbb{R}^* t \rightarrow |t|^x \text{sgn}(x)^a$. The standard modules $I(x_8, (\bar{a}, b), (x, 0))$ are the same, with antiholomorphic in place of holomorphic.

If $x \geq b$ then γ is dominant. If $x < b$ it makes sense to apply (d) of Section 2. Using the fact that for x_5, x_6 , $\theta(x, y) = (x, -y)$, with the obvious notation we have

$$\begin{aligned} I(x_7, (\bar{a}, b), (x, 0)) &= I(x_5, (b, \bar{a}), (0, x)) \\ I(x_8, (\bar{a}, b), (x, 0)) &= I(x_6, (b, \bar{a}), (0, x)) \end{aligned}$$

Cartan 3 This is the split Cartan subgroup, with $x = x_{10}$, $\theta = -1$, $(1 - \theta)X^* = 2\mathbb{Z} \times 2\mathbb{Z}$, $\lambda = (\bar{a}, \bar{b}) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\nu = (x, y) \in \mathbb{Q}^2$. The standard modules are

$$I(x_{10}, (\bar{a}, \bar{b}), (x, y)) \quad (x \geq y \geq 0)$$

with infinitesimal character (x, y) . The *final* condition is

$$\begin{aligned} y = 0 &\Rightarrow \bar{b} = 1 \\ x - y = 0 &\Rightarrow \bar{a} - \bar{b} = 1 \end{aligned}$$

These standard modules are always nonzero.

3 nblock

The `nblock` command is essential to these computations. The translation between `nblock` coordinates and human ones is tricky. The main point of this section is to explain the `nblock` command, and give a translation for $Sp(4, \mathbb{R})$ between `nblock` and human coordinates.

Here is an overview of the `nblock` command, which (among other things) allows the user to input a general standard module. Recall a standard module is determined by a parameter (x, λ, ν) . See Section 2.

The user first inputs a Cartan, from the `atlas` list `0..n`. The software then gives a list of `kgb` elements x in the distinguished fiber for this Cartan. In particular this specifies a specific Cartan involution θ of H .

The user chooses one of the elements x .

Next the user defines λ . The user inputs $(\lambda - \rho)_{in} \in X^*$ in the “software” coordinates. Suppose G is semisimple. If it is entered as `sc` these are fundamental weight coordinates, while `ad` gives simple root coordinates. The

software computes

$$\lambda = (\lambda - \rho)_{in} + \rho.$$

Only the image of λ in $\rho + X^*/(1 - \theta)X^*$ matters.

Next the user defines $\nu \in (X^* \otimes_{\mathbb{Z}} \mathbb{Q})^{-\theta} = \mathfrak{a}(\mathbb{Q})^*$. The user inputs an arbitrary element $\nu_{in} \in X^* \otimes_{\mathbb{Z}} \mathbb{Q}$ (in software coordinates), and the software defines

$$\nu = \frac{1}{2}(1 - \theta)\nu_{in} \in \mathfrak{a}(\mathbb{Q})^*$$

Let

$$\lambda_0 = \frac{1}{2}(1 + \theta)\lambda.$$

Then (cf. (2.1)) the infinitesimal character is computed as

$$\gamma = \lambda_0 + \nu = \frac{1}{2}(1 + \theta)\lambda + \frac{1}{2}(1 - \theta)\nu_{in}.$$

NB: Only $\nu = (1 - \theta)\nu_{in}$ matters, and for the infinitesimal character only $\lambda_0 = (1 + \theta)\lambda$ matters. However, λ has extra (torsion) information not contained in λ_0 . For example, if the Cartan is split $\lambda_0 = 0$ but λ is an element of $\rho + X^*/2X^*$.

If γ is dominant the triple (x, λ, ν) defines a standard module. If the infinitesimal character is singular it may be reducible or 0. If γ is not dominant on the integral roots, the software uses complex cross actions to replace this triple with (x', λ', ν') such that $\gamma' = \lambda'_0 + \nu'$ is dominant, and conjugate to γ .

4 Standard Modules for $Sp(4, \mathbb{R})$ using `nblock`

We show how to construct the standard modules of Section 2.1 using `nblock`. This is primarily just a change of coordinates.

In each case the user will enter a Cartan, `kgb` element, $(\lambda - \rho)_{in} = [\tilde{a}, \tilde{b}]$, and $\nu_{in} = [\tilde{x}, \tilde{y}]$.

(a) Cartan #0

This case is clear. If $0 \leq k \leq 3$:

$$I(x_k, (a, b)) = I(x_k, [\tilde{a}, \tilde{b}], [\tilde{x}, \tilde{y}])$$

provided $a = \tilde{a} + \tilde{b} + 2, b = \tilde{b} + 1$ (\tilde{x}, \tilde{y} are irrelevant).

(b) Cartan #1

This is the \mathbb{C}^* Cartan, with $M \simeq GL(2, \mathbb{R})$.

As in Section 2.1, there is only one kgb element \mathbf{x}_9 in the distinguished fiber, $\theta = 2, 1, 2$, $\theta(x, y) = (-y, -x)$. In fundamental weight coordinates $\theta[x, y] = [x, -x - y]$.

The coroot $[1, 0] = (1, -1)$ is imaginary, so the software requires $\tilde{a} \geq -1$. Since $(1 - \theta)X^* = [0, \mathbb{Z}]$, \tilde{b} has no effect (the Cartan is connected).

If $\nu = [\tilde{x}, \tilde{y}]$ then $\frac{1}{2}(1 - \theta)\nu = [0, \tilde{y} + \frac{\tilde{x}}{2}]$. It is convenient to take $\tilde{x} = 0$, so $\frac{1}{2}(1 - \theta)\nu = \nu$.

So the module

$$I(x_9, (a, b) \bmod(c, c), (x, x)) \quad (a - b \geq 0, x \geq 0)$$

of Section 2.1 is given in these coordinates by

$$I(x_9, [\tilde{a}, 0] \bmod(0, \mathbb{Z}), [0, \tilde{x}]) \quad (\tilde{a} \geq -1, \tilde{x} \geq 0)$$

with $\tilde{a} = a - b - 1 \geq -1$ and $\tilde{x} = x$. In particular the infinitesimal characters match up:

$$(x + \frac{a - b}{2}, x - \frac{a - b}{2}) = (\tilde{x} + \frac{\tilde{a} + 1}{2}, \tilde{x} - \frac{\tilde{a} + 1}{2}).$$

In terms of the `nblock` interaction, here is what happens (dropping the tildes). Choose x_9 ,

$$\begin{aligned} (\lambda - \rho)_{in} &= [a, b] = (a + b, b) \\ \nu &= \nu_{in} = [0, x] = (x, x) \end{aligned}$$

If $x \geq \frac{a+1}{2}$ then the software computes:

$$\begin{aligned} \lambda &= [a + 1, b + 1] = (a + b + 2, b + 1) \\ \lambda_0 &= [a + 1, -\frac{a + 1}{2}] = (\frac{a + 1}{2}, -\frac{a + 1}{2}) \\ \gamma &= [a + 1, x - \frac{a + 1}{2}] = (x + \frac{a + 1}{2}, x - \frac{a + 1}{2}) \end{aligned}$$

and γ is dominant.

If $x < \frac{a+1}{2}$, γ is not dominant, so (provided that $x - \frac{a+1}{2}$ is an integer) the software conjugates λ, ν by s_2 . The new Cartan involution is $\theta(x, y) = (y, x)$,

and we get:

$$\begin{aligned}\lambda &= [a + 2b + 3, -b - 1] = (a + b + 2, -b - 1) \\ \lambda_0 &= [0, \frac{a+1}{2}] = (\frac{a+1}{2}, \frac{a+1}{2}) \\ \nu &= [2x, -x] = (x, -x) \\ \gamma &= [2x, -x + \frac{a+1}{2}] = (x + \frac{a+1}{2}, -x + \frac{a+1}{2})\end{aligned}$$

(c) Cartan #2

This is the Cartan $S^1 \times \mathbb{R}^*$, with $M = SL(2, \mathbb{R}) \times \mathbb{R}^*$. As in Section 2.1 the distinguished fiber has $\theta = 1, 2, 1$, so $\theta(x, y) = (-x, y)$. In fundamental weight coordinates $\theta[a, b] = [-a - 2b, b]$. The coroot $[0, 1] = (0, 1)$ is imaginary, so we have the condition $\tilde{b} \geq -1$. Since $(1 - \theta)X^* = [2\mathbb{Z}, 0]$, only the image of $a \in Z/2\mathbb{Z}$ matters, so we could take $\tilde{a} = 0, 1$.

Since $\nu_{in} = [\tilde{x}, \tilde{y}]$ gives $\nu = \frac{1}{2}(1 - \theta)\nu_{in} = [\tilde{x} + \tilde{y}, 0]$, we may as well take

$$\nu_{in} = [\tilde{x}, 0] = (\tilde{x}, 0)$$

in which case $\nu = \nu_{in}$.

The translation is, with $k = 7, 8$,

$$I(x_k, (\bar{a}, b), (x, 0)) \quad (b, x \geq 0)$$

is given in the new coordinates by

$$I(x_k, [\tilde{a}, \tilde{b}], [\tilde{x}, 0]) \quad (\tilde{b} \geq -1, \tilde{x} \geq 0)$$

provided $\tilde{a} + \tilde{b} \equiv \bar{a} \pmod{2}$, $\tilde{b} + 1 = b$ and $\tilde{x} = x$.

In terms of the software interaction (dropping the tildes), if $x \geq b + 1$:

$$\begin{aligned}\nu &= [x, 0] = (x, 0) \\ \lambda &= [a + 1, b + 1] = (a + b + 2, b + 1) \\ \lambda_0 &= [-b - 1, b + 1] = (0, b + 1) \\ \gamma &= [x - b - 1, b + 1] = (x, b + 1)\end{aligned}$$

and γ is dominant.

If $x < b + 1$ γ is not dominant, so we conjugate everything by s_1 . The new Cartan involution is $s_2 : (x, y) = (x, -y)$.

$$\begin{aligned}
\lambda &= [-a - 1, a + b + 2] = (b + 1, a + b + 2) \\
\lambda_0 &= [b + 1, 0] = (b + 1, 0) \\
\nu &= [-x, x] = (0, x) \\
\gamma &= [b + 1 - x, x] = (b + 1, x)
\end{aligned}$$

(d) **Split Cartan**

$$I(x_{10}, (\bar{a}, \bar{b}), (x, y))$$

is the same as

$$I(x_{10}, [\tilde{a}, \tilde{b}], [\tilde{x}, \tilde{y}])$$

provided $\bar{a} = \tilde{a} + \tilde{b} \pmod{2}$, $\bar{b} = \tilde{b} + 1 \pmod{2}$, and $\tilde{x} + \tilde{y} = x, \tilde{y} = y$.

5 Standard K-Representations

Suppose $I(x, \lambda, 0)$ is a nonzero, final standard limit representation, with $\nu = 0$. As in Section 2 its restriction to K is $I_K(x, \lambda, 0)$. This is an important object so we give it a name:

Definition 5.1 $I_K(x, \lambda)$ is the restriction of the nonzero, final standard limit module $I(x, \lambda, 0)$ to K .

These are a basis of the Grothendieck group of K . Here is a list of these modules for $Sp(4, \mathbb{R})$. The last column gives the highest weight of the lowest K -types.

$$\begin{aligned}
(5.2) \quad & I_K(x_0, (a, b)) \quad a \geq b \geq 0 \quad (a + 1, -b) \\
& I_K(x_1, (a, b)) \quad a \geq b \geq 0 \quad (b, -a - 1) \\
& I_K(x_2, (a, b)) \quad a > b \geq 0 \quad (a + 1, b + 2) \\
& I_K(x_3, (a, b)) \quad a > b \geq 0 \quad (-b - 2, -a - 1) \\
& I_K(x_9, (c, 0)) \quad c > 0 \text{ odd} \quad \left(\frac{c+1}{2}, -\frac{c+1}{2}\right) \\
& I_K(x_7, (\bar{1}, b)) \quad b \geq 0 \quad (b + 1, 1) \\
& I_K(x_8, (\bar{1}, b)) \quad b \geq 0 \quad (-1, -b - 1) \\
& I_K(x_{10}, (\bar{0}, \bar{1})) \quad (0, 0)
\end{aligned}$$

We will need some Hecht-Schmid identities for some non-final parameters. For example $I(x_9, (c, 0), (0, 0))$ is not final if c is even. To see this using

the software, use the `Ktypeform` command. In this example $\lambda = (2, 0)$, $\lambda - \rho = (0, -1)$ which equals $[1, -1]$ in fundamental weight coordinates.

```
real: Ktypeform
Choose KGB element: 9
2rho = [ 2, 2 ]
Give lambda-rho: 1 -1
Representation [ 4, 4 ]@(0,0)#1 is not final, as witnessed by coroot [1,2].
```

This is because

$$(5.3)(a) \quad I(x_9, c, 0) = J(x_0, \frac{1}{2}(c, c)) + J(x_1, \frac{1}{2}(c, c))$$

and so

$$(5.3)(b) \quad I_K(x_9, (c, 0)) = I_K(x_0, \frac{1}{2}(c, c)) + I_K(x_1, \frac{1}{2}(c, c))$$

with lowest K -types

$$(5.3)(c) \quad (\frac{c}{2} + 1, -\frac{c}{2}), (\frac{c}{2}, -\frac{c}{2} - 1)$$

Similarly if $b \geq 0$:

$$(5.4)(a) \quad \begin{aligned} I_K(x_7, \bar{0}, b) &= I_K(x_0, (b, 0)) + I_K(x_2, (b, 0)) \\ I_K(x_8, \bar{0}, b) &= I_K(x_1, (b, 0)) + I_K(x_3, (b, 0)) \end{aligned}$$

with lowest K -types

$$(5.4)(b) \quad \begin{aligned} I_K(x_7, (\bar{0}, b)) & \quad (b + 1, 0), (b + 1, 2) \\ I_K(x_8, (\bar{0}, b)) & \quad (0, -b - 1), (-2, -b - 1) \end{aligned}$$

without the second term if $b = 0$.

Question: What is the best way to see this using the software?

6 Reducibility: Real Roots (parity condition)

Suppose $I(x, \lambda, \nu)$ is a standard module. A real root α satisfies the *parity condition* if

$$(6.1)(a) \quad \lambda(m_\alpha) = -(-1)^{\langle \nu + \rho_r, \alpha^\vee \rangle}$$

or equivalently

$$(6.1)(b) \quad \langle \lambda + \nu + \rho_r, \alpha^\vee \rangle \in 2\mathbb{Z} + 1.$$

This holds if and only if $I(x, \lambda, \nu)$ has some reducibility accounted for by $SL(2, \mathbb{R})_\alpha$.

Note Here is a mnemonic for remembering this condition. The trivial representation of $SL(2, \mathbb{R})$ is given by the holomorphic character ρ : $\rho(z) = z$ ($z \in \mathbb{C}^*$). The character ρ is given by $(\lambda, \nu) = (\rho, \rho)$, and note that $\rho = \rho_r$. So (b) says $\langle 3\rho, \alpha^\vee \rangle \in 2\mathbb{Z} + 1$ which is true; this standard module is reducible.

Note that *no real root satisfies the parity condition* if and only if

$$(6.1)(c) \quad \langle \lambda + \nu, \alpha^\vee \rangle \in 2\mathbb{Z} + 1 \text{ for all real-simple roots } \alpha.$$

In particular

$$(6.1)(d) \quad I(x, \lambda, \nu) \text{ irreducible} \Rightarrow (6.1)(c) \text{ holds.}$$

An important special case is $\nu = 0$:

$$(6.1)(e) \quad I(x, \lambda, 0) \text{ irreducible} \Rightarrow \langle \lambda, \alpha^\vee \rangle \in 2\mathbb{Z} + 1 \text{ for all real-simple roots } \alpha.$$

This is the *final* condition of Section 2. Also see the help file for the `Ktypeform` command.

Cartan 1 There is a single real root $\alpha = (1, 1)$, and the parity condition for $I(x_9, (a, b) \pmod{(c, c)}, (x, x))$ is:

$$a + b + 2x + 1 \in 2\mathbb{Z} + 1$$

Cartan 2 The real root is $(2, 0)$, and the parity condition for $I(x_k, (\bar{a}, b), (x, 0))$, $k = 7, 8$, is

$$\bar{a} + x + 1 \in 2\mathbb{Z} + 1$$

Cartan 3 All roots are real. The parity conditions on $I(x_{10}, (\bar{a}, \bar{b}), (x, y))$ are

$$(6.2) \quad \begin{aligned} \langle \lambda + \nu + \rho_r, \alpha^\vee \rangle &\in 2\mathbb{Z} + 1. \\ \alpha_1 = (1, -1) &: \bar{a} - \bar{b} + x - y + 1 \in 2\mathbb{Z} + 1 \\ \alpha_2 = (0, 2) &: \bar{b} + y + 1 \in 2\mathbb{Z} + 1 \\ (1, 1) &: \bar{a} + \bar{b} + x + y + 3 \in 2\mathbb{Z} + 1 \\ (2, 0) &: \bar{a} + x + 2 \in 2\mathbb{Z} + 1 \end{aligned}$$

For example for $I(x_{10}, (2, 1), (2, 1))$, α_1, α_2 satisfy the parity condition. On the other hand for $I(x_{10}, (0, 0), (2, 1))$, no root satisfies the parity condition.

We collect these conditions in a table, using the alternate parametrization of (2.1.2)(b) for Cartan 1.

Parity for $Sp(4, \mathbb{R})$

Cartan	Module	root	parity condition
0	no real roots		
1	$I(x_9, (r \pm s, 0), \frac{1}{2}(r \mp s, r \mp s))$	(1, 1)	$2r + 1 \in 2\mathbb{Z} + 1$
2	$I(x_k, (\bar{a}, b), (x, 0))$	(2, 0)	$\bar{a} + x + 1 \in 2\mathbb{Z} + 1$
3	$I(x_{10}, (\bar{a}, \bar{b}), (x, y))$	(1, -1)	$\bar{a} - \bar{b} + x - y + 1 \in 2\mathbb{Z} + 1$
3		(0, 2)	$\bar{b} + y + 1 \in 2\mathbb{Z} + 1$
3		(2, 0)	$\bar{a} + x + 2 \in 2\mathbb{Z} + 1$
3		(1, 1)	$\bar{a} + \bar{b} + x + y + 3 \in 2\mathbb{Z} + 1$

On Cartan 1 recall α is integral if $2r \in \mathbb{Z}$.

It might be better to write the conditions on the split Cartan (the last four entries) as follows:

$$\begin{aligned} (1, -1) & \quad (\bar{a} + x) - (\bar{b} + y) \in 2\mathbb{Z} \\ (0, 2) & \quad \bar{b} + y \in 2\mathbb{Z} \\ (2, 0) & \quad (\bar{a} + x) \in 2\mathbb{Z} + 1 \\ (1, 1) & \quad (\bar{a} + x) + (\bar{b} + y) \in 2\mathbb{Z} \end{aligned}$$

7 Reducibility: Complex Roots

Suppose $I(x, \lambda, \nu)$ is standard, with regular infinitesimal character. An integral complex root α contributes to reducibility of $I(x, \lambda, \nu)$ if and only if $\theta_x(\alpha) < 0$; equivalently $\alpha \in \tau(I)$, or α is of type \mathbf{C}^- .

For $Sp(4, \mathbb{R})$ only Cartans 1, 2 have complex roots.

Cartan 1 Consider $I(x_9, (r \pm s, 0), \frac{1}{2}(r \mp s, r \mp s))$ with $r, s \geq 0, r \pm s \in \mathbb{Z}$. Assume $\gamma = (r, \mp s)$ is regular, i.e. $r > s > 0$.

Consider the positive complex root $\alpha = (2, 0)$. Assume this is integral, i.e. $r \in \mathbb{Z}$. Then $\theta(\alpha) = (0, -2)$, and this is positive or negative depending on the sign. Thus:

$$\begin{aligned} I(x_9, (r - s, 0), \frac{1}{2}(r + s, r + s)) &\rightarrow \alpha \text{ is type C-} \\ I(x_9, (r + s, 0), \frac{1}{2}(r - s, r - s)) &\rightarrow \alpha \text{ is type C+} \end{aligned}$$

Cartan 2 Consider $I(x_k, (\bar{a}, b), (x, 0))$ with $k = 7, 8$, and $x, b \geq 0$. so $\gamma = (x, b)$.

Assume γ is regular, i.e. $x \neq b$, and $x, b \neq 0$, and $\alpha = (1, 1)$ is integral, i.e. $x + b \in \mathbb{Z}$. Recall $\theta(a, b) = (-a, b)$. Thus $\alpha > 0$ is complex, and $\theta(\alpha) = (-1, 1) > 0$ if $x < b$, and < 0 otherwise. Therefore

$$\alpha \text{ is type C-} \Leftrightarrow x > b.$$

8 Integral blocks of $Sp(4, \mathbb{R})$

Here is the big block:

0(0,6):	0	[i1,i1]	1	2	(4, *)	(5, *)	0	e
1(1,6):	0	[i1,i1]	0	3	(4, *)	(6, *)	0	e
2(2,6):	0	[ic,i1]	2	0	(*, *)	(5, *)	0	e
3(3,6):	0	[ic,i1]	3	1	(*, *)	(6, *)	0	e
4(4,5):	1	[r1,C+]	4	9	(0, 1)	(*, *)	1	1
5(5,4):	1	[C+,r1]	7	5	(*, *)	(0, 2)	2	2
6(6,4):	1	[C+,r1]	8	6	(*, *)	(1, 3)	2	2
7(7,3):	2	[C-,i1]	5	8	(*, *)	(10, *)	2	1,2,1
8(8,3):	2	[C-,i1]	6	7	(*, *)	(10, *)	2	1,2,1
9(9,2):	2	[i2,C-]	9	4	(10,11)	(*, *)	1	2,1,2
10(10,0):	3	[r2,r1]	11	10	(9, *)	(7, 8)	3	2,1,2,1
11(10,1):	3	[r2,rn]	10	11	(9, *)	(*, *)	3	2,1,2,1

Fix an integral regular infinitesimal character (a, b) with $a > b > 0$. Here are the parameters realized using only one fiber on each Cartan, using the cross action to replace x_4 with x_9 , and $x_{4,5}$ with $x_{7,8}$:

number	Cartan	length	x	λ	ν
0	0	0	0	(a, b)	*
1	0	0	1	(a, b)	*
2	0	0	2	(a, b)	*
3	0	0	3	(a, b)	*
4	1	1	9	$(a + b, 0)$	$(\frac{1}{2}(a - b), \frac{1}{2}(a - b))$
5	2	1	7	(\bar{b}, a)	$(b, 0)$
6	2	1	8	(\bar{b}, a)	$(b, 0)$
7	2	2	7	(\bar{a}, b)	$(a, 0)$
8	2	2	8	(\bar{a}, b)	$(a, 0)$
9	1	2	9	$(a - b, 0)$	$(\frac{1}{2}(a + b), \frac{1}{2}(a + b))$
10	3	3	10	(\bar{a}, \bar{b})	(a, b)
11	3	3	10	$(\bar{a} + 1, \bar{b} + 1)$	(a, b)

Note that 10 is the trivial representation, given by $\lambda = \nu = \rho$.

Alternatively, using all x_i , so the x -values match up with the output of `block`:

number	Cartan	length	x	λ	ν
0	0	0	0	(a, b)	*
1	0	0	1	(a, b)	*
2	0	0	2	(a, b)	*
3	0	0	3	(a, b)	*
4	1	1	4	$(a + b, 0)$	$(\frac{1}{2}(a - b), -\frac{1}{2}(a - b))$
5	2	1	5	(a, \bar{b})	$(0, b)$
6	2	1	6	(a, \bar{b})	$(0, b)$
7	2	2	7	(\bar{a}, b)	$(a, 0)$
8	2	2	8	(\bar{a}, b)	$(a, 0)$
9	1	2	9	$(a - b, 0)$	$(\frac{1}{2}(a + b), \frac{1}{2}(a + b))$
10	3	3	10	(\bar{a}, \bar{b})	(a, b)
11	3	3	10	$(\bar{a} + 1, \bar{b} + 1)$	(a, b)

Here is the block dual to $SO(4, 1)$:

0(5,2):	1	[C+,rn]	2	0	(*,*)	(*,*)	2	2
1(6,2):	1	[C+,rn]	3	1	(*,*)	(*,*)	2	2
2(7,1):	2	[C-,i1]	0	3	(*,*)	(4,*)	2	1,2,1
3(8,1):	2	[C-,i1]	1	2	(*,*)	(4,*)	2	1,2,1
4(10,0):	3	[rn,r1]	4	4	(*,*)	(2,3)	3	2,1,2,1

number	Cartan	length	x	λ	ν
0	2	1	7	$(\bar{b} + 1, a)$	$(b, 0)$
1	2	1	8	$(\bar{b} + 1, a)$	$(b, 0)$
2	2	2	7	$(\bar{a} + 1, b)$	$(a, 0)$
3	2	2	8	$(\bar{a} + 1, b)$	$(a, 0)$
4	3	3	10	$(\bar{a} + 1, \bar{b})$	(a, b)

The principal series $I(x_{10}, (\bar{a}, \bar{b} + 1), (a, b))$ is irreducible, dual to $SO(5, 0)$.

9 Orientation Numbers for $Sp(4, \mathbb{R})$

See [4, Section 5].

Definition 9.1 Suppose $\gamma = (x, \lambda, \nu)$ is a parameter and α is a nonintegral real root. Define

$$(9.2) \quad t_\alpha(\gamma) = \langle \nu - (\lambda - \rho_r), \alpha^\vee \rangle \pmod{2\mathbb{Z}}$$

Since α is not integral $t_\alpha \notin \mathbb{Z}$; think of it as an element of $(0, 1) \cup (1, 2)$.

We say a real nonintegral root is oriented (with respect to γ) if $0 < t_\alpha(\gamma) < 1$.

Definition 9.3 Suppose $\gamma = (x, \lambda, \nu)$ is a parameter. The orientation number $\ell_0(\gamma)$ is the sum of

the number of pairs of complex nonintegral positive roots $\{\alpha, -\theta\alpha\}$

and

the number of real nonintegral oriented roots that are positive on γ .

Here are the orientation numbers for $Sp(4, \mathbb{R})$.

On Cartan 0 all roots are imaginary and $\ell_0 = 0$.

9.1 Cartan 1

Consider $(x, \lambda, \nu) = (x_9, c, x) = (x_9, (c, 0), \frac{1}{2}(x, x))$, with infinitesimal character $\gamma = \frac{1}{2}(x + c, x - c)$. Here $c \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}_{\geq 0}$. Recall $(1, 1)$ is the unique positive real root, and $(2, 0), (0, 2)$ are complex.

The values of the roots are $\frac{1}{2}(x \pm c), x, c$. Assume γ is not integral, i.e. c, x are not integers of the same parity.

Case 1

Assume $x \in \mathbb{Z}$, but not of the same parity as c . The integral root system is of type $D_2 = \{(\pm 1, \pm 1)\}$. and the nonintegral roots are $(2, 0), (0, 2)$ (complex),

Table 9.1.1		
c, x	complex pair $(2, 0), (0, 2)$	ℓ_0
$c > x$		0
$c < x$	Y	1

Case 2

Assume $x \notin \mathbb{Z}$. The only positive integral root is $(1, -1)$, the integral root system is type A_1 , and the nonintegral roots are $(2, 0), (0, 2)$ (complex) and $(1, 1)$ (real).

Compute

$$\nu - (\lambda - \rho_r) = \frac{1}{2}(x, x) - (c, 0) + \left(\frac{1}{2}, \frac{1}{2}\right) = \left(-c + \frac{x-1}{2}, \frac{x-1}{2}\right)$$

and

$$t_\alpha = -c + x + 1$$

Therefore α is oriented if and only if $1 < x - c < 2 \pmod{2\mathbb{Z}}$, or equivalently

$$0 < c - x < 1 \pmod{2\mathbb{Z}}$$

So:

Table 9.1.2				
c, x	range $\pmod{2\mathbb{Z}}$	complex pair $(2, 0), (0, 2)$	$(1, 1)$	ℓ_0
$c > x$	$0 < c - x < 1 \pmod{2\mathbb{Z}}$		Y	1
	$1 < c - x < 2 \pmod{2\mathbb{Z}}$			0
$c < x$	$0 < c - x < 1 \pmod{2\mathbb{Z}}$	Y	Y	2
	$1 < c - x < 2 \pmod{2\mathbb{Z}}$	Y		1

Note that if we write the infinitesimal character as (γ_1, γ_2) then the table becomes

c, x	range (mod $2\mathbb{Z}$)	complex pair $(2, 0), (0, 2)$	$(1, 1)$	ℓ_0
$\gamma_2 < 0$	$0 < \gamma_2 < \frac{1}{2} \pmod{\mathbb{Z}}$			0
	$\frac{1}{2} < \gamma_2 < 1 \pmod{\mathbb{Z}}$		Y	1
$\gamma_2 > 0$	$0 < \gamma_2 < \frac{1}{2} \pmod{\mathbb{Z}}$	Y		1
	$\frac{1}{2} < \gamma_2 < 1 \pmod{\mathbb{Z}}$	Y	Y	2

which agrees with [4, ?]

9.2 Cartan 2

Consider $(x, \lambda, \nu) = (x_k, (\bar{a}, b), (x, 0))$ with infinitesimal character (x, b) . If $x \in \mathbb{Z}$ this is integral and $\ell_0 = 0$, so assume $x \notin \mathbb{Z}$.

Compute

$$\nu - (\lambda - \rho_r) = (x, 0) - (\bar{a}, b) + (1, 0) = (x - \bar{a} + 1, -b)$$

and for α the real root $(2, 0)$, $t_\alpha = x - \bar{a} + 1$. So α is oriented if and only if

$$1 < x - \bar{a} < 2 \pmod{2\mathbb{Z}}$$

x, b	\bar{a}	range (mod $2\mathbb{Z}$)	complex pair $(1, \pm 1)$	$(2, 0)$	ℓ_0
$x < b$	0	$0 < x < 1$			0
$x < b$	0	$1 < x < 2$		Y	1
$x < b$	1	$0 < x < 1$		Y	1
$x < b$	1	$1 < x < 2$			0
$x > b$	0	$0 < x < 1$	Y		1
$x > b$	0	$1 < x < 2$	Y	Y	2
$x > b$	1	$0 < x < 1$	Y	Y	2
$x > b$	1	$1 < x < 2$	Y		1

9.3 Cartan 3

Consider $I(x_{10}, (\bar{a}, \bar{b}), (x, y))$ with $x > y > 0$. The nonintegral cases are as follows (not including the case with no integral roots). The third column gives the type of the integral root system, and the last column gives a consequence of the conditions in the first column.

Case	x, y	type	integral roots	
1	$x \in \mathbb{Z}, y \notin \mathbb{Z}$	A_1	$(2, 0)$	
2	$x \notin \mathbb{Z}, y \in \mathbb{Z}$	A_1	$(0, 2)$	
3	$x + y \in \mathbb{Z}; x - y \notin \mathbb{Z}$	A_1	$(1, 1)$	$x, y \notin \mathbb{Z} + \frac{1}{2}$
4	$x - y \in \mathbb{Z}; x + y \notin \mathbb{Z}$	A_1	$(1, -1)$	$x, y \notin \mathbb{Z} + \frac{1}{2}$
5	$x \pm y \in \mathbb{Z}; x, y \notin \mathbb{Z}$	D_2	$(1, \pm 1)$	$x, y \in \mathbb{Z} + \frac{1}{2}$

We have $\rho = \rho_r = (2, 1)$ and

$$(9.3.4) \quad \nu - (\lambda - \rho_r) = (x - a + 2, y - b + 1) \equiv (x - a, y - b + 1).$$

Cartan 3, Case 1: $x \in \mathbb{Z}, y \notin \mathbb{Z}$

The nonintegral roots are $(1, \pm 1)$ and $(0, 2)$. They are oriented if:

$$(9.3.5) \quad \begin{aligned} (1, 1) &: 1 < x + y - (\bar{a} + \bar{b}) < 2 \pmod{2\mathbb{Z}} \\ (1, -1) &: 1 < x - y - (\bar{a} + \bar{b}) < 2 \pmod{2\mathbb{Z}} \\ (0, 2) &: 1 < y - \bar{b} < 2 \pmod{2\mathbb{Z}} \end{aligned}$$

It is easy to see that exactly one of $(1, \pm 1)$ is an oriented root.

\bar{b}	range (mod $2\mathbb{Z}$)	$(1, 1)$ xor $(1, -1)$	$(0, 2)$	ℓ_0
0	$0 < y < 1$	Y		1
	$1 < y < 2$	Y	Y	2
1	$0 < y < 1$	Y	Y	2
	$1 < y < 2$	Y		1

Cartan 3, Case 2: $x \notin \mathbb{Z}, y \in \mathbb{Z}$

The nonintegral roots are $(1, \pm 1)$ and $(2, 0)$. Now both $(1, \pm 1)$ are oriented, or neither are. The condition on $(2, 0)$ is

$$(9.3.6) \quad (2, 0) : 0 < x - \bar{a} < 1 \pmod{2\mathbb{Z}}$$

\bar{a}	$\bar{a} + \bar{b}$	$x \pmod{2\mathbb{Z}}$	$x + y \pmod{2\mathbb{Z}}$	$(1, 1)$ and $(1, -1)$	$(2, 0)$	ℓ_0	
0	0	$0 < x < 1$	$0 < x + y < 1$		Y	1	
			$1 < x + y < 2$	Y	Y	3	
		$1 < x < 2$	$0 < x + y < 1$				0
			$1 < x + y < 2$	Y	Y	2	
1	0	$0 < x < 1$	$0 < x + y < 1$			0	
			$1 < x + y < 2$	Y		2	
		$1 < x < 2$	$0 < x + y < 1$		Y	1	
			$1 < x + y < 2$	Y	Y	3	
0	1	$0 < x < 1$	$0 < x + y < 1$	Y	Y	3	
			$1 < x + y < 2$		Y	1	
		$1 < x < 2$	$0 < x + y < 1$	Y		2	
			$1 < x + y < 2$			0	
1	1	$0 < x < 1$	$0 < x + y < 1$	Y		2	
			$1 < x + y < 2$			0	
		$1 < x < 2$	$0 < x + y < 1$	Y	Y	3	
			$1 < x + y < 2$		Y	1	

Cartan 3, Case 3: $x + y \in \mathbb{Z}, x - y \notin \mathbb{Z}$

An example is $(x, y) = (\frac{3}{4}, \frac{1}{4})$. The nonintegral roots are $(2, 0)$, $(0, 2)$ and $(1, -1)$. The oriented conditions are:

$$\begin{aligned}
 (2, 0) &: 0 < x - \bar{a} < 1 \pmod{2\mathbb{Z}} \\
 (9.3.7)(a) \quad (0, 2) &: 1 < y - \bar{b} < 2 \pmod{2\mathbb{Z}} \\
 (1, -1) &: 1 < (x - \bar{a}) - (y - \bar{b}) < 2 \pmod{2\mathbb{Z}}
 \end{aligned}$$

Recall $x + y \in \mathbb{Z}$. For example, suppose $x + y$ is even, so $x = -y \pmod{2\mathbb{Z}}$, so replace x with $-y$ everywhere:

$$\begin{aligned}
 (2, 0) &: 0 < -y - \bar{a} < 1 \pmod{2\mathbb{Z}} \\
 (9.3.7)(b) \quad (0, 2) &: 1 < y - \bar{b} < 2 \pmod{2\mathbb{Z}} \\
 (1, -1) &: 1 < -2y - \bar{a} + \bar{b} < 2 \pmod{2\mathbb{Z}}.
 \end{aligned}$$

After a few manipulations, including dividing the last line by 2, this is equiv-

alent to

$$(9.3.7)(c) \quad \begin{aligned} (2, 0) &: 1 < y + \bar{a} < 2 \pmod{2\mathbb{Z}} \\ (0, 2) &: 1 < y - \bar{b} < 2 \pmod{2\mathbb{Z}} \\ (1, -1) &: 0 < y + \frac{1}{2}(\bar{a} - \bar{b}) < \frac{1}{2} \pmod{\mathbb{Z}}. \end{aligned}$$

If $x + y$ is odd replace x with $-y + 1 \pmod{2\mathbb{Z}}$, and something similar happens. Here is the answer.

If $x + y \in 2\mathbb{Z}, x - y \notin \mathbb{Z}$:

Table 9.3.3					
(\bar{a}, \bar{b})	range $\pmod{2\mathbb{Z}}$	$(2, 0)$	$(0, 2)$	$(1, -1)$	ℓ_0
$(0, 0)$	$0 < y < \frac{1}{2}$			Y	1
	$\frac{1}{2} < y < 1$				0
	$1 < y < \frac{3}{2}$	Y	Y	Y	3
	$\frac{3}{2} < y < 2$	Y	Y		2
$(1, 0)$	$0 < y < \frac{1}{2}$	Y			1
	$\frac{1}{2} < y < 1$	Y		Y	2
	$1 < y < \frac{3}{2}$		Y		1
	$\frac{3}{2} < y < 2$		Y	Y	2
$(0, 1)$	$0 < y < \frac{1}{2}$		Y		1
	$\frac{1}{2} < y < 1$		Y	Y	2
	$1 < y < \frac{3}{2}$	Y			1
	$\frac{3}{2} < y < 2$	Y		Y	2
$(1, 1)$	$0 < y < \frac{1}{2}$	Y	Y	Y	3
	$\frac{1}{2} < y < 1$	Y	Y		2
	$1 < y < \frac{3}{2}$			Y	1
	$\frac{3}{2} < y < 2$				0

If $x + y \in 2\mathbb{Z} + 1, x - y \notin \mathbb{Z}$:

Table 9.3.4					
(\bar{a}, \bar{b})	range (mod $2\mathbb{Z}$)	$(2, 0)$	$(0, 2)$	$(1, -1)$	ℓ_0
$(0, 0)$	$0 < y < \frac{1}{2}$	Y			1
	$\frac{1}{2} < y < 1$	Y		Y	2
	$1 < y < \frac{3}{2}$		Y		1
	$\frac{3}{2} < y < 2$		Y	Y	2
$(1, 0)$	$0 < y < \frac{1}{2}$			Y	1
	$\frac{1}{2} < y < 1$				0
	$1 < y < \frac{3}{2}$	Y	Y	Y	3
	$\frac{3}{2} < y < 2$	Y	Y		2
$(0, 1)$	$0 < y < \frac{1}{2}$	Y	Y	Y	3
	$\frac{1}{2} < y < 1$	Y	Y		2
	$1 < y < \frac{3}{2}$			Y	1
	$\frac{3}{2} < y < 2$				0
$(1, 1)$	$0 < y < \frac{1}{2}$		Y		1
	$\frac{1}{2} < y < 1$		Y	Y	2
	$1 < y < \frac{3}{2}$	Y			1
	$\frac{3}{2} < y < 2$	Y		Y	2

Cartan 3, Case 4: $x + y \notin \mathbb{Z}, x - y \in \mathbb{Z}$

An example is $(x, y) = (\frac{4}{3}, \frac{1}{3})$.

The nonintegral roots are $(2, 0)$, $(0, 2)$ and $(1, 1)$. The oriented conditions are

$$\begin{aligned}
 (9.3.8)(a) \quad & (2, 0) : 0 < x - \bar{a} < 1 \pmod{2\mathbb{Z}} \\
 & (0, 2) : 1 < y - \bar{b} < 2 \pmod{2\mathbb{Z}} \\
 & (1, 1) : 1 < (x - \bar{a}) + (y - \bar{b}) < 2 \pmod{2\mathbb{Z}}
 \end{aligned}$$

Suppose $x - y = n \in \mathbb{Z}$. As in Case 3, (a) becomes

$$\begin{aligned}
 (9.3.8)(b) \quad & (2, 0) : 0 < y + n - \bar{a} < 1 \pmod{2\mathbb{Z}} \\
 & (0, 2) : 1 < y - \bar{b} < 2 \pmod{2\mathbb{Z}} \\
 & (1, 1) : \frac{1}{2} < y + \frac{1}{2}(n - \bar{a} - \bar{b}) < 1 \pmod{\mathbb{Z}}.
 \end{aligned}$$

If $x - y \in 2\mathbb{Z}, x + y \notin \mathbb{Z}$:

(\bar{a}, \bar{b})	range (mod $2\mathbb{Z}$)	$(2, 0)$	$(0, 2)$	$(1, 1)$	ℓ_0
$(0, 0)$	$0 < y < \frac{1}{2}$	Y			1
	$\frac{1}{2} < y < 1$	Y		Y	2
	$1 < y < \frac{3}{2}$		Y		1
	$\frac{3}{2} < y < 2$		Y	Y	2
$(1, 0)$	$0 < y < \frac{1}{2}$			Y	1
	$\frac{1}{2} < y < 1$				0
	$1 < y < \frac{3}{2}$	Y	Y	Y	3
	$\frac{3}{2} < y < 2$	Y	Y		2
$(0, 1)$	$0 < y < \frac{1}{2}$	Y	Y	Y	3
	$\frac{1}{2} < y < 1$	Y	Y		2
	$1 < y < \frac{3}{2}$			Y	1
	$\frac{3}{2} < y < 2$				0
$(1, 1)$	$0 < y < \frac{1}{2}$		Y		1
	$\frac{1}{2} < y < 1$		Y	Y	2
	$1 < y < \frac{3}{2}$	Y			1
	$\frac{3}{2} < y < 2$	Y		Y	2

If $x - y \in 2\mathbb{Z} + 1, x + y \notin \mathbb{Z}$:

(\bar{a}, \bar{b})	range (mod $2\mathbb{Z}$)	$(2, 0)$	$(0, 2)$	$(1, 1)$	ℓ_0
$(0, 0)$	$0 < y < \frac{1}{2}$			Y	1
	$\frac{1}{2} < y < 1$				0
	$1 < y < \frac{3}{2}$	Y	Y	Y	3
	$\frac{3}{2} < y < 2$	Y	Y		2
$(1, 0)$	$0 < y < \frac{1}{2}$	Y			1
	$\frac{1}{2} < y < 1$	Y		Y	2
	$1 < y < \frac{3}{2}$		Y		1
	$\frac{3}{2} < y < 2$		Y	Y	2
$(0, 1)$	$0 < y < \frac{1}{2}$		Y		1
	$\frac{1}{2} < y < 1$		Y	Y	2
	$1 < y < \frac{3}{2}$	Y			1
	$\frac{3}{2} < y < 2$	Y		Y	2
$(1, 1)$	$0 < y < \frac{1}{2}$	Y	Y	Y	3
	$\frac{1}{2} < y < 1$	Y	Y		2
	$1 < y < \frac{3}{2}$			Y	1
	$\frac{3}{2} < y < 2$				0

Cartan 3, Case 5: $x \pm y \in \mathbb{Z}, x, y \notin \mathbb{Z}$

In other words $x, y \in \mathbb{Z} + \frac{1}{2}$. An example is $(x, y) = (\frac{3}{2}, \frac{1}{2})$.

The nonintegral roots are $(2, 0), (0, 2)$. The oriented conditions are

$$(9.3.9)(a) \quad \begin{aligned} (2, 0) &: 0 < x - \bar{a} < 1 \pmod{2\mathbb{Z}} \\ (0, 2) &: 1 < y - \bar{b} < 2 \pmod{2\mathbb{Z}} \end{aligned}$$

$x \pm y \in \mathbb{Z}; x, y \notin \mathbb{Z}$

Table 9.3.7				
(\bar{a}, \bar{b})	$x \pmod{2\mathbb{Z}}$	$y \pmod{2\mathbb{Z}}$	oriented	ℓ_0
$(\bar{0}, \bar{0})$	$0 < x < 1$	$0 < y < 1$	$(2, 0)$	1
	$0 < x < 1$	$1 < y < 2$	$(2, 0), (0, 2)$	2
	$1 < x < 2$	$0 < y < 1$		0
	$1 < x < 2$	$1 < y < 2$	$(0, 2)$	1
$(\bar{1}, \bar{0})$	$0 < x < 1$	$0 < y < 1$		0
	$0 < x < 1$	$1 < y < 2$	$(0, 2)$	1
	$1 < x < 2$	$0 < y < 1$	$(2, 0)$	1
	$1 < x < 2$	$1 < y < 2$	$(2, 0), (0, 2)$	2
$(\bar{0}, \bar{1})$	$0 < x < 1$	$0 < y < 1$	$(2, 0), (0, 2)$	2
	$0 < x < 1$	$1 < y < 2$	$(2, 0)$	1
	$1 < x < 2$	$0 < y < 1$	$(0, 2)$	1
	$1 < x < 2$	$1 < y < 2$		0
$(\bar{1}, \bar{1})$	$0 < x < 1$	$0 < y < 1$	$(0, 2)$	1
	$0 < x < 1$	$1 < y < 2$		0
	$1 < x < 2$	$0 < y < 1$	$(2, 0), (0, 2)$	2
	$1 < x < 2$	$1 < y < 2$	$(2, 0)$	1

10 Composition Series: Integral Infinitesimal Character

Assume the infinitesimal character γ is regular, so $\gamma = (a, b)$ with $a > b > 0$.

Cartan 1 There are two standard modules $I(x_9, (a \mp b, 0), \frac{1}{2}(a \pm b, a \pm b))$; necessarily $a \mp b \in \mathbb{Z}$ (depending on which of the two modules we consider).

By the preceding sections the real root $(1, 1)$ gives reducibility if and only if $a \in \mathbb{Z}$, and the complex root $(2, 0)$ gives reducibility only of $I(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b))$, again only if $a \in \mathbb{Z}$. So we may assume $a, b \in \mathbb{Z}$.

First consider $I(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b))$. Then (cf. (2.1.3))

$$I(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) = I(x_4, (a + b, 0), \frac{1}{2}(a - b, -a + b));$$

this is the module #4 in the output of `block`. Using `klbasis` we see this standard module has three composition factors, including the two large discrete series representations at this infinitesimal character.

Remark 10.1 *Recall that in order to compute the composition factors of the standard modules using `klbasis`, we need to invert the matrix obtained from the values of the KLV polynomials at 1 with the signs coming from the lengths, as in (11.1).*

So

$$(10.2)(a) \quad \begin{aligned} I(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) = \\ J(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) \\ + J(x_0, (a, b)) + J(x_1, (a, b)). \end{aligned}$$

Next, $I(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b))$ is block element 9, so `klbasis` says J has irreducible modules 9, 0, 1, 4, 5, 6 as composition factors (with multiplicity one).

Then:

$$(10.2)(b) \quad \begin{aligned} I(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b)) = \\ J(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b)) \\ + J(x_0, (a, b)) + J(x_1, (a, b)) \\ + J(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) \\ + J(x_7, (\bar{b}, a), (b, 0)) \\ + J(x_8, (\bar{b}, a), (b, 0)) \end{aligned}$$

Cartan 2

The standard modules are $I(x_k, (\epsilon, b), (a, 0))$ and $I(x_k, (\epsilon, a), (b, 0))$ with $k = 7, 8$ and $\epsilon = 0, 1$. We continue to assume $a > b > 0$.

The real root $(2, 0)$ gives reducibility $I(x_k, (\bar{a}, b), (a, 0))$ and $I(x_k, (\bar{b}, a), (b, 0))$ provided $a, b \in \mathbb{Z}$. The complex root $(1, 1)$ contributes to reducibility of $I(x_k, (\epsilon, b), (a, 0))$, again only if $a, b \in \mathbb{Z}$. So the infinitesimal character is necessarily integral. These representations are in one of two blocks, depending on ϵ .

Using `klbasis` we compute the following composition series.

Here are the formulas in the big block.

$I(x_7, (\bar{b}, a), (b, 0))$ (standard module 5):

$$(10.2)(c) \quad I(x_7, (\bar{b}, a), (b, 0)) = J(x_7, (\bar{b}, a), (b, 0)) + J(x_0, (a, b)) + J(x_2, (a, b)).$$

$I(x_8, (\bar{b}, a), (b, 0))$ (standard module 6):

(10.2)(d)

$$I(x_8, (\bar{b}, a), (b, 0)) = J(x_8, (\bar{b}, a), (b, 0)) + J(x_1, (a, b)) + J(x_3, (a, b)).$$

$I(x_7, (\bar{a}, b), (a, 0))$ (standard module 7):

$$(10.2)(e) \quad \begin{aligned} I(x_7, (\bar{a}, b), (a, 0)) &= J(x_7, (\bar{a}, b), (a, 0)) + J(x_0, (a, b)) \\ &+ J(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) + J(x_7, (\bar{b}, a), (b, 0)) \end{aligned}$$

$I(x_8, (\bar{a}, b), (a, 0))$ (standard module 8):

$$(10.2)(f) \quad \begin{aligned} I(x_8, (\bar{a}, b), (a, 0)) &= J(x_8, (\bar{a}, b), (a, 0)) + J(x_1, (a, b)) \\ &+ J(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) + J(x_8, (\bar{b}, a), (b, 0)) \end{aligned}$$

Here are the formulas in the block dual to $SO(4, 1)$.

By the preceding $I(x_k, (\bar{b} + 1, a), (b, 0))$ with $k = 7, 8$ are both irreducible. These are standard modules $\mathbf{0}, \mathbf{1}$ from this block.

$I(x_7, (\bar{a} + 1, b), (a, 0))$ (standard module 2):

$$(10.2)(g) \quad \begin{aligned} I(x_7, (\bar{a} + 1, b), (a, 0)) &= J(x_7, (\bar{a} + 1, b), (a, 0)) \\ &+ J(x_7, (\bar{b} + 1, a), (b, 0)) \end{aligned}$$

$I(x_8, (\bar{a} + 1, b), (a, 0))$ (standard module 3):

$$(10.2)(h) \quad \begin{aligned} I(x_8, (\bar{a} + 1, b), (a, 0)) &= J(x_8, (\bar{a} + 1, b), (a, 0)) \\ &+ J(x_8, (\bar{b} + 1, a), (b, 0)) \end{aligned}$$

Cartan 3

These are the modules $I(x_{10}, (\delta, \epsilon), (a, b))$ with $\delta, \epsilon = 0, 1$ and $a > b > 0$.

First consider the representations in the big block, so $a, b \in \mathbb{Z}$.

$I(x_{10}, (\bar{a} + 1, \bar{b} + 1), (a, b))$ This is block element 11, which has irreducible modules 11, 0, 1, 2, 3, 4, 5, 6, 9, all of multiplicity one.

$$(10.2)(i) \quad \begin{aligned} I(x_{10}, (\bar{a} + 1, \bar{b} + 1), (a, b)) &= J(x_{10}, (\bar{a} + 1, \bar{b} + 1), (a, b)) \\ &+ J(x_0, (a, b)) + J(x_1, (a, b)) + J(x_2, (a, b)) + J(x_3, (a, b)) \\ &+ J(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) \\ &+ J(x_7, (\bar{b}, a), (b, 0)) + J(x_8, (\bar{b}, a), (b, 0)) \\ &+ J(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b)) \end{aligned}$$

$I(x_{10}, (\bar{a}, \bar{b}), (a, b))$ This is block element 10, which has irreducible modules 10, 0, 1, 4, 5, 6, 7, 8, 9, all of multiplicity one except 4 has multiplicity 2. See Section 15.

$$(10.2)(j) \quad \begin{aligned} I(x_{10}, (\bar{a}, \bar{b}), (a, b)) &= J(x_{10}, (\bar{a}, \bar{b}), (a, b)) \\ &+ J(x_0, (a, b)) + J(x_1, (a, b)) \\ &+ 2 \times J(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) \\ &+ J(x_7, (\bar{b}, a), (b, 0)) + J(x_8, (\bar{b}, a), (b, 0)) \\ &+ J(x_7, (\bar{a}, b), (a, 0)) + J(x_8, (\bar{a}, b), (a, 0)) \\ &+ J(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b)) \end{aligned}$$

Next, consider the integral block dual to $SO(4, 1)$. There is one representation on the split Cartan, number 4.

$$\begin{aligned}
(10.2)(k) \quad I(x_{10}, (\bar{a} + 1, \bar{b}), (a, b)) &= J(x_{10}, (\bar{a} + 1, \bar{b}), (a, b)) \\
&+ J(x_7, (\bar{b} + 1, a), (b, 0)) + J(x_8, (\bar{b} + 1, a), (b, 0)) \\
&+ J(x_7, (\bar{a} + 1, b), (a, 0)) + J(x_8, (\bar{a} + 1, b), (a, 0))
\end{aligned}$$

The principal series $I(x_{10}, (\bar{a}, \bar{b}+1), (a, b))$ is irreducible (dual to $SO(5, 0)$).

11 Character Formulas: Integral Infinitesimal Character

Assume the infinitesimal character γ is regular, so $\gamma = (a, b)$ with $a > b > 0$.

Suppose $\mu = (x, \lambda, \nu)$. Use `nblock` to compute the block of $I(\mu)$, and `klbasis` to compute $P_{\mu, \delta}$ for all δ in the block. Then:

$$(11.1) \quad J(\mu) = \sum (-1)^{\ell(\mu) - \ell(\delta)} P_{\delta, \mu}(1) I(\delta)$$

Here are character formulas; the numbering is parallel to that of (10.2)(a-k).

Cartan 1

$$\begin{aligned}
(11.2)(a) \quad J(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) &= \\
I(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) & \\
- I(x_0, (a, b)) - I(x_1, (a, b)). &
\end{aligned}$$

$$\begin{aligned}
(11.2)(b) \quad J(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b)) &= \\
I(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b)) & \\
+ I(x_0, (a, b)) + I(x_1, (a, b)) + I(x_2, (a, b)) + I(x_3, (a, b)) & \\
- I(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) & \\
- I(x_7, (\bar{b}, a), (b, 0)) - I(x_8, (\bar{b}, a), (b, 0)) &
\end{aligned}$$

Cartan 2

Here are the formulas on the big block.

$$(11.2)(c) \quad J(x_7, (\bar{b}, a), (b, 0)) = I(x_7, (\bar{b}, a), (b, 0)) - I(x_0, (a, b)) - I(x_2, (a, b))$$

$$(11.2)(d) \quad J(x_8, (\bar{b}, a), (b, 0)) = I(x_8, (\bar{b}, a), (b, 0)) - I(x_1, (a, b)) - I(x_3, (a, b))$$

$$(11.2)(e) \quad \begin{aligned} J(x_7, (\bar{a}, b), (a, 0)) &= I(x_7, (\bar{a}, b), (a, 0)) \\ &+ I(x_0, (a, b)) + I(x_1, (a, b)) + I(x_2, (a, b)) \\ &- I(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) - I(x_7, (\bar{b}, a), (b, 0)) \end{aligned}$$

$$(11.2)(f) \quad \begin{aligned} J(x_8, (\bar{a}, b), (a, 0)) &= I(x_8, (\bar{a}, b), (a, 0)) \\ &+ I(x_0, (a, b)) + I(x_1, (a, b)) + I(x_3, (a, b)) \\ &- I(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) - I(x_8, (\bar{b}, a), (b, 0)) \end{aligned}$$

Here are the formulas on the block dual to $SO(4, 1)$:

$$(11.2)(g) \quad J(x_7, (\bar{a} + 1, b), (a, 0)) = I(x_7, (\bar{a} + 1, b), (a, 0)) - I(x_7, (\bar{b} + 1, a), (b, 0))$$

$$(11.2)(h) \quad J(x_8, (\bar{a} + 1, b), (a, 0)) = I(x_8, (\bar{a} + 1, b), (a, 0)) - I(x_8, (\bar{b} + 1, a), (b, 0))$$

Cartan 3

First consider the big block, with integral infinitesimal character.

$$(11.2)(i) \quad \begin{aligned} J(x_{10}, (\bar{a} + 1, \bar{b} + 1), (a, b)) &= I(x_{10}, (\bar{a} + 1, \bar{b} + 1), (a, b)) - I(x_2, (a, b)) - I(x_3, (a, b)) \\ &- I(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b)) \end{aligned}$$

See Section 15.

$$\begin{aligned}
(11.2)(j) \quad & J(x_{10}, (\bar{a}, \bar{b}), (a, b)) = I(x_{10}, (\bar{a}, \bar{b}), (a, b)) \\
& - I(x_0, (a, b)) - I(x_1, (a, b)) - I(x_2, (a, b)) - I(x_3, (a, b)) \\
& + I(x_9, (a+b, 0), \frac{1}{2}(a-b, a-b)) \\
& + I(x_7, (\bar{b}, a), (b, 0)) + I(x_8, (\bar{b}, a), (b, 0)) \\
& - I(x_7, (\bar{a}, b), (a, 0)) - I(x_8, (\bar{a}, b), (a, 0)) \\
& - I(x_9, (a-b, 0), \frac{1}{2}(a+b, a+b))
\end{aligned}$$

On the block dual to $SO(4, 1)$ there is a single formula on the split Cartan:

$$(11.2)(k) \quad \begin{aligned}
J(x_{10}, (\bar{a}+1, \bar{b}), (a, b)) &= I(x_{10}, (\bar{a}+1, \bar{b}), (a, b)) - \\
& J(x_7, (\bar{a}+1, b), (a, 0)) - J(x_8, (\bar{a}+1, b), (a, 0))
\end{aligned}$$

12 Composition Series and Character Formulas: Non-Integral Infinitesimal Character

Write $\gamma = (a, b)$ with $a > b > 0$. Let $\Delta(\gamma)$ be the integral root system: $\Delta(\gamma) = \{\alpha \mid \langle \gamma, \alpha^\vee \rangle \in \mathbb{Z}\}$. If this is empty then the standard module is irreducible. Here are the remaining cases.

12.1 $\Delta(\gamma) = \{(\pm 1, \pm 1)\}$ (type D_2)

If $a, b \in \mathbb{Z} + \frac{1}{2}$ then $\Delta(\gamma) = \{(\pm 1, \pm 1)\}$ is of type D_2 .

On the compact Cartan subgroup or Cartan 2 at least one of the roots $2e_i$ is imaginary, hence integral, so this case doesn't arise. So we're on the \mathbb{C}^* or split Cartan.

Recall ((2.1.4)(a)) $I(x_9, c, x)$ has infinitesimal character $\gamma = \frac{1}{2}(x+c, x-c)$, with $c \in \mathbb{Z}$, which gives $D(\gamma)$ of type D_2 in the following cases:

$$(12.1.1) \quad I(x_9, c, x) \quad (x, c \geq 0, \text{ integers of opposite parity})$$

For example the representations with infinitesimal character $(\frac{3}{2}, \frac{1}{2})$ are

$$\begin{aligned} I(x_9, 1, 2) &= I(x_9, (1, 0), (1, 1)) \\ I(x_9, 2, 1) &= I(x_9, (2, 0), (\frac{1}{2}, \frac{1}{2})) \end{aligned}$$

These are irreducible:

Suppose $x > c$. For example take $c = 1, x = 2$, i.e. $I(x_9, 1, 2) = I(x_9, (1, 0), (1, 1))$. Using `nblock` (see Section 4) this is $\lambda - \rho = (-1, -1) = [0, -1]$ and $\nu = (1, 1) = [0, 1]$. The infinitesimal character is $\gamma = [1, \frac{1}{2}] = (\frac{3}{2}, \frac{1}{2})$.

```
real: nblock
choose Cartan class (one of 0,1,2,3): 1
Choose a KGB element from the following canonical fiber:
  9:  2  [n,C]    9  4   10  *  (0,0)#1 2,1,2
KGB number: 9
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[1,0] (>=-1)
Give lambda-rho: 0 -1
denominator for nu: 1
numerator for nu: 0 1
Name an output file (return for stdout, ? to abandon):
x = 9, gamma = [2,1]/2, lambda = [1,0]/1
Subsystem on dual side is of type A1.A1, with roots 4,6.
Given parameters define element 0 of the following block:
0(0,2):  0  [i2,rn]  0  0  (1,2)  (*,*)  *( 9,  [1,0]= rho+  [0,-1])  1  1
1(1,0):  1  [r2,rn]  2  1  (0,*)  (*,*)  *(10, [1,0]= rho+  [0,-1])  0  e
2(1,1):  1  [r2,rn]  1  2  (0,*)  (*,*)  *(10, [1,-1]= rho+  [0,-2])  0  e
KL polynomials (-1)^(1(0)-1(x))*P_{x,0}:
0: 1
```

The KLV polynomial information tells us that this module (module 0) is irreducible.

The two other standard modules here, on the split Cartan subgroup, are $I(x_{10}, (\bar{1}, \bar{0}), (\frac{3}{2}, \frac{1}{2}))$ and $I(x_{10}, (\bar{0}, \bar{1}), (\frac{3}{2}, \frac{1}{2}))$. Using the `nblock` command for these two (principal series) representations, we see that they have composition series

$$\begin{aligned} I(x_{10}, (\bar{1}, \bar{0}), (\frac{3}{2}, \frac{1}{2})) &= J(x_{10}, (\bar{1}, \bar{0}), (\frac{3}{2}, \frac{1}{2})) + J(x_9, 1, 2) \\ I(x_{10}, (\bar{0}, \bar{1}), (\frac{3}{2}, \frac{1}{2})) &= J(x_{10}, (\bar{0}, \bar{1}), (\frac{3}{2}, \frac{1}{2})) + J(x_9, 1, 2) \end{aligned}$$

and character formulas

$$J(x_{10}, (\bar{1}, \bar{0}), (\frac{3}{2}, \frac{1}{2})) = I(x_{10}, (\bar{1}, \bar{0}), (\frac{3}{2}, \frac{1}{2})) - I(x_9, 1, 2)$$

$$J(x_{10}, (\bar{0}, \bar{1}), (\frac{3}{2}, \frac{1}{2})) = I(x_{10}, (\bar{0}, \bar{1}), (\frac{3}{2}, \frac{1}{2})) - I(x_9, 1, 2)$$

More generally for $x > c > 0$, x, c integers of opposite parity, the standard modules are

$$I(x_9, c, x) \quad (x > c \geq 0, \text{ integers of opposite parity})$$

(12.1.2)(a)

$$I(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x+c, x-c))$$

$$I(x_{10}, (\bar{1}, \bar{1+c}), \frac{1}{2}(x+c, x-c))$$

and the character formulas are

(12.1.2)(b)

$$J(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x+c, x-c)) = I(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x+c, x-c)) - I(x_9, c, x)$$

$$J(x_{10}, (\bar{1}, \bar{1+c}), \frac{1}{2}(x+c, x-c)) = I(x_{10}, (\bar{1}, \bar{1+c}), \frac{1}{2}(x+c, x-c)) - I(x_9, c, x)$$

Remark 12.1.3 *If we choose $x > c = 0$ with x an odd integer, say $x = 1$, we get*

```

real: nblock
choose Cartan class (one of 0,1,2,3): 1
Choose a KGB element from the following canonical fiber:
  9:  2 [n,C]   9   4   10   * (0,0)#1 2,1,2
KGB number: 9
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[1,0] (>=-1)
Give lambda-rho: -1 -1
denominator for nu: 2
numerator for nu: 0 1
Name an output file (return for stdout, ? to abandon):
x = 9, gamma = [0,1]/2, lambda = [0,0]/1
Subsystem on dual side is of type A1.A1, with roots 4,6.
Given parameters define element 0 of the following block:
0(0,2):  0 [i2,rn]  0  0  (1,2)  (*,*)  *( 9, [0,0]= rho+ [-1,-1])  1
1(1,0):  1 [r2,rn]  2  1  (0,*)  (*,*)  (10, [0,0]= rho+ [-1,-1])  e
2(1,1):  1 [r2,rn]  1  2  (0,*)  (*,*)  (10, [0,-1]= rho+ [-1,-2])  e
(cumulated) KL polynomials (-1)^{1(0)-1(x)}*P_{x,0}:
0: 1

```

In this case, the block contains only the module $I(x_9, 0, x)$; the parameters for $I(x_{10}, (\bar{0}, \bar{0}), \frac{1}{2}(x, x))$ and $I(x_{10}, (\bar{1}, \bar{1}), \frac{1}{2}(x, x))$ are not final; so the irreducible modules $J(x_{10}, (\bar{0}, \bar{0}), \frac{1}{2}(x, x))$ and $J(x_{10}, (\bar{1}, \bar{1}), \frac{1}{2}(x, x))$ are zero. The full principal series representations associated to these parameters are actually irreducible, and both equivalent to $J(x_9, 0, x)$.

If $0 \leq x < c$, we get the same character formulas. For example $x = 1, c = 2$ gives $I(x_9, 2, 1) = I(x_9, (2, 0), \frac{1}{2}(1, 1))$ with $\gamma = (\frac{3}{2}, -\frac{1}{2})$. In `nblock` coordinates this is $\lambda = (2, 0)$, $\lambda - \rho = (0, -1) = [1, -1]$, $\nu = (\frac{1}{2}, \frac{1}{2}) = [0, \frac{1}{2}]$.

```

real: nblock
choose Cartan class (one of 0,1,2,3): 1
Choose a KGB element from the following canonical fiber:
  9:  2  [n,C]    9  4    10  *  (0,0)#1 2,1,2
KGB number: 9
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[1,0] (>=-1)
Give lambda-rho: 1 -1
denominator for nu: 2
numerator for nu: 0 1
Name an output file (return for stdout, ? to abandon):
x = 9, gamma = [4,-1]/2, lambda = [2,0]/1
Subsystem on dual side is of type A1.A1, with roots 4,6.
Given parameters define element 0 of the following block:
0(0,2):  0  [i2,rn]  0  0  (1,2)  (*,*)  *( 9, [2,-2]= rho+  [1,-3])  1
1(1,0):  1  [r2,rn]  2  1  (0,*)  (*,*)  *(10, [2,-2]= rho+  [1,-3])  e
2(1,1):  1  [r2,rn]  1  2  (0,*)  (*,*)  *(10, [2,-1]= rho+  [1,-2])  e
KL polynomials (-1)^{1(0)-1(x)}*P_{x,0}:
0: 1

```

The infinitesimal character is not dominant on the coroot $(0, 1)$; however, it is dominant on the integral coroots. The standard modules and character formulas are exactly as in (12.1.2)(a) and (12.1.2)(b). Using

$$(12.1.4) \quad I(x_{10}(\bar{a}, \bar{b}), (x, y)) \cong I(x_{10}(\bar{a}, \bar{b}), (x, -y)),$$

we write these formulas

$$(12.1.5)(a) \quad \begin{array}{l} I(x_9, c, x) \quad (c > 0, x \geq 0, \text{ integers of opposite parity}) \\ I(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x+c, -x+c)) \\ I(x_{10}, (\bar{1}, \bar{1}+c), \frac{1}{2}(x+c, -x+c)) \end{array}$$

and the character formulas

$$(12.1.5)(b) \quad \begin{array}{l} J(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x+c, -x+c)) = I(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x+c, -x+c)) - I(x_9, c, x) \\ J(x_{10}, (\bar{1}, \bar{1}+c), \frac{1}{2}(x+c, -x+c)) = I(x_{10}, (\bar{1}, \bar{1}+c), \frac{1}{2}(x+c, -x+c)) - I(x_9, c, x) \end{array}$$

In all remaining cases $\Delta(\gamma)$ is of type A_1 .

12.2 $\Delta(\gamma) = \{\pm(1, -1)\}$ or $\{\pm(1, 1)\}$

As in (12.1) we are on the \mathbb{C}^* or split Cartan. On the \mathbb{C}^* Cartan take

$$I(x_9, c, x) = I(x_9, (c, 0), \frac{1}{2}(x, x)) \quad (x \notin \mathbb{Z}, x \geq 0, c > 0).$$

with

$$\gamma = \frac{1}{2}(x+c, x-c)$$

for which $(1, -1)$ is the unique positive integral root (and is imaginary). (If $x < c$ then we can conjugate to the fiber of \mathbf{x}_4 to make the infinitesimal character dominant; then $(1, 1)$ will be the unique positive integral root.)

The standard module is irreducible.

Here is an example; compare Section 12.1.

```
real: nblock
choose Cartan class (one of 0,1,2,3): 1
Choose a KGB element from the following canonical fiber:
 9:  2 [n,C]   9  4   10  * (0,0)#1 2,1,2
KGB number: 9
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[1,0] (>=-1)
Give lambda-rho: 0 -1
```

```

denominator for nu: 8
numerator for nu: 0 5
Name an output file (return for stdout, ? to abandon):
x = 9, gamma = [8,1]/8, lambda = [1,0]/1
Subsystem on dual side is of type A1, with roots 4.
Given parameters define element 0 of the following block:
0(0,2): 0 [i2] 0 (1,2) *( 9, [1,0]= rho+ [0,-1]) 1 1
1(1,0): 1 [r2] 2 (0,*) *(10, [1,0]= rho+ [0,-1]) 0 e
2(1,1): 1 [r2] 1 (0,*) *(10, [1,-1]= rho+ [0,-2]) 0 e
KL polynomials (-1)^{l(0)-l(x)}*P_{x,0}:
0: 1

```

Recall the situation is $x \notin \mathbb{Z}, x, c > 0$. Compare (12.1.5)(a) and (12.1.5)(b), which is the same except that $x \in \mathbb{Z}$ of opposite parity to c . The standard modules are

$$(12.2.6)(a) \quad \begin{array}{l} I(x_9, c, x) = I(x_9, (c, 0), \frac{1}{2}(x, x)) \quad (x \notin \mathbb{Z}, x > c) \\ I(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x + c, x - c)) \\ I(x_{10}, (\bar{1}, \bar{1} + c), \frac{1}{2}(x + c, x - c)) \end{array}$$

and the character formulas are as in (12.1.5)(b)

(12.2.6)(b)

$$\begin{array}{l} J(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x + c, x - c)) = I(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x + c, x - c)) - I(x_9, c, x) \\ J(x_{10}, (\bar{1}, \bar{1} + c), \frac{1}{2}(x + c, x - c)) = I(x_{10}, (\bar{1}, \bar{1} + c), \frac{1}{2}(x + c, x - c)) - I(x_9, c, x) \end{array}$$

As in (12.1.5)(a) and (12.1.5)(b), if $x < c$, it may be convenient to use the formula (12.1.4) to write

(12.2.7)

$$\begin{array}{l} J(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x + c, -x + c)) = I(x_{10}, (\bar{0}, \bar{c}), \frac{1}{2}(x + c, -x + c)) - I(x_9, c, x) \\ J(x_{10}, (\bar{1}, \bar{1} + c), \frac{1}{2}(x + c, -x + c)) = I(x_{10}, (\bar{1}, \bar{1} + c), \frac{1}{2}(x + c, -x + c)) - I(x_9, c, x) \end{array}$$

instead.

12.3 $\Delta(\gamma) = \{\pm(0, 2)\}$

Consider

$$I(x_k, (\bar{a}, b), (x, 0)) \quad (k = 7, 8; x \notin \mathbb{Z}; x > b)$$

with $\gamma = (x, b)$ dominant. This standard module is irreducible. On the split Cartan we have

$$(12.3.8) \quad I(x_{10}, (\bar{a}, \bar{b}), (x, b)).$$

The character formula is (recall $x > b$):

(12.3.9)

$$J(x_{10}, (\bar{a}, \bar{b}), (x, b)) = I(x_{10}, (\bar{a}, \bar{b}), (x, b)) - I(x_7, (\bar{a}, b), (x, 0)) - I(x_8, (\bar{a}, b), (x, 0))$$

Remark 12.3.10 *If we choose $b = 0$, nblock yields:*

```

real: nblock
choose Cartan class (one of 0,1,2,3): 2
Choose a KGB element from the following canonical fiber:
  7:  2  [C,n]   5  8      *  10  (0,0)#2 1,2,1
  8:  2  [C,n]   6  7      *  10  (0,1)#2 1,2,1
KGB number: 7
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[0,1] (>=-1)
Give lambda-rho: -1 -1
denominator for nu: 3
numerator for nu: 0 2
Name an output file (return for stdout, ? to abandon):
x = 7, gamma = [2,0]/3, lambda = [0,0]/1
Subsystem on dual side is of type A1, with roots 5.
Given parameters define element 1 of the following block:
0(0,1):  0  [i1]  1  (2,*)  *( 8,  [0,0]= rho+  [-1,-1])  2
1(1,1):  0  [i1]  0  (2,*)  *( 7,  [0,0]= rho+  [-1,-1])  2
2(2,0):  1  [r1]  2  (1,0)  (10,  [0,0]= rho+  [-1,-1])  e
(cumulated) KL polynomials (-1)^(1(1)-1(x))*P_{x,1}:
1: 1

```

The block contains only the two modules which are related by a cross action through the imaginary root. Once again, the parameter attached to Cartan 3 is not final; the corresponding principal series representation is the direct sum of the two modules 0 and 1.

12.4 $\Delta(\gamma) = \{\pm(2, 0)\}$

Consider

$$I(x_k, (\bar{a}, b), (x, 0)) \quad (k = 7, 8; x \notin \mathbb{Z}; x < b).$$

Now $\gamma = (x, b)$ is not dominant, but conjugate to (b, x) ; this is dominant, and gives integral roots $\pm(2, 0)$. Probably it is better to think of this as

$$(12.4.11) \quad I(x_\ell, (b, \bar{a}), (0, x)) \quad (\ell = 5, 6; x \notin \mathbb{Z}; x < b).$$

which makes it clear that $(2, 0)$ is an imaginary root.

These standard modules are irreducible. There is also a standard module on the split Cartan:

$$(12.4.12) \quad I(x_{10}, (\bar{a}, \bar{b}), (x, b))$$

The character formula is precisely as in (12.3.9) (recall that now $x < b$):

$$(12.4.13)$$

$$J(x_{10}, (\bar{a}, \bar{b}), (x, b)) = I(x_{10}, (\bar{a}, \bar{b}), (x, b)) - I(x_7, (\bar{a}, b), (x, 0)) - I(x_8, (\bar{a}, b), (x, 0))$$

However it is probably clearer to use (12.4.11) (and rule (c) in Section 2 for conjugation by an element of W_r) and write this as (for $x < b$):

$$(12.4.14)$$

$$J(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x)) = I(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x)) - I(x_5, (b, \bar{a}), (0, x)) - I(x_6, (b, \bar{a}), (0, x))$$

13 c-invariant Hermitian Forms

Every irreducible representation J has a distinguished c-invariant Hermitian form. We think of this as a virtual K -representation with coefficients in $\mathbb{W} = \mathbb{Z}[s]$ where $s^2 = 1$. A term $(p + qs)\mu$ means that the K -type μ has multiplicity $p + q$, and occurs p times with a plus sign, and q with a minus. In other words the μ -isotypic space is $V_\mu \otimes V[\mu]$ where V_μ is the space of μ equipped with a positive definite form, and $V[\mu] = \mathbb{C}^{p+q}$ with a form of signature (p, q) .

Definition 13.1 (a) $J(x, \lambda, \nu)_c$ is the irreducible representation $J(x, \lambda, \nu)$, equipped with its canonical c -invariant Hermitian form, normalized to be positive on the lowest K -types.

(b) Write $J_K(x, \lambda, \nu)_c$ for the restriction of $J(x, \lambda, \nu)$ to K . I won't always make this distinction.

(c) Suppose $I_K(x, \lambda)$ is a nonzero standard final limit K -representation. Let $I_K(x, \lambda)_c$ denote this module equipped with the canonical c -invariant Hermitian form which is positive on the (unique) lowest K -type.

The $I_K(x, \lambda)_c$ form a basis of the Grothendieck group of K -representations, equipped with a c -invariant Hermitian form.

We want to compute formulas of the form

$$(13.2)(a) \quad J_K(x, \lambda, \nu)_c = \sum_{x', \lambda'} a(x', \lambda') I_K(x', \lambda')_c$$

where the sum is over nonzero final standard limit K -data, and $a(x', \lambda') \in \mathbb{W}$. This is an identity in the Grothendieck group of K -modules with a c -invariant form.

This proceeds in two steps. We first write $J(x, \lambda, \nu)_c$ as a linear combination of $I(x', \lambda', \nu')_c$ with coefficients in \mathbb{W} . Recall (11.1) there is a character formula

$$(13.2)(b) \quad J(x, \lambda, \nu) = \sum_{(x', \lambda', \nu')} (-1)^{\ell(x, \lambda, \nu) - \ell(x', \lambda', \nu')} P_{((x', \lambda', \nu'), (x, \lambda, \nu))}(1) I(x', \lambda', \nu')$$

This is an identity in the Grothendieck group of (\mathfrak{g}, K) -modules and the sum is over the block containing $J(x, \lambda, \nu)$.

If $I(x, \lambda, \nu)$ is a standard module, $\text{gr}I(x, \lambda, \nu)$ has a distinguished, nondegenerate, c -invariant form, obtained by deforming ν in the outward direction so it becomes irreducible. We denote this $I(x, \lambda, \nu)_c$. Note that $I(x, \lambda, \nu)_c$ is lower semi-continuous in ν . As usual write $I_K(x, \lambda, \nu)_c$ to denote the restriction to K .

In the equal rank case there is a simple generalization of (b):

$$(13.2)(c) \quad J(x, \lambda, \nu)_c = \sum_{(x', \lambda', \nu')} \epsilon(s) P_{((x', \lambda', \nu'), (x, \lambda, \nu))}(s) I(x', \lambda', \nu')_c$$

where

$$(13.2)(d) \quad \epsilon(s) = s^{\frac{\ell_0(x,\lambda,\nu) - \ell_0(x',\lambda',\nu')}{2}} (-1)^{\ell(x,\lambda,\nu) - \ell(x',\lambda',\nu')}.$$

This is an identity in the Grothendieck group of (\mathfrak{g}, K) -modules equipped with a c -invariant form. The sum is over the block containing $J(x, \lambda, \nu)$; taking $s = 1$ gives (b). The integers ℓ_0 are the orientation numbers of Section 9.

The second step is to write $I_K(x', \lambda', \nu')_c$ in terms of $I_K(x'', \lambda'')_c$. This is by *deforming* ν to 0, which we defer to Section 14.

Here is how to compute (13.2)(c). Use `nblock` to define $J(x, \lambda, \nu)$ and compute its block; this is at possibly nonintegral or singular infinitesimal character γ . Each parameter in the output of `nblock` may have an asterisk, indicating which of the terms are nonzero at γ . The output also includes a computation of the $P_{\delta, \mu}$ for $\mu = (x, \lambda, \nu)$.

Note that the character formula (b) gives the c -invariant form formula (c) provided $P_{*,*}$ is constant for all terms occuring, and all orientation numbers are 0. All orientation numbers are 0 for integral infinitesimal character.

13.1 c-Invariant Forms: Integral Infinitesimal Character

Since the orientation numbers are all 0 the character formulas of Section 11 hold as stated unless some $P_{\mu, \delta}$ is not a constant. The only case in which this happens is formula (11.2)(i) in which the terms 2, 3 have a coefficient of q .

Therefore formulas (11.2)(a-h,j,k) are all valid as formulas for the c -invariant form, except that (11.2)(i) should read:

$$(13.1.5) \quad \begin{aligned} J(x_{10}, (\bar{a} + 1, \bar{b} + 1), (a, b))_c &= I(x_{10}, (\bar{a}, \bar{b}), (a, b))_c \\ &\quad - sI(x_2, (a, b))_c - sI(x_3, (a, b))_c \\ &\quad - I(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b))_c \end{aligned}$$

13.2 c-Invariant Forms: Nonintegral Infinitesimal Character

In the case of nonintegral infinitesimal character, the integral root system is type D_2 or A_1 , and all the P polynomials are constant. Therefore the only

way for a c-invariant form formula to differ from the character formula is from a difference of orientation numbers being odd.

So we have to consider character formulas (12.1.2)(b), (12.1.5)(b), (12.2.6)(b), (12.3.9), and (12.4.13) and see which of them require a correction due to the orientation numbers.

Cases (12.1.2)(b), (12.1.5)(b): $I(x_9, c, x)$ with x, c integers of opposite parity.

Combining (12.1.2)(b) and (12.1.5)(b) gives, for $x, y \in \mathbb{Z} + \frac{1}{2}$, $x > y > 0$,

(13.2.6)

$$J(x_{10}, (\bar{a}, \bar{b}), (x, y)) = I(x_{10}, (\bar{a}, \bar{b}), (x, y)) - \begin{cases} I(x_9, x - y, x + y) & x + y = \bar{a} + \bar{b} + 1 \pmod{2\mathbb{Z}} \\ I(x_9, x + y, x - y) & x + y = \bar{a} + \bar{b} \pmod{2\mathbb{Z}} \end{cases}$$

The orientation numbers for $I(x_{10}, (*, *), \frac{1}{2}(x+c, x-c))$ are given in Table 9.3.7, and for $I(x_9, c, x)$ in Table 9.1.1.

A little monkeying around shows the following.

If $x + y = \bar{a} + \bar{b} + 1 \pmod{2\mathbb{Z}}$ then

$$\ell_0(x_{10}, (\bar{a}, \bar{b}), (x, y)) = 1, \ell_0(x_9, x - y, x + y) = 1$$

so there is no contribution from the orientation numbers, and (12.1.2)(b) holds as a formula for c-invariant forms as follows.

Assume $x + y = a + b + 1 \pmod{2\mathbb{Z}}$, with $x, y \in \mathbb{Z} + \frac{1}{2}$, $x > y > 0$. Then (13.2.7)

$$\boxed{J(x_{10}, (\bar{a}, \bar{b}), (x, y))_c = I(x_{10}, (\bar{a}, \bar{b}), (x, y))_c - I(x_9, x - y, x + y)_c.}$$

Now suppose $x + y = \bar{a} + \bar{b} \pmod{2\mathbb{Z}}$. Then

$$\ell_0(x_9, x + y, x - y) = 0.$$

and on the other hand

$$\ell_0(x_{10}, (\bar{a}, \bar{b}), (x, y)) = \begin{cases} 0 & 0 < y - \bar{b} < 1 \\ 2 & 1 < y - \bar{b} < 2 \end{cases}$$

So this gives the first case of a nontrivial orientation number.

So: $x + y = \bar{a} + \bar{b} \pmod{2\mathbb{Z}}$ implies

$$(13.2.8) \quad \boxed{J(x_{10}, (\bar{a}, \bar{b}), (x, y))_c = I(x_{10}, (\bar{a}, \bar{b}), (x, y))_c - \begin{cases} I(x_9, x + y, x - y)_c & 0 < y - \bar{b} < 1 \\ sI(x_9, x + y, x - y)_c & 1 < y - \bar{b} < 2 \end{cases}}$$

Case (12.2.6)(b): $x \notin \mathbb{Z}$. This is very similar to cases (12.1.2)(b), (12.1.5)(b); see (13.2.7) and (13.2.8).

Suppose $x - y \in \mathbb{Z}, x + y \notin \mathbb{Z}$. If $x - y = \bar{a} + \bar{b} \pmod{2\mathbb{Z}}$ then

$$(13.2.9)(a) \quad J(x_{10}, (\bar{a}, \bar{b}), (x, y)) = I(x_{10}, (\bar{a}, \bar{b}), (x, y)) - I(x_9, x - y, x + y)$$

If $x - y \neq \bar{a} + \bar{b} \pmod{2\mathbb{Z}}$ then $I(x_{10}, (\bar{a}, \bar{b}), (x, y))$ is irreducible.

Suppose $x + y \in \mathbb{Z}, x - y \notin \mathbb{Z}$. If $x + y = \bar{a} + \bar{b} \pmod{2\mathbb{Z}}$ then

$$(13.2.9)(b) \quad J(x_{10}, (\bar{a}, \bar{b}), (x, y)) = I(x_{10}, (\bar{a}, \bar{b}), (x, y)) - I(x_9, x + y, x - y)$$

If $x + y \neq \bar{a} + \bar{b} \pmod{2\mathbb{Z}}$ then $I(x_{10}, (\bar{a}, \bar{b}), (x, y))$ is irreducible.

In (a) all terms have $\ell_0 = 1$ if $0 < y < \frac{1}{2} \pmod{\mathbb{Z}}$, and $\ell_0 = 2$ if $\frac{1}{2} < y < 1 \pmod{\mathbb{Z}}$, so this holds as a formula for c-invariant forms. In other words (13.2.7) still holds here: $x - y \in \mathbb{Z}, x + y \notin \mathbb{Z}, x - y = \bar{a} + \bar{b} \pmod{2\mathbb{Z}}$ implies

$$(13.2.10) \quad \boxed{J(x_{10}, (\bar{a}, \bar{b}), (x, y))_c = I(x_{10}, (\bar{a}, \bar{b}), (x, y))_c - I(x_9, x - y, x + y)_c}$$

Similarly (13.2.8) still holds here: $x + y \in \mathbb{Z}, x - y \notin \mathbb{Z}, x + y = \bar{a} + \bar{b} \pmod{2\mathbb{Z}}$ implies

$$(13.2.11) \quad \boxed{J(x_{10}, (\bar{a}, \bar{b}), (x, y))_c = I(x_{10}, (\bar{a}, \bar{b}), (x, y))_c - \begin{cases} I(x_9, x + y, x - y)_c & 0 < y - \bar{b} < 1 \\ sI(x_9, x + y, x - y)_c & 1 < y - \bar{b} < 2 \end{cases}}$$

Case (12.3.9):

Now $x > y = b > 0$ with $b \in \mathbb{Z}$. The character formula in this case is:

(13.2.12)

$$J(x_{10}, (\bar{a}, \bar{b}), (x, b)) = I(x_{10}, (\bar{a}, \bar{b}), (x, b)) - I(x_7, (\bar{a}, b), (x, 0)) - I(x_8, (\bar{a}, b), (x, 0))$$

The orientation numbers ℓ_0 of all terms are the same, either 1 or 2, so this holds as a formula for c-invariant forms:

(13.2.13)

$$J(x_{10}, (\bar{a}, \bar{b}), (x, b))_c = I(x_{10}, (\bar{a}, \bar{b}), (x, b))_c - I(x_7, (\bar{a}, b), (x, 0))_c - I(x_8, (\bar{a}, b), (x, 0))_c.$$

Case (12.4.13):

This is analogous to the last case, with $x \notin \mathbb{Z}$ and $b \in \mathbb{Z}$, except that now $x < b$. The character formula is

$$\begin{aligned} & J(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x)) \\ &= I(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x)) - I(x_7, (\bar{a}, b), (x, 0)) - I(x_8, (\bar{a}, b), (x, 0)). \end{aligned}$$

If $1 < x - a < 2 \pmod{2\mathbb{Z}}$ then the orientation numbers of all terms are $\ell_0 = 1$; if $0 < x - a < 1 \pmod{2\mathbb{Z}}$ then $I(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x))$ has orientation number 2, while the other terms have $\ell_0 = 0$. So we have

(13.2.14)

$$\begin{aligned} J(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x))_c &= I(x_{10}, (\overline{b+1}, \overline{a+1}), (b, x))_c \\ &\begin{cases} -sI(x_7, (\bar{a}, b), (x, 0))_c - sI(x_8, (\bar{a}, b), (x, 0))_c & 0 < x - \bar{a} < 1 \\ -I(x_7, (\bar{a}, b), (x, 0))_c - I(x_8, (\bar{a}, b), (x, 0))_c & 1 < x - \bar{a} < 2 \end{cases} \end{aligned}$$

14 Deforming ν to 0

In the previous section we reduced the computation of $J(x, \lambda, \nu)_c$ to computing $I(x', \lambda', \nu')_c$ (see (13.2)(c)). In this section we discuss how to write

$$(14.1) \quad I(x, \lambda, \nu)_c = \sum_{x', \lambda'} b(x', \lambda') I_K(x', \lambda')_c$$

for $b(x', \lambda') \in \mathbb{W}$, and the sum is over nonzero final standard limit K -data. This proceeds by deformation to $\nu = 0$, and by induction, which requires using (13.2)(c) along the way.

Fix (x, λ, ν) and consider $I(x, \lambda) = I(x, \lambda, 0)$. Assume for the moment that $I(x, \lambda)$ is final (and nonzero), so $I_K(x, \lambda)$ is a final limit standard K -representation.

After deforming ν if necessary, we can assume $I(x, \lambda, t_k \nu)$ is reducible for finitely many $0 < t_1 < \dots < t_n \leq 1$. This implies

$$I(x, \lambda, t\nu)_c = I_K(x, \lambda)_c \quad (t < t_1)$$

Assume we've computed $I(x, \lambda, t\nu)_c$ for $t_{k-1} \leq t < t_k$.

Write $\gamma = (x, \lambda, t_k \nu)$.

First compute the composition factors $J(\gamma')$ of $I(\gamma)$, and the polynomials $Q(\gamma', \gamma)$. (Recall these are the polynomials satisfying $I(\gamma) = \sum_{\gamma'} Q(\gamma', \gamma)(1)J(\gamma')$. Currently `nblock` computes $P(\gamma', \gamma)$. See Section 15.)

The c -invariant form changes sign on the odd levels of the Jantzen filtration. What this amounts to is the following.

For each γ' with $Q(\gamma', \gamma) \neq 0$ write

$$(14.10)(a) \quad Q(\gamma', \gamma)(q) = \sum_{n=0}^{\ell(\gamma) - \ell(\gamma')} a_n(\gamma', \gamma) q^{\frac{\ell(\gamma) - \ell(\gamma') - n}{2}}$$

Note that $a_n(\gamma', \gamma) = 0$ unless $\ell(\gamma) - \ell(\gamma') = n \pmod{2\mathbb{Z}}$, so

$$(14.10)(b) \quad Q(\gamma', \gamma)(q) = \sum_{\substack{n=0 \\ n \equiv \ell(\gamma) - \ell(\gamma')}}^{\ell(\gamma) - \ell(\gamma')} a_n(\gamma', \gamma) q^{\frac{\ell(\gamma) - \ell(\gamma') - n}{2}}$$

Then a_n is the multiplicity of $J(\gamma')$ in level n of the Jantzen filtration. Note that γ' can occur in an odd level of the Jantzen filtration of $I(\gamma)$ only if $\ell(\gamma) - \ell(\gamma')$ is odd. Therefore

$$(14.11) \quad \begin{aligned} I(x, \lambda, t_k \nu)_c &= I(x, \lambda, t_{k-1} \nu)_c \\ &+ (1-s) \sum_{\substack{\gamma' \\ \ell(\gamma') - \ell(\gamma) \text{ odd}}} \sum_{n \text{ odd}} s^{\frac{\ell(\gamma) - \ell(\gamma') - n}{2}} s^{\frac{\ell_0(\gamma) - \ell_0(\gamma')}{2}} a_n(\gamma', \gamma) J(\gamma')_c \end{aligned}$$

By (14.10)(b) the inner sum, (after pulling out the ℓ_0 term), is just $Q(\gamma', \gamma)(s)$. So:

$$(14.12) \quad \boxed{\begin{aligned} I(x, \lambda, t_k \nu)_c &= I(x, \lambda, t_{k-1} \nu)_c \\ &+ (1-s) \sum_{\substack{\gamma' \\ \ell(\gamma') - \ell(\gamma) \text{ odd}}} s^{\frac{\ell_0(\gamma) - \ell_0(\gamma')}{2}} Q(\gamma', \gamma)(s) J(\gamma')_c \end{aligned}}$$

Here is the big block for $Sp(4, \mathbb{R})$.

length 0

Of course there is nothing to do here:

$$(14.13) \quad J(x_k, (a, b))_c = I_K(x_k, (a, b))_c \quad (0 \leq k \leq 3)$$

length 1, Cartan 1

Consider $I(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b))$. It is easier to use the other coordinates $I(x_9, c, x) = I(x_9, (c, 0), \frac{1}{2}(x, x))$ (2.1.4)(a). Notice that length 1 implies $c > x$.

The real root $(1, 1)$ gives reducibility if and only if $x = c \pmod{2}$. The complex root $(2, 0)$ gives reducibility if and only if $x = c \pmod{2}$ and $x \geq c$.

Here is how to pass back and forth. If $c > x$:

$$I(x_9, c, x) = I(x_9, (a + b, 0), \frac{1}{2}(a - b, a - b)) \quad (a = \frac{1}{2}(c + x), b = \frac{1}{2}(c - x))$$

and these have length 1, while if $c \leq x$:

$$I(x_9, c, x) = I(x_9, (a - b, 0), \frac{1}{2}(a + b, a + b)) \quad (a = \frac{1}{2}(c + x), b = \frac{1}{2}(x - c))$$

of length 2.

Suppose c is even. Then

$$(14.14)(a) \quad I(x_9, c, x)_c = I_K(x_9, c)_c \quad (0 \leq x < 2)$$

(This module is not final, so we can write it as a sum of two limits of discrete series...but ignore this for this calculation.) If $c > 2$ then, taking $x = 2$, (10.2)(a) gives

$$(14.14)(b) \quad I(x_9, c, 2) = J(x_9, c, 2) + J(x_0, \frac{1}{2}(c+2, c-2)) + J(x_1, \frac{1}{2}(c+2, c-2))$$

Apply (14.12) (or (14.11)). Since the reducibility points are at integral infinitesimal character the orientation numbers are 0, and for $2 \leq x < 4$:

(14.14)(c)

$$I(x_9, c, x)_c = I_K(x_9, c)_c + (1 - s)[I_K(x_0, \frac{1}{2}(c + 2, c - 2))_c + I_K(x_1, \frac{1}{2}(c + 2, c - 2))_c]$$

Similarly if $c > 4$ then, taking $x = 4$, (10.2)(a) gives

$$(14.14)(d) \quad I(x_9, c, 4) = J(x_9, c, 4) + J(x_0, \frac{1}{2}(c+4, c-4)) + J(x_1, \frac{1}{2}(c+4, c-4))$$

and by (14.12) for $(4 \leq x < 6)$:

(14.14)(e)

$$\begin{aligned} \mathcal{I}(x_9, c, x)_c &= I_K(x_9, c)_c \\ &\quad + (1-s)[I_K(x_0, \frac{1}{2}(c+2, c-2))_c + I_K(x_1, \frac{1}{2}(c+2, c-2))_c] \\ &\quad + (1-s)[I_K(x_0, \frac{1}{2}(c+4, c-4))_c + I_K(x_1, \frac{1}{2}(c+4, c-4))_c] \end{aligned}$$

By induction on x we see for c even, $x < c$:

(14.14)(f)

$$\begin{aligned} I(x_9, c, x)_c &= I_K(x_9, c)_c \\ &\quad + (1-s) \sum_{k=1}^{\lfloor \frac{x}{2} \rfloor} [I_K(x_0, \frac{1}{2}(c+2k, c-2k))_c + I_K(x_1, \frac{1}{2}(c+2k, c-2k))_c] \end{aligned}$$

Similarly for c odd, $x < c$:

$$\begin{aligned} (14.14)(g) \quad I(x_9, c, x)_c &= I_K(x_9, c)_c \\ &\quad + (1-s) \sum_{k=1}^{\frac{x+1}{2}} [I_K(x_0, \frac{1}{2}(c+(2k-1), c-(2k-1)))_c \\ &\quad \quad \quad + I_K(x_1, \frac{1}{2}(c+(2k-1), c-(2k-1)))_c] \end{aligned}$$

In particular at ρ take $c = 3, x = 1$:

(14.14)(h)

$$I(x_9, 3, 1)_c = I_K(x_9, 3)_c + (1-s)[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c]$$

length 1, Cartan 2

Consider the representations

$$I(x_k, (\delta, a), (x, 0)) \quad (k = 7, 8; a \in \mathbb{Z}_{\geq 0})$$

which have length 1 if $x < a$. Such a representation is reducible if and only if $x = \delta \pmod{2}$.

Take $k = 7$ and $\delta = 0$. There will be reducibility at $x = 0, 2, \dots$. We start with

$$(14.15)(a) \quad I(x_7, (\bar{0}, a), (0, 0)) = I_K(x_7, (\bar{0}, a)).$$

For $0 \leq x < 2 \leq a$, we get

$$I(x_7, (\bar{0}, a), (x, 0))_c = I_K(x_7, (\bar{0}, a))_c.$$

If $2 < a$ then taking $x = 2$ in (11.2)(c) gives

$$I(x_7, (\bar{0}, a), (2, 0)) = J(x_7, (\bar{0}, a), (2, 0)) \\ + J(x_0, (a, 2)) + J(x_2, (a, 2))$$

so by (14.12), if $(2 \leq x < 4)$:

$$I(x_7, (\bar{0}, a), (x, 0))_c = I_K(x_7, (\bar{0}, a), (x, 0))_c \\ + (1 - s)[J(x_0, (a, 2)) + J(x_2, (a, 2))_c]$$

Repeating this as in the previous case we conclude, for $x < a$:

$$(14.15)(b) \quad \boxed{I(x_7, (\bar{0}, a), (x, 0))_c = I_K(x_7, (\bar{0}, a))_c \\ + (1 - s) \sum_{k=1}^{\lfloor \frac{x}{2} \rfloor} [I_K(x_0, (a, 2k))_c + I_K(x_2, (a, 2k))_c]}$$

Similarly using (10.2)(d)

$$(14.15)(c) \quad \boxed{I(x_8, (\bar{0}, a), (x, 0))_c = I_K(x_8, (\bar{0}, a))_c \\ + (1 - s) \sum_{k=1}^{\lfloor \frac{x}{2} \rfloor} [I_K(x_1, (a, 2k))_c + I_K(x_3, (a, 2k))_c]}$$

The cases $k = 7, 8$, $\delta = \bar{1}$ are similar. The results are:

$$(14.15)(d) \quad \boxed{I(x_7, (\bar{1}, a), (x, 0))_c = I_K(x_7, (\bar{1}, a))_c \\ + (1 - s) \sum_{k=1}^{\lfloor \frac{x+1}{2} \rfloor} [I_K(x_0, (a, 2k - 1))_c + I_K(x_2, (a, 2k - 1))_c]}$$

$$(14.15)(e) \quad \boxed{I(x_8, (\bar{1}, a), (x, 0))_c = I_K(x_8, (\bar{1}, a))_c \\ + (1 - s) \sum_{k=1}^{\lfloor \frac{x+1}{2} \rfloor} [I_K(x_1, (a, 2k - 1))_c + I_K(x_3, (a, 2k - 1))_c]}$$

Consider the case of infinitesimal character $\rho = (2, 1)$. Formulas (14.15)(b-e) specialize as follows.

Take $a = 2, x = 1$ in (14.15)(b,c):

$$(14.16)(a) \quad I(x_7, (\bar{0}, 2), (1, 0))_c = I_K(x_7, (\bar{0}, 2))_c$$

$$(14.16)(b) \quad I(x_8, (\bar{0}, 2), (1, 0))_c = I_K(x_8, (\bar{0}, 2))_c$$

Take $a = 2, x = 1$ in (14.15)(d,e):

(14.16)(c)

$$I(x_7, (\bar{1}, 2), (1, 0))_c = I_K(x_7, (\bar{1}, 2))_c + (1 - s)[J(x_0, (2, 1))_c + J(x_2, (2, 1))_c]$$

(14.16)(d)

$$I(x_8, (\bar{1}, 2), (1, 0))_c = I_K(x_8, (\bar{1}, 2))_c + (1 - s)[J(x_1, (2, 1))_c + J(x_3, (2, 1))_c]$$

This completes the length 1 portion of our program. From now on we'll only include some special cases of a and b .

length 2, Cartan 1

These are the modules $I(x_9, c, x) = I(x_9, (c, 0), \frac{1}{2}(x, x))$ with $c \leq x$. This is reducible due to the real root $(1, 1)$ if $x = c \pmod{2}$. It is reducible due to the complex root $(2, 0)$ provided this is integral, which is again the condition $x = c \pmod{2}$. So if c is odd this is reducible at $x = 1, 3, \dots, c, c + 2, \dots$

Take $c = 1$. Then $x = 3$ gives infinitesimal character $(\frac{x+c}{2}, \frac{x-c}{2}) = (2, 1) = \rho$.

Starting at $x = 0$:

$$(14.17)(a) \quad I(x_9, 1, x) = I_K(x_9, 1) \quad (x < 1).$$

Reducibility at $x = 1$:

$$(14.17)(b) \quad I(x_9, 1, 1) = J(x_9, 1, 1) + J(x_0, (1, 0)) + J(x_1, (1, 0))$$

The last two terms are limits of large discrete series. By (14.12) (since we're at integral infinitesimal character all orientation numbers are 0):

$$(14.17)(c) \quad I(x_9, 1, 1)_c = I_K(x_9, 1)_c + (1 - s)[I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c]$$

Reducibility at $x = 3$; the infinitesimal character is $\frac{1}{2}(3+1, 3-1) = (2, 1) = \rho$, so this is directly from (10.2)(b):

$$(14.17)(d) \quad \begin{aligned} I(x_9, 1, 3) &= J(x_9, 1, 3)^2 \\ &\quad + J(x_0, (2, 1))^0 + J(x_1, (2, 1))^0 \\ &\quad + J(x_9, 3, 1)^1 \\ &\quad + J(x_7, (\bar{1}, 2), (1, 0))^1 + J(x_8, (\bar{1}, 2), (1, 0))^1 \end{aligned}$$

with lengths denoted by superscripts. Therefore (no orientation numbers here; and recall that only modules with odd length difference occur in the sum (14.12)):

$$(14.17)(e) \quad \begin{aligned} I(x_9, 1, 3)_c &= I(x_9, 1, 1)_c \\ &\quad + (1-s)[J(x_9, 3, 1)_c + J(x_7, (\bar{1}, 2), (1, 0))_c + J(x_8, (\bar{1}, 2), (1, 0))_c] \end{aligned}$$

We'll plug in (c) for the first term.

Now for the first time we need to express the c-invariant form on an irreducible (each of the three in the last line) in terms of c-invariant forms on standards, using Section 13, which goes back to the character formulas of Section 11 in this case. Thus by (11.2)(a)

$$(14.17)(f) \quad J(x_9, 3, 1)_c = I(x_9, 3, 1)_c - J(x_0, (2, 1))_c - J(x_1, (2, 1))_c$$

and plug in (14.14)(h) for $I(x_9, 3, 1)$ to give:

$$(14.17)(g) \quad \begin{aligned} J(x_9, 3, 1)_c &= I_K(x_9, 3)_c \\ &\quad + (1-s)[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c] \\ &\quad - I_K(x_0, (2, 1))_c - I_K(x_1, (2, 1))_c \end{aligned}$$

which simplifies to

$$(14.17)(h) \quad \boxed{J(x_9, 3, 1)_c = I_K(x_9, 3)_c - s[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c]}$$

Similarly (11.2)(c) says:

$$(14.17)(i) \quad \begin{aligned} I(x_7, (\bar{1}, 2), (1, 0))_c &= I(x_7, (\bar{1}, 2), (1, 0))_c - I(x_0, (2, 1))_c - I(x_2, (2, 1))_c. \end{aligned}$$

Use (14.16)(c) to expand

$$(14.17)(j) \quad \begin{aligned} I(x_7, (\bar{1}, 2), (1, 0))_c &= I_K(x_7, (\bar{1}, 2))_c + (1-s)[J(x_0, (2, 1))_c + J(x_2, (2, 1))_c] \end{aligned}$$

and so

(14.17)(k)

$$J(x_7, (\bar{1}, 2), (1, 0))_c = I_K(x_7, (\bar{1}, 2))_c - s[I_K(x_0, (2, 1))_c + I_K(x_2, (2, 1))_c]$$

Finally (11.2)(d) says:

(14.17)(l)

$$J(x_8, (\bar{1}, 2), (1, 0))_c = I(x_8, (\bar{1}, 2), (1, 0))_c - I(x_1, (2, 1))_c - I(x_3, (2, 1))_c.$$

and using (14.16)(d) to expand

(14.17)(m)

$$I(x_8, (\bar{1}, 2), (1, 0))_c = I_K(x_8, (\bar{1}, 2))_c + (1 - s)[J(x_1, (2, 1))_c + J(x_3, (2, 1))_c]$$

gives

(14.17)(n)

$$I(x_8, (\bar{1}, 2), (1, 0))_c = I_K(x_8, (\bar{1}, 2))_c - s[I_K(x_1, (2, 1))_c + I_K(x_3, (2, 1))_c].$$

Plugging (c), (h), (k) and (n) into (e) gives:

(14.17)(o)

$$\begin{aligned} I(x_9, 1, 3)_c &= I_K(x_9, 1)_c + (1 - s)[I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c] \\ &\quad + (1 - s)\{I_K(x_9, 3)_c - s[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c]\} \\ &\quad + (1 - s)\{I_K(x_7, (\bar{1}, 2))_c - s[I_K(x_0, (2, 1))_c + I_K(x_2, (2, 1))_c]\} \\ &\quad + (1 - s)\{I_K(x_8, (\bar{1}, 2))_c - s[I_K(x_1, (2, 1))_c + I_K(x_3, (2, 1))_c]\}. \end{aligned}$$

Grouping terms finally gives:

(14.17)(p)

$$\begin{aligned} I(x_9, 1, 3)_c &= I_K(x_9, 1)_c + (1 - s)[I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c] \\ &\quad + (1 - s)\{2 \times I_K(x_0, (2, 1))_c + 2 \times I_K(x_1, (2, 1))_c + I_K(x_2, (2, 1))_c + I_K(x_3, (2, 1))_c\} \\ &\quad + (1 - s)\{I_K(x_9, 3)_c + I_K(x_7, (\bar{1}, 2))_c + I_K(x_8, (\bar{1}, 2))_c\}. \end{aligned}$$

length 2, Cartan 2 We compute $I(x_7, (\bar{2}, 1), (2, 0))_c$ and $I(x_8, (\bar{2}, 1), (2, 0))_c$.

For $k = 7, 8$, let's deform $I(x_k, (\bar{2}, 1), (2, 0))$ to $I(x_k, (\bar{2}, 1), (0, 0))$. By the parity condition $I(x_k, (\bar{2}, 1), (x, 0))$ is reducible only if $x = \bar{2} \pmod{2\mathbb{Z}}$. We're taking $0 \leq x \leq 2$, so this occurs only at the endpoints $x = 0, x = 2$.

At $x = 0$ this has to do with the fact that $I(x_k, (\bar{2}, 1), (0, 0))$ is not final. Ignoring this for the moment, consider $I(x_k, (\bar{2}, 1), (2, 0))$, at infinitesimal character $(2, 1)$.

The composition series at $(2, 1)$ are given by (10.2)(e,f) (with lengths given by superscripts)

$$\begin{aligned}
(14.18)(a) \quad & I(x_7, (\bar{2}, 1), (2, 0))^2 = J(x_7, (\bar{2}, 1), (2, 0))^2 + J(x_0, (2, 1))^0 \\
& \quad + J(x_9, 3, 1)^1 + J(x_7, (\bar{1}, 2), (1, 0))^1 \\
& I(x_8, (\bar{2}, 1), (2, 0)) = J(x_8, (\bar{2}, 1), (2, 0))^2 + J(x_1, (2, 1))^0 \\
& \quad + J(x_9, 3, 1)^1 + J(x_8, (\bar{1}, 2), (1, 0))^1
\end{aligned}$$

Considering terms of odd length, with the orientation numbers being all 0, (14.12) gives

$$\begin{aligned}
(14.18)(b) \quad & I(x_7, (\bar{2}, 1), (2, 0))_c = I(x_7, (\bar{2}, 1), (0, 0))_c \\
& \quad + (1-s)[J(x_9, 3, 1)_c + J(x_7, (\bar{1}, 2), (1, 0))_c] \\
& I(x_8, (\bar{2}, 1), (2, 0))_c = I(x_8, (\bar{2}, 1), (0, 0))_c \\
& \quad + (1-s)[J(x_9, 3, 1)_c + J(x_8, (\bar{1}, 2), (1, 0))_c]
\end{aligned}$$

Here we are using that $I(x_k, (\bar{2}, 1), (2-\epsilon, 0))_c = I(x_k, (\bar{2}, 1), (0, 0))_c$, since there is no reducibility for $0 < x < 2$. We know $J(x_9, 3, 1)_c$, $J(x_7, (\bar{1}, 2), (1, 0))_c$, $J(x_8, (\bar{1}, 2), (1, 0))_c$ from (14.17)(h),(k) and (n), respectively. Also use (5.4)(a) to eliminate the terms $I(x_k, (\bar{2}, 1), (0, 0))_c$ with $k = 7, 8$. Plugging these in gives:

$$\begin{aligned}
(14.18)(c) \quad & I(x_7, (\bar{2}, 1), (2, 0))_c = I(x_0, (1, 0))_c + I(x_2, (1, 0))_c \\
& \quad + (1-s)[\{I_K(x_9, 3)_c - s[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c]\} \\
& \quad \quad + \{I_K(x_7, (\bar{1}, 2))_c - s[I_K(x_0, (2, 1)) + I_K(x_2, (2, 1))]\}]
\end{aligned}$$

and this can be rewritten (recall that $(1-s)(-s) = 1-s$)

$$\begin{aligned}
(14.18)(d) \quad & \boxed{I(x_7, (\bar{2}, 1), (2, 0))_c = I(x_0, (1, 0))_c + I(x_2, (1, 0))_c} \\
& \quad + (1-s)[I_K(x_9, 3)_c + I_K(x_7, (\bar{1}, 2))_c \\
& \quad \quad + 2 \times I(x_0, (2, 1))_c + I(x_1, (2, 1))_c + I(x_2, (2, 1))_c].
\end{aligned}$$

Similarly

$$\begin{aligned}
(14.18)(e) \quad & \boxed{I(x_8, (\bar{2}, 1), (2, 0))_c = I(x_1, (1, 0))_c + I(x_3, (1, 0))_c} \\
& \quad + (1-s)[I_K(x_9, 3)_c + I_K(x_8, (\bar{1}, 2))_c \\
& \quad \quad + I(x_0, (2, 1))_c + 2 \times I(x_1, (2, 1))_c + I(x_3, (2, 1))_c].
\end{aligned}$$

15 Digression: computing the Q polynomials

In the setting of the `atlas` software, suppose we have a block, with elements 0 - n . The `klbasis` command gives the polynomials $P_{\gamma,\gamma'}(q)$ on the block.

For example consider this part of the output for the big block of $Sp(4, \mathbb{R})$:

```
10:  0:  1
      1:  1
      2:  1
      3:  1
      4:  1
      5:  1
      6:  1
      7:  1
      8:  1
      9:  1
     10:  1
```

This says the polynomials $P_{0,10}(q), P_{1,10}(q), \dots, P_{10,10}(q)$ are all 1. Recall this means

$$J(10) = \sum_k (-1)^{\ell(10) - \ell(k)} P_{k,10}(q) I(k).$$

Taking lengths into account (from the `block` command) gives

$$J(10) = I(10) - I(9) - I(8) - I(7) + I(6) + I(5) + I(4) - I(3) - I(2) - I(1) - I(0)$$

This is the character formula for the trivial representation. See (11.2)(j).

On the other hand we're interested in the Q polynomials, which satisfy

$$I(n) = \sum_k Q_{k,n}(1) J(k)$$

Here's how to get these from `klbasis`.

Make G the dual group, and G^\vee the group. Then

$$Q_{k,n}(q) = P_{n^\vee, k^\vee}(q)$$

where $k \rightarrow k^\vee$ is the duality map, given by `dualmap`.

For example take $G = SO(3, 2)$, $G^\vee = Sp(4, \mathbb{R})$. Consider this part of the output of `klbasis`:

7: 0: 2
 1: 1
 2: 1
 3: 1
 4: 1
 7: 1

Also dualmap gives

[10, 11, 9, 7, 8, 5, 6, 4, 0, 1, 2, 3]

The first line says $P_{0,7}(q) = 2$ for $SO(3, 2)$, and dualmap says $0^\vee = 10, 7^\vee = 4$, so

$$2 = P_{0,7}^{SO}(q) = Q_{4,10}^{Sp}(q),$$

which says $J(4)$ has multiplicity 2 in $I(10)$ for $Sp(4, \mathbb{R})$. See (10.2)(j).

16 Second Digression: An alternative version of the calculation

Recall we used (14.12) to write the form on a standard module $I(x, \lambda, \nu)$ in terms of forms on a standard module with smaller ν parameter, and forms on more tempered irreducible modules $J(\gamma')$:

$$(16.1) \quad I(x, \lambda, t_k \nu)_c = I(x, \lambda, t_{k-1} \nu)_c + (1-s) \sum_{\substack{\gamma' \\ \ell(\gamma') - \ell(\gamma) \text{ odd}}} s^{\frac{\ell_0(\gamma) - \ell_0(\gamma')}{2}} Q_{\gamma', \gamma}(s) J(\gamma')_c$$

We proceeded by induction, assuming we'd already computed the terms $J(\gamma')_c$.

Here is an equivalent formulation, which combines the two steps, and is easier from a computational point of view. We restate this in a self-contained, single step.

Inductive algorithm

Suppose we are given $I(\gamma) = I(x, \lambda, \nu)$. Typically this is at singular, and possibly nonintegral, infinitesimal character. For small $\epsilon > 0$, $I(x, \lambda, (1+\epsilon)\nu)$

is irreducible. By (14.12) we have

$$(16.2)(a) \quad \begin{aligned} I(\gamma)_c &= I(x, \lambda, (1 - \epsilon)\nu)_c \\ &+ (1 - s) \sum_{\substack{\gamma' < \gamma \\ \ell(\gamma) - \ell(\gamma') \text{ odd}}} s^{\frac{\ell_0(\gamma) - \ell_0(\gamma')}{2}} Q_{\gamma', \gamma}(s) J(\gamma')_c \end{aligned}$$

By (13.2)(c), for each γ' , write

$$(16.2)(b) \quad J(\gamma')_c = \sum_{\delta' \leq \gamma'} (-1)^{\ell(\gamma') - \ell(\delta')} s^{\frac{\ell_0(\gamma') - \ell_0(\delta')}{2}} P_{\delta', \gamma'}(s) I(\delta')_c$$

This formula is computed at (possibly singular, non-integral) infinitesimal character. Plug it in to give

$$(16.2)(c) \quad \begin{aligned} I(\gamma)_c &= I(x, \lambda, (1 - \epsilon)\nu)_c \\ &+ (1 - s) \sum_{\substack{\delta' \leq \gamma' < \gamma \\ \ell(\gamma) - \ell(\gamma') \text{ odd}}} (-1)^{\ell(\gamma') - \ell(\delta')} s^{\frac{\ell_0(\gamma) - \ell_0(\delta')}{2}} P_{\delta', \gamma'}(s) Q_{\gamma', \gamma}(s) I(\delta')_c \end{aligned}$$

or, spelling it out more explicitly:

(16.2)(d)

$$\begin{aligned} I(\gamma)_c &= I(x, \lambda, (1 - \epsilon)\nu)_c \\ &+ (1 - s) \sum_{\substack{\delta' < \gamma \\ \delta' \leq \gamma' < \gamma \\ \ell(\gamma) - \ell(\gamma') \text{ odd}}} s^{\frac{\ell_0(\gamma) - \ell_0(\delta')}{2}} \left[\sum_{\substack{\delta' \leq \gamma' < \gamma \\ \ell(\gamma) - \ell(\gamma') \text{ odd}}} (-1)^{\ell(\gamma') - \ell(\delta')} P_{\delta', \gamma'}(s) Q_{\gamma', \gamma}(s) \right] I(\delta')_c \end{aligned}$$

17 Invariant Forms

Suppose J is a representation of $Sp(4, \mathbb{R})$ with a central character, and μ is a K -type of π . Identify μ with its highest weight (r, s) .

The element τ defining $Sp(4, \mathbb{R})$ is $\text{diag}(i, i, -i, -i)$, which is central in K , and has square $-I$. It acts in μ by the scalar i^{r+s} . Note that if $-I$ acts in J by $\epsilon = \pm 1$ then $(i^{r+s})^2 = \epsilon$, i.e.

$$r + s \text{ is } \begin{cases} \text{even} & \epsilon = 1 \\ \text{odd} & \epsilon = -1. \end{cases}$$

Now assume J is irreducible and $\mu = (r, s)$ is a lowest K -type of J . Write \langle , \rangle for the *invariant* form on J , and \langle , \rangle_c for the c -invariant form as usual.

Lemma 17.1 *Suppose we have a formula as in (13.2)(a)*

$$(17.2) \quad J(x, \lambda, \nu)_c = \sum_{x', \lambda'} a(x', \lambda') I_K(x', \lambda')_c.$$

Suppose $-I$ acts by ϵ in $J(x, \lambda, \nu)$ and choose $\zeta^2 = \epsilon$. Write the (unique) lowest K -type of $I_K(x', \lambda')$ as $(r(x', \lambda'), s(x', \lambda'))$. Define

$$\delta(x', \lambda') = \begin{cases} 1 & \zeta i^{r(x', \lambda') + s(x', \lambda')} = 1 \\ s & \zeta i^{r(x', \lambda') + s(x', \lambda')} = -1. \end{cases}$$

Then an invariant form on J is given by

$$(17.3) \quad J(x, \lambda, \nu)_0 = \sum_{x', \lambda'} \delta(x', \lambda') a(x', \lambda') I_K(x', \lambda')_0.$$

where $I_K(x', \lambda')_0$ is the unique positive definite invariant form. There is one other invariant form, $-$ this one.

In particular an invariant form on J is definite if and only if, for all x', λ' appearing in (17.2),

$$(17.4)(a) \quad \delta(x', \lambda') a(x', \lambda') \in \mathbb{Z} \text{ for all } (x', \lambda')$$

or

$$(17.4)(b) \quad \delta(x', \lambda') a(x', \lambda') \in s\mathbb{Z} \text{ for all } (x', \lambda')$$

17.1 Some Invariant Forms on Irreducibles for $Sp(4, \mathbb{R})$

We have some formulas for c -invariant forms on some irreducible representations: (14.17)(h,k,n) on $J(x_9, 3, 1)$, $J(x_7, (\bar{1}, 2), (1, 0))$ and $J(x_8, (\bar{1}, 2), (1, 0))$, all at ρ . Let's convert these to invariant forms.

These representations all have trivial central character. Therefore, we may take $\zeta = 1$ in Lemma 17.1. If (r, s) is a lowest K -type then the sign in the lemma is $(-1)^{\frac{r+s}{2}}$.

First consider $J(x_8, (\bar{1}, 2), (1, 0))$.

Equation (14.17)(n) says

(17.1.5)(a)

$$\boxed{J(x_8, (\bar{1}, 2), (1, 0))_c = I_K(x_8, (\bar{1}, 2))_c - s[I_K(x_1, (2, 1))_c + I_K(x_3, (2, 1))_c]}$$

By Section 5 the lowest K -types on the right hand side are $(-1, -3)$, $(1, -3)$ and $(-3, -3)$, respectively, which have $(-1)^{\frac{r+s}{2}} = 1, -1, -1$, respectively. Therefore

(17.1.5)(b)

$$\boxed{J(x_8, (\bar{1}, 2), (1, 0))_0 = I_K(x_8, (\bar{1}, 2))_0 - [I_K(x_1, (2, 1))_0 + I_K(x_3, (2, 1))_0]}.$$

This representation is unitary. This is representation 6 from the output of `block`.

Of course, using (14.17)(k), $J(x_7, (\bar{1}, 2), (1, 0))_0$ is essentially the same:

(17.1.5)(c)

$$\boxed{J(x_7, (\bar{1}, 2), (1, 0))_0 = I_K(x_7, (\bar{1}, 2))_0 - [I_K(x_0, (2, 1))_0 + I_K(x_2, (2, 1))_0]}.$$

and this is unitary. This is representation 5 from the output of `block`.

Finally (14.17)(h) says

$$(17.1.5)(d) \quad J(x_9, 3, 1)_c = I_K(x_9, 3) - s[I_K(x_0, (2, 1)) + I_K(x_1, (2, 1))]$$

and the invariant formula is

$$(17.1.5)(e) \quad \boxed{J(x_9, 3, 1)_0 = I_K(x_9, 3)_0 - [I_K(x_0, (2, 1))_0 + I_K(x_1, (2, 1))_0]}$$

which is unitary. This is representation 4 from the output of `block`.

Remark 17.1.6 *Using the `blocku` command we can confirm the unitarity. These representations are indeed $A_q(\lambda)$ modules attached to theta stable parabolic subalgebras with Levi factors $L = U(1) \times SL(2, \mathbb{R})$ and $L = U(1, 1)$, respectively.*

18 c-Invariant Forms on Irreducible Representations

We already have some formulas for c-invariant forms on some irreducible representations: (14.17)(h,k,n) on $J(x_9, 3, 1)$, $J(x_7, (\bar{1}, 2), (1, 0))$ and $J(x_8, (\bar{1}, 2), (1, 0))$, all at ρ .

18.1 The c-invariant form on $J(x_9, 1, 3)$

Let's use (14.17)(p) to get a formula for $J(x_9, 1, 3)_c$. Recall $J(x_9, 1, 3) = J(x_9, (1, 0), \frac{1}{2}(3, 3))$, irreducible representation **9** from the output of **block** (see Section 8). The formula for $J(x_9, 1, 3)_c$ is given by (11.2)(b) with $a = 2, b = 1$:

$$(18.1.1) \quad \begin{aligned} J(x_9, 1, 3)_c = & I(x_9, 1, 3)_c \\ & + I(x_0, (2, 1))_c + I(x_1, (2, 1))_c + I(x_2, (2, 1))_c + I(x_3, (2, 1))_c \\ & - I(x_9, 3, 1)_c \\ & - I(x_7, (\bar{1}, 2), (1, 0))_c - I(x_8, (\bar{1}, 2), (1, 0))_c \end{aligned}$$

The terms on the right hand side are given by (14.17)(p), (14.13) (4 times), (14.14)(h) and (14.15)(d,e), respectively. The last three are also given in (14.16)(a,d,e). This gives:

$$(18.1.2) \quad \begin{aligned} J(x_9, 1, 3)_c = & \{ I_K(x_9, 1)_c + (1-s)[I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c] \\ & + (1-s)\{2I_K(x_0, (2, 1))_c + 2I_K(x_1, (2, 1))_c\} \\ & + (1-s)\{I_K(x_2, (2, 1))_c + I_K(x_3, (2, 1))_c\} \\ & + (1-s)\{I_K(x_9, 3)_c + I_K(x_7, \bar{1}, 2)_c + I_K(x_8, \bar{1}, 2)_c\} \} \\ & + I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c + I_K(x_2, (2, 1))_c + I_K(x_3, (2, 1))_c \\ & - \{ I_K(x_9, 3)_c + (1-s)[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c] \} \\ & - \{ I_K(x_7, (\bar{1}, 2))_c + (1-s)[I_K(x_0, (2, 1))_c + I_K(x_2, (2, 1))_c] \} \\ & - \{ I_K(x_8, (\bar{1}, 2))_c + (1-s)[I_K(x_1, (2, 1))_c + I_K(x_3, (2, 1))_c] \} \end{aligned}$$

Here is a table of the terms:

k	λ	coefficients	total	(r, s)	δ
9	1	1	1	(1, -1)	1
0	(1, 0)	$1 - s$	$1 - s$	(2, 0)	s
1	(1, 0)	$1 - s$	$1 - s$	(0, -2)	s
0	(2, 1)	$2(1 - s) + 1 - (1 - s) - (1 - s)$	1	(3, -1)	s
1	(2, 1)	$2(1 - s) + 1 - (1 - s) - (1 - s)$	1	(1, -3)	s
2	(2, 1)	$(1 - s) + 1 - (1 - s)$	1	(3, 3)	s
3	(2, 1)	$(1 - s) + 1 - (1 - s)$	1	(-3, -3)	s
9	3	$(1 - s) - 1$	$-s$	(2, -2)	1
7	($\bar{1}$, 2)	$(1 - s) - 1$	$-s$	(3, 1)	1
8	($\bar{1}$, 2)	$(1 - s) - 1$	$-s$	(-1, -3)	1

And the answer is:

(18.1.3)

$$\begin{aligned}
J(x_9, 1, 3)_c &= I_K(x_9, 1)_c + \\
&\quad + (1 - s) [I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c] \\
&\quad + [I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c + I_K(x_2, (2, 1))_c + I_K(x_3, (2, 1))_c] \\
&\quad - s [I_K(x_9, 3)_c + I_K(x_7, (\bar{1}, 2))_c + I_K(x_8, (\bar{1}, 2))_c]
\end{aligned}$$

Note that if $s = 1$ this gives (11.2)(b) again, as it must.

Using Section 17, the invariant form is given as follows.

(18.1.4)

$$\begin{aligned}
J(x_9, 1, 3)_0 &= I_K(x_9, 1)_0 + \\
&\quad + (s - 1) [I_K(x_0, (1, 0))_0 + I_K(x_1, (1, 0))_0] \\
&\quad + s [I_K(x_0, (2, 1))_0 + I_K(x_1, (2, 1))_0 + I_K(x_2, (2, 1))_0 + I_K(x_3, (2, 1))_0] \\
&\quad - s [I_K(x_9, 3)_0 + I_K(x_7, (\bar{1}, 2))_0 + I_K(x_8, (\bar{1}, 2))_0]
\end{aligned}$$

This is not unitary. Note that we can tell this from the previous expression because of the hyperbolic term $(1 - s)$.

18.2 The c-invariant form on $J(x_{7,8}, (\bar{2}, 1), (2, 0))$

The character formula for $J(x_7, (\bar{2}, 1), (2, 0))$ is (11.2)(e), and this holds as a formula for c-invariant forms:

$$\begin{aligned}
(18.2.5) \quad J(x_7, (\bar{2}, 1), (2, 0))_c &= I(x_7, (\bar{2}, 1), (2, 0))_c \\
&+ I(x_0, (2, 1))_c + I(x_1, (2, 1))_c + I(x_2, (2, 1))_c \\
&- I(x_9, (3, 1))_c - I(x_7, (\bar{1}, 2), (1, 0))_c
\end{aligned}$$

The non-discrete series terms on the right hand side are given by (14.18)(d), (14.14)(h) and (14.16)(c). The result is:

(18.2.6)

$$\begin{aligned}
J(x_7, (\bar{2}, 1), (2, 0))_c &= -s[I(x_9, 3)_c + I(x_7, (\bar{1}, 2))_c] \\
&+ [I(x_0, (1, 0))_c + I(x_2, (1, 0))_c] \\
&+ [I(x_0, (2, 1))_c + I(x_1, (2, 1))_c + I(x_2, (2, 1))_c]
\end{aligned}$$

The lowest K -types of these terms are: $(2, -2), (3, 1), (2, 0), (2, 2), (3, -1), (1, -3), (3, 3)$. We multiply by $1, 1, s, 1, s, s, s$ respectively, to give the invariant form:

(18.2.7)

$$\begin{aligned}
J(x_7, (\bar{2}, 1), (2, 0))_0 &= -s[I(x_9, 3)_0 + I(x_7, (\bar{1}, 2))_0] \\
&+ [sI(x_0, (1, 0))_0 + I(x_2, (1, 0))_0] \\
&+ s[I(x_0, (2, 1))_0 + I(x_1, (2, 1))_0 + I(x_2, (2, 1))_0]
\end{aligned}$$

This representation is *not* unitary: the signs differ on the two lowest K -types $(2, 2)$ and $(2, 0)$.

Similarly:

(18.2.8)

$$\begin{aligned}
J(x_8, (\bar{2}, 1), (2, 0))_c &= -s[I(x_9, 3)_c + I(x_8, (\bar{1}, 2))_c] \\
&+ [I(x_1, (1, 0))_c + I(x_3, (1, 0))_c] \\
&+ [I(x_0, (2, 1))_c + I(x_1, (2, 1))_c + I(x_3, (2, 1))_c]
\end{aligned}$$

19 The Trivial Representation

Let's prove the trivial representation $J(x_{10}, (\bar{2}, \bar{1}), (2, 1))$ is unitary.

The deformation will go as follows. Let $\nu = (1 + \epsilon)(2, 1)$ for $\epsilon > 0$ small. Deforming ν to 0 we have the following reducibility points, up to small deformations.

$$(19.1)(a) \quad \nu = (2, 1), \left(\frac{4}{3}, \frac{2}{3}\right), \left(1, \frac{1}{2}\right), \left(\frac{2}{3}, \frac{1}{3}\right)$$

At the first point the integral root system is C_2 ; at all the others it is of type A_1 .

Here is the reducibility at $\nu = \left(\frac{2}{3}, \frac{1}{3}\right)$, obtained using `nblock`.

The input is $\lambda = \rho = (2, 1)$, so $\lambda - \rho = (0, 0)$.

```

real: nblock
choose Cartan class (one of 0,1,2,3): 3
Choosing the unique KGB element for the Cartan class:
10: 3 [r,r] 10 10 * * (0,0)#3 1^2x1^e
rho = [1,1]/1
Give lambda-rho: 0 0
denominator for nu: 3
numerator for nu: 1 1
x=10, lambda=[1,1]/1, gamma=[1,1]/3.
Name an output file (return for stdout, ? to abandon):
Subsystem on dual side is of type A1, with roots 6.
Given parameters define element 1 of the following block:
0(0,2): 0 [i2] 0 (1,2) *(x= 4, nu= [2,-1]/6,,lam=rho+ [-2,0]) 2,1,2
1(1,0): 1 [r2] 2 (0,*) *(x=10, nu= [1,1]/3,,lam=rho+[-2,-2]) e
2(1,1): 1 [r2] 1 (0,*) *(x=10, nu= [1,1]/3,,lam=rho+[-2,-1]) e
KL polynomials (-1)^[1(1)-1(x)]*P_{x,1}:
0: -1
1: 1

```

This says:

$$(19.1)(b) \quad I(x_{10}, (\bar{2}, \bar{1}), \left(\frac{2}{3}, \frac{1}{3}\right)) = J(x_{10}, (\bar{2}, \bar{1}), \left(\frac{2}{3}, \frac{1}{3}\right)) + J(x_9, 1, \frac{1}{3}).$$

Dangerous Bend: Here come some orientation numbers.

Applying (14.12) as usual, for the first time we have some nontrivial orientation numbers.

By Table 9.3.4, $(\bar{a} = 0, \bar{b} = 1, y = \frac{1}{3})$, $\ell_0(x_{10}, (\bar{2}, \bar{1}), \left(\frac{2}{3}, \frac{1}{3}\right)) = 3$. On the other hand by Table (9.1).2, with $(c = 1, x = \frac{1}{3})$, $\ell_0(x_9, 1, \frac{1}{3}) = 1$. The orientation number contribution to (14.12) is $s^{\frac{1}{2}(3-1)} = s$, so:

$$(19.1)(c) \quad I(x_{10}, (\bar{2}, \bar{1}), (\frac{2}{3}, \frac{1}{3}))_c = I_K(x_{10}, (\bar{2}, \bar{1}))_c + (1-s)sI_K(x_9, 1)_c \\ = I_K(x_{10}, (\bar{2}, \bar{1}))_c + (s-1)I_K(x_9, 1)_c.$$

The next reducibility point is $(1, \frac{1}{2})$. We're interested in $I(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2}))$. This is of type A_{sn}^I in the notation of [4, Section 6]. Here is `nblock`:

```
real: nblock
choose Cartan class (one of 0,1,2,3): 3
Choosing the unique KGB element for the Cartan class:
10: 3 [r,r] 10 10 * * (0,0)#3 1^2x1^e
rho = [1,1]/1
Give lambda-rho: 0 0
denominator for nu: 2
numerator for nu: 1 1
x=10, lambda=[1,1]/1, gamma=[1,1]/2.
Name an output file (return for stdout, ? to abandon):
Subsystem on dual side is of type A1, with roots 7.
Given parameters define element 2 of the following block:
0(0,1): 0 [i1] 1 (2,*) *(x= 5, nu= [-1,1]/2, lam=rho+ [0,-1]) 1,2,1
1(1,1): 0 [i1] 0 (2,*) *(x= 6, nu= [-1,1]/2, lam=rho+ [0,-1]) 1,2,1
2(2,0): 1 [r1] 2 (0,1) *(x=10, nu= [1,1]/2, lam=rho+[-2,-2]) e
KL polynomials (-1)^(1(2)-1(x))*P_{x,2}:
0: -1
1: -1
2: 1
```

This says

$$(19.1)(d) \quad I(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2})) = J(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2})) + J(x_5, (1, \bar{0}), (0, \frac{1}{2})) + J(x_6, (1, \bar{0}), (0, \frac{1}{2}))$$

or alternatively

$$(19.1)(e) \quad I(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2})) = J(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2})) + J(x_7, (\bar{0}, 1), (\frac{1}{2}, 0)) + J(x_8, (\bar{0}, 1), (\frac{1}{2}, 0))$$

Now apply (14.12). There is an orientation number here: $\ell_0(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2})) = 2$ (Table 9.3.1) and $\ell_0(x_{7,8}, (\bar{0}, 1), (\frac{1}{2}, 0)) = 0$ (Table 9.2.1). So:

$$(19.1)(f) \quad I(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2}))_c = I(x_{10}, (\bar{2}, \bar{1}), (\frac{2}{3}, \frac{1}{3}))_c \\ + (1-s)s[I_K(x_7, (\bar{0}, 1))_c + I_K(x_8, (\bar{0}, 1))_c]$$

Plugging in (c) gives

$$(19.1)(g) \quad I(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2}))_c = I_K(x_{10}, (\bar{2}, \bar{1}))_c + (s-1)I_K(x_9, 1)_c \\ + (s-1)[I_K(x_7, (\bar{0}, 1))_c + I_K(x_8, (\bar{0}, 1))_c]$$

However there is one more step: $I(x_7, (\bar{0}, 1))$ and $I(x_8, (\bar{0}, 1))$ are not final. Use (5.4)(a) to give

$$(19.1)(h) \quad I(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2}))_c = I_K(x_{10}, (\bar{2}, \bar{1}))_c + (s-1)I_K(x_9, 1)_c \\ + (s-1)[I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c + I_K(x_2, (1, 0))_c + I_K(x_3, (1, 0))_c]$$

Let's move on to $(\frac{4}{3}, \frac{2}{3})$, i.e. $I(x_{10}, (\bar{2}, \bar{1}), (\frac{4}{3}, \frac{2}{3}))$. This standard module is irreducible. The integral real root is $(1, 1)$. By the table in Section 6 this fails the parity condition, so doesn't give reducibility. This is also what `nblock` says:

```
real: nblock
choose Cartan class (one of 0,1,2,3): 3
Choosing the unique KGB element for the Cartan class:
10: 3 [r,r] 10 10 * * (0,0)#3 1^2x1^e
rho = [1,1]/1
Give lambda-rho: 0 0
denominator for nu: 3
numerator for nu: 2 2
x=10, lambda=[1,1]/1, gamma=[2,2]/3.
Name an output file (return for stdout, ? to abandon):
Subsystem on dual side is of type A1, with roots 6.
Given parameters define element 0 of the following block:
0(0,0): 0 [rn] 0 (*,*) *(x=10, nu= [2,2]/3, lam=rho+[-2,-2]) e
KL polynomials (-1)^{l(0)-l(x)}*P_{x,0}:
0: 1
```

Note that in [4, Section 6], case A_{sn}^I , this example doesn't appear (meaning it is irreducible); in that notation we'd be considering $J(x, \bar{2}e, (\frac{4}{3}, \frac{2}{3}))$, but what appears in the table is $J(x, \bar{2}o, (\frac{4}{3}, \frac{2}{3}))$.

Finally we come to $\nu = (2, 1)$. By (10.2)(j) with $a = 2, b = 1$:

$$\begin{aligned}
(19.1)(i) \quad I(x_{10}, (\bar{2}, \bar{1}), (2, 1))^3 = & \\
& J(x_{10}, (\bar{2}, \bar{1}), (2, 1))^3 \\
& + J(x_0, (2, 1))^0 + J(x_1, (2, 1))^0 \\
& + 2 \times J(x_9, 3, 1)^1 \\
& + J(x_7, (\bar{1}, 2), (1, 0))^1 + J(x_8, (\bar{1}, 2), (1, 0))^1 \\
& + J(x_7, (\bar{2}, 1), (2, 0))^2 + J(x_8, (\bar{2}, 1), (2, 0))^2 \\
& + J(x_9, 1, 3)^2
\end{aligned}$$

Remember $J(x_9, c, x) = J(x_9, (c, 0), \frac{1}{2}(x, x))$, with infinitesimal character $\frac{1}{2}(x + c, x - c)$.

We're at integral infinitesimal character, so there are no orientation numbers, so (14.12) gives

$$\begin{aligned}
(19.1)(j) \quad I(x_{10}, (\bar{2}, \bar{1}), (2, 1))_c = & I(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2}))_c \\
& + (1 - s) [I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c] \\
& + (1 - s) [J(x_7, (\bar{2}, 1), (2, 0))_c + J(x_8, (\bar{2}, 1), (2, 0))_c + J(x_9, 1, 3)_c]
\end{aligned}$$

and plugging in (h) for $J(x_{10}, (\bar{2}, \bar{1}), (1, \frac{1}{2}))_c$ gives

$$\begin{aligned}
(19.1)(k) \quad I(x_{10}, (\bar{2}, \bar{1}), (2, 1))_c = & I_K(x_{10}, (\bar{2}, \bar{1}))_c \\
& + (s - 1)I_K(x_9, 1)_c \\
& + (s - 1) [I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c + I_K(x_2, (1, 0))_c + I_K(x_3, (1, 0))_c] \\
& + (1 - s) [I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c] \\
& + (1 - s) [J(x_7, (\bar{2}, 1), (2, 0))_c + J(x_8, (\bar{2}, 1), (2, 0))_c + J(x_9, 1, 3)_c]
\end{aligned}$$

We still have to deal with the terms $J(x_7, (\bar{2}, 1), (2, 0))$, $J(x_8, (\bar{2}, 1), (2, 0))$ and $J(x_9, 1, 3)$.

These are available from (18.2.6), (18.2.8) and (18.1.3). So:

$$\begin{aligned}
(19.1)(1) \\
I(x_{10}, (\bar{2}, \bar{1}), (2, 1))_c &= I_K(x_{10}, (\bar{2}, \bar{1}))_c \\
&+ (s-1)I_K(x_9, 1)_c \\
&+ (s-1)[I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c + I_K(x_2, (1, 0))_c + I_K(x_3, (1, 0))_c] \\
&+ (1-s)[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c] \\
&+ (1-s)\{ \\
&- s[I(x_9, 3)_c + I(x_7, (\bar{1}, 2))_c] \\
&+ I(x_0, (1, 0))_c + I(x_2, (1, 0))_c \\
&+ I(x_0, (2, 1))_c + I(x_1, (2, 1))_c + I(x_2, (2, 1))_c \\
&- s[I(x_9, 3)_c + I(x_8, (\bar{1}, 2))_c] \\
&+ I(x_1, (1, 0))_c + I(x_3, (1, 0))_c \\
&+ I(x_0, (2, 1))_c + I(x_1, (2, 1))_c + I(x_3, (2, 1))_c \\
&+ I_K(x_9, 1)_c + (1-s)[I_K(x_0, (1, 0))_c + I_K(x_1, (1, 0))_c] \\
&+ I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c + I_K(x_2, (2, 1))_c + I_K(x_3, (2, 1))_c \\
&- s[I_K(x_9, 3)_c + I_K(x_7, (\bar{1}, 2))_c + I_K(x_8, \bar{1}, 2)_c] \quad \}
\end{aligned}$$

Here is a table of the terms:

k	λ	coefficients	total
10	$(\bar{2}, \bar{1})$	1	1
9	1	$(s-1) + (1-s)$	0
9	3	$(1-s) + (1-s) + (1-s)$	$3(1-s)$
8	$(\bar{1}, 2)$	$(1-s) + (1-s)$	$2(1-s)$
7	$(\bar{1}, 2)$	$(1-s) + (1-s)$	$2(1-s)$
3	$(2, 1)$	$(1-s) + (1-s)$	$2(1-s)$
2	$(2, 1)$	$(1-s) + (1-s)$	$2(1-s)$
1	$(2, 1)$	$(1-s) + (1-s) + (1-s) + (1-s)$	$4(1-s)$
0	$(2, 1)$	$(1-s) + (1-s) + (1-s) + (1-s)$	$4(1-s)$
3	$(1, 0)$	$(s-1) + (1-s)$	0
2	$(1, 0)$	$(s-1) + (1-s)$	0
1	$(1, 0)$	$(s-1) + (1-s) + (1-s)^2$	$2(1-s)$
0	$(1, 0)$	$(s-1) + (1-s) + (1-s)^2$	$2(1-s)$

The character formula for the trivial representation is (11.2)(j) with $a = 2, b = 1$, and this holds without change for c-invariant forms (see Section 13.1):

(19.1)(m)

$$\begin{aligned}
J(x_{10}, (\bar{2}, \bar{1}), (2, 1))_c &= I(x_{10}, (\bar{2}, \bar{1}), (2, 1))_c \\
&\quad - \{I(x_9, 1, 3)_c + I(x_7, (\bar{2}, 1), (2, 0))_c + I(x_8, (\bar{2}, 1), (2, 0))_c\} \\
&\quad + \{I(x_7, (\bar{1}, 2), (1, 0))_c + I(x_8, (\bar{1}, 2), (1, 0))_c + I(x_9, 3, 1)_c\} \\
&\quad - \{I(x_3, (2, 1))_c + I(x_2, (2, 1))_c + I(x_1, (2, 1))_c + I(x_0, (2, 1))_c\}
\end{aligned}$$

We know the terms on the right hand side:

(1) $I(x_{10}, (\bar{2}, \bar{1}), (2, 1))_c$: (19.1)(l)

(2) $I(x_9, 1, 3)_c$: (14.17)(p)

(3) $I(x_9, 3, 1)_c$: (14.14)(h)

(4) $I(x_k, (\bar{1}, 2), (1, 0))_c$ ($k = 7, 8$): (14.16)

(5) $I(x_k, (\bar{2}, 1), (2, 0))_c$ ($k = 7, 8$): (14.18)(d) and (e)

Of course the last four terms require no further comment.

So let's tabulate everything. The next table has a column for each standard module on the right hand side of (19.1)(m), each row is a limit K -representation $I_K(x, \lambda)$, and the columns give the multiplicities in the expression of the c-invariant form on the standard module.

	$I(x_{10}, (\bar{2}, \bar{1}), (2, 1))_c$	$I(x_9, 1, 3)_c$	$I(x_7, (\bar{2}, 1), (2, 0))_c$	$I(x_8, (\bar{2}, 1), (2, 0))_c$
$I_K(x_{10}, (\bar{2}, \bar{1}))_c$	1			
$I_K(x_9, 1)_c$	0	-1		
$I_K(x_9, 3)_c$	$3(1-s)$	$-(1-s)$	$-(1-s)$	$-(1-s)$
$I_K(x_8, (\bar{1}, 2))_c$	$2(1-s)$	$-(1-s)$		$-(1-s)$
$I_K(x_7, (\bar{1}, 2))_c$	$2(1-s)$	$-(1-s)$	$-(1-s)$	
$I_K(x_3, (2, 1))_c$	$2(1-s)$	$-(1-s)$		$-(1-s)$
$I_K(x_2, (2, 1))_c$	$2(1-s)$	$-(1-s)$	$-(1-s)$	
$I_K(x_1, (2, 1))_c$	$4(1-s)$	$-2(1-s)$	$-(1-s)$	$-2(1-s)$
$I_K(x_0, (2, 1))_c$	$4(1-s)$	$-2(1-s)$	$-2(1-s)$	$-(1-s)$
$I_K(x_3, (1, 0))_c$				-1
$I_K(x_2, (1, 0))_c$			-1	
$I_K(x_1, (1, 0))_c$	$2(1-s)$	$-(1-s)$		-1
$I_K(x_0, (1, 0))_c$	$2(1-s)$	$-(1-s)$	-1	

	$I(x_7, (\bar{1}, 2), (1, 0))_c$	$I(x_8, (\bar{1}, 2), (1, 0))_c$	$I(x_9, 3, 1)_c$	DS	Sum
$I_K(x_{10}, (\bar{2}, \bar{1}))_c$					1
$I_K(x_9, 1)_c$					-1
$I_K(x_9, 3)_c$			1		1
$I_K(x_8, (\bar{1}, 2))_c$		1			1
$I_K(x_7, (\bar{1}, 2))_c$	1				1
$I_K(x_3, (2, 1))_c$		$(1-s)$		-1	$-s$
$I_K(x_2, (2, 1))_c$	$(1-s)$			-1	$-s$
$I_K(x_1, (2, 1))_c$		$(1-s)$	$(1-s)$	-1	$-s$
$I_K(x_0, (2, 1))_c$	$(1-s)$		$(1-s)$	-1	$-s$
$I_K(x_3, (1, 0))_c$					-1
$I_K(x_2, (1, 0))_c$					-1
$I_K(x_1, (1, 0))_c$					$-s$
$I_K(x_0, (1, 0))_c$					$-s$

So:

$$\begin{aligned}
(19.1)(n) \quad & J(x_{10}, (\bar{2}, \bar{1}), (2, 1))_c = I_K(x_{10}, (\bar{2}, \bar{1}))_c \\
& + [I_K(x_9, 3)_c - I_K(x_9, 1)_c] \\
& + [I_K(x_7, (\bar{1}, 2))_c + I_K(x_8, (\bar{1}, 2))_c] \\
& - s[I_K(x_0, (2, 1))_c + I_K(x_1, (2, 1))_c + I_K(x_2, (2, 1))_c + I_K(x_3, (2, 1))_c] \\
& - [sI_K(x_0, (1, 0))_c + sI_K(x_1, (1, 0))_c + I_K(x_2, (1, 0))_c + I_K(x_3, (1, 0))_c]
\end{aligned}$$

To get the invariant form, we give the lowest K -types and the corresponding correction factors:

$I_K(x_{10}, (\bar{2}, \bar{1}))$	$(0, 0)$	1
$I_K(x_9, 1)$	$(1, -1)$	1
$I_K(x_9, 3)$	$(2, -2)$	1
$I_K(x_8, (\bar{1}, 2))$	$(-1, -3)$	1
$I_K(x_7, (\bar{1}, 2))$	$(3, 1)$	1
$I_K(x_3, (2, 1))$	$(-3, -3)$	s
$I_K(x_2, (2, 1))$	$(3, 3)$	s
$I_K(x_1, (2, 1))$	$(1, -3)$	s
$I_K(x_0, (2, 1))$	$(3, -1)$	s
$I_K(x_3, (1, 0))$	$(-2, -2)$	1
$I_K(x_2, (1, 0))$	$(2, 2)$	1
$I_K(x_1, (1, 0))$	$(0, -2)$	s
$I_K(x_0, (1, 0))$	$(2, 0)$	s

Taking this into account we compute the invariant form on the trivial representation $J(x_{10}, (\bar{2}, \bar{1}), (2, 1))$:

$$\begin{aligned}
(19.1)(o) \quad & J(x_{10}, (\bar{2}, \bar{1}), (2, 1))_0 = I_K(x_{10}, (\bar{2}, \bar{1})) \\
& + [I_K(x_9, 3) - I_K(x_9, 1)] \\
& + [I_K(x_7, (\bar{1}, 2)) + I_K(x_8, (\bar{1}, 2))] \\
& - [I_K(x_0, (2, 1)) + I_K(x_1, (2, 1)) + I_K(x_2, (2, 1)) + I_K(x_3, (2, 1))] \\
& - [I_K(x_0, (1, 0)) + I_K(x_1, (1, 0)) + I_K(x_2, (1, 0)) + I_K(x_3, (1, 0))].
\end{aligned}$$

Therefore the trivial representation of $Sp(4, \mathbb{R})$ is unitary. If anyone knows an easier proof please let me know.

References

- [1] Jeffrey Adams Some notes on parametrizing representations. www.liegroups.org/papers/basepoint.pdf.
- [2] Annegret Paul Assigning Representation Parameters to Atlas Block Output. www.liegroups.org/papers/Dictionary08.pdf.
- [3] David A. Vogan, Jr. Branching example: $Sp(4, \mathbb{R})$. www-math.mit.edu/~dav/sp4ex.pdf.
- [4] David A. Vogan, Jr. Forms example: $Sp(4, \mathbb{R})$. www.liegroups.org/papers/sp4forms.pdf.