Strong real forms and the Kac classification

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This paper is expository. It is a mild generalization of the Kac classification of real forms of a simple Lie group to *strong* real forms. The basic reference for strong real forms in this language is [1]. For the Kac classification we follow [6]. There is also a treatment in [3], in slightly different terms.

1 Real forms and strong real forms

Let G be a reductive algebraic group. We will occasionally identify algebraic groups with their complex points. We have the standard exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1 \tag{1.1}$$

where $\mathrm{Int}(G) \simeq G/Z(G)$ is the group of inner automorphisms of G, $\mathrm{Aut}(G)$ is the automorphims of G, and $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$.

Definition 1.2 1. *A real form of G is an equivalence class of involutions in $Aut(G)$, where equivalence is by conjugation by G, i.e. the action of $Int(G)$.*

2. *A traditional real form of G is an equivalence class of involutions, where equivalence is by the action of $Aut(G)$.*

The real form defined by θ has a maximal compact subgroup whose complexification is $K = G^\theta$.

Remark 1.3 A real forms are also defined by an antiholomorphic involution σ of $G(\mathbb{C})$, i.e. $G(\mathbb{R}) = G(\mathbb{C})^\sigma$. Given θ choose an antiholomorphic involution σ_0 so that $\theta\sigma_0 = \sigma_0\theta$ and $G(\mathbb{R})_0^\sigma$ is compact. Then the real form defined by θ is given by $\sigma = \theta\sigma_0$. See [4, Section VI.2].

We say two involutions $\theta, \theta' \in \mathrm{Aut}(G)$ are inner to each other, or in the same inner class, if they have the same image in $\mathrm{Out}(G)$. Such a class is determined by an involution $\gamma \in \mathrm{Out}(G)$, and we refer this inner class as the real forms of (G, γ).

We will work entirely in a fixed inner class, so fix an involution $\gamma \in \mathrm{Out}(G)$.

Fix a splitting datum for the exact sequence (1.1). This is a set $(H, B, \{X_\alpha\})$ consisting of a Cartan subgroup H, a Borel subgroup B containing H, and a set of simple root vectors. This induces a splitting $\mathrm{Out}(G) \to \mathrm{Aut}(G)$ of (1.1), and we let θ be the image of γ in $\mathrm{Aut}(G)$. Thus θ is an involution of G, it corresponds to the "most compact" real form in the given inner class. We let $K = G^\theta$.

Remark 1.4 Suppose G is simple and simply connected. It does not necessarily follow that K is simply connected; it is not simply connected if and only if the real form $G = G(\mathbb{R})$ of G corresponding to K has a non-linear cover. Since θ is the most compact inner form of (G, γ) K has a chance to be simply connected. In fact this holds unless $G = SL(2n+1)$, in which case $K = SO(2n+1)$ and $\pi_1(K) = \mathbb{Z}/2\mathbb{Z}$. This exception is due to the fact that Δ_θ (cf. Lemma B.1) is not reduced in this case. See the table in Section 3.1.

For most of these notes G will be semisimple, or even simple. Let $\Delta = \Delta(G, H)$ be the root system of H in G, and let $D = D(\Delta)$ be the Dynkin diagram of Δ. A choice of splitting datum induces an isomorphism

$$\mathrm{Out}(\mathfrak{g}) \simeq \mathrm{Aut}(D) \tag{1.5}$$

Furthermore $\mathrm{Out}(G) \subset \mathrm{Out}(\mathfrak{g})$, with equality if G is simply connected or adjoint. Thus γ is given by an involution of D.

Let

$$G^\Gamma = G \rtimes \langle \delta \rangle$$

where $\delta^2 = 1$ and $\delta g \delta^{-1} = \theta(g)$.

Definition 1.6 *A strong real form of* (G, γ) *is an equivalence class of elements* $x \in G^\Gamma$*, satisfying* $x \notin G$*, and* $x^2 \in Z(G)$*, where equivalence is by conjugation by* G*.*

The map $x \to \theta_x = int(x)$ is a surjection from the strong real forms of (G, γ) to the real forms of (G, γ). Let

$$H^\Gamma = H \rtimes \langle \delta \rangle \subset G^\Gamma.$$

Let T be the identity component of H^θ, and A be the identity component of $H^{-\theta} = \{h \in H \mid \theta(h) = h^{-1}\}$. Then $H = TA$. Let

$$T^\Gamma = T \times \langle \delta \rangle \subset H^\Gamma.$$

Remark 1.7 We may write

$$H \simeq \mathbb{C}^{*a} \times \mathbb{C}^{*b} \times (\mathbb{C}^* \times \mathbb{C}^*)^c \tag{1.8}$$

where θ acts trivially on the first a factors, by inverse on the next b, and $\theta(z, w) = (w, z)$ on each of the last c terms. Note that if $b \neq 0$ then T is a proper subset of H^θ. This happens, for example, in $SO(3, 1)$.

A key observation is that every element of $H\delta$ is conjugate to an element of $T\delta$, since A acts by conjugation on $H\delta$ by multiplication by A. That is for $a \in A, h \in H$, $a(h\delta)a^{-1} = ah(\delta a^{-1}\delta)\delta = ah\theta(a^{-1})\delta = a^2h\delta$. Since A is connected every element has a square root, so this gives multiplication by an arbitary element of A. Therefore $ta\delta$ is conjugate to $t\delta \in T\delta$. (In fact we could replace T with $H/A = T/T \cap A$, see ?.)

Lemma 1.9 *Suppose* $x \in G\delta$ *is a semi-simple element (i.e.* $x = g\delta$ *with* $g \in G$ *semisimple). Then* x *is* G*-conjugate to an element of* $T\delta$*.*

Proof. Write $x = g\delta$ and choose a Cartan subgroup H' containing g. Write $H' = T'A'$ as usual and $g = ta$ accordingly. As above we may assume $a = 1$. Since T is a Cartan subgroup of K we may choose $k \in K$ so that $ktk^{-1} \in T$, and then $k(t\delta)k^{-1} \in T\delta$. ∎

Let $W = \mathrm{Norm}_G(H)/H$. Then θ acts on W, and we let W^θ be its fixed points. Note that W^θ acts naturally on T, A and $T \cap A$.

Lemma 1.10

$$W(K^0, T) \simeq W(G, H)^\theta \tag{1.11}$$

Remark 1.12 In almost all cases K is connected, and T is a Cartan subgroup of K. if K is not connected hen H^θ is a Cartan subgroup of K. In this case $H^\theta = TZ(G)$, and $W(K, H^\theta) \simeq W(K^0, T)$. This is the case, for example, if $G = SO(2n)$ and $K = S[O(2n-1) \times O(1)] \simeq O(2n-1)$.

Remark 1.13 One consequence of Lemma 1.10 is this: if $w \in W^\theta$ we may choose a representative $g \in \mathrm{Norm}_G(H)$ of w to be in K.

Lemma 1.14 *Suppose $x, x' \in T\delta$ are G-conjugate. Then there exists $g \in Norm_G(T\delta)$ so that $gxg^{-1} = x'$.*

Thus G-conjugacy of elements of $T\delta$ is controlled by the group W_δ of the next definition.

Definition 1.15

$$W_\delta = Norm_G(T\delta)/Cent_G(T\delta) \tag{1.16}$$

It is well known that $\mathrm{Cent}_G(T) = H$. Therefore $\mathrm{Norm}_{K^0}(T) \subset \mathrm{Norm}_G(H)$ and we obtain a map

$$W(K^0, T) \hookrightarrow W(G, T)$$

whose image is contained in $W(G, T)^\theta$.

Proposition 1.17

$$W_\delta \simeq W^\theta \ltimes (A \cap T). \tag{1.18}$$

The subgroup W^θ is the stabilizer of δ in W_δ, and acts on T via its natural action. The subgroup $A \cap T$ acts on $T\delta$ by multiplication.

Proof. It is well known that $\mathrm{Cent}_G(T) = H$ (every root $\alpha \in \Delta(G, H)$ is non-trivial on T, since there are no real roots). Therefore $\mathrm{Norm}_G(T) = \mathrm{Norm}_G(H)$. Thus

$$\mathrm{Norm}_G(T\delta) = \{g \in \mathrm{Norm}_G(H) \mid g\delta g^{-1} \in T\}. \tag{1.19)(a}$$

It is also clear that

$$\mathrm{Cent}_G(T) = \{g \in H \mid g\delta g^{-1} = \delta\} = H^\theta \tag{1.19)(b}$$

Therefore

(1.19)(c) $$W_\delta = \{g \in \mathrm{Norm}_G(H) \mid g\delta g^{-1} \in T\}/H^\theta.$$

We also have

(1.19)(d) $$W^\theta = \{g \in \mathrm{Norm}_G(H) \mid g\delta g^{-1} \in H\}/H$$

If $g\delta g^{-1} = ta \in H$, choose $b \in H$ so that $b^2 = a$. Then $(bg)\delta(bg)^{-1} = t \in T$. It follows that the natural map $W_\delta \to W^\theta$ is a surjection. The kernel is

(1.20) $$\{h = ta \in H \mid a^2 \in T\}/H^\theta$$

Let $A_0 = \{a \in A \mid a^2 \in T\}$, so the kernel is

(1.21) $$TA_0/H^\theta = TA_0/TA^\theta = A_0/A^\theta.$$

Now the map $a \to a^2$ takes A_0 onto $A \cap T$ and there is an exact sequence

(1.22) $$1 \to A^\theta \to A_0 \to A \cap T \to 1$$

Therefore $A_0/A^\theta \simeq A \cap T$. See Remark 1.24.

Putting this together we have an exact sequence

(1.23) $$1 \to A \cap T \to W_\delta \to W_\theta \to 1$$

Define a splitting of (1.23) by taking $w \in W^\theta$ to the unique preimage in W_δ fixing δ. This exists by Lemma 1.10: given $w \in W^\theta$ there exists $g \in \mathrm{Norm}_K(H) \subset \mathrm{Norm}_G(H)$ representing w. It is easy to see this is a well defined splitting.

The action of W^θ on $T\delta$ is clear. For $a \in A \cap T$ choose $b \in A_0$ so that $b^2 = a$. Then $b(t\delta)b^{-1} = bt\theta(b)^{-1}\delta = b^2t\delta = a(t\delta)$, so $A \cap T$ acts by multiplication. ■

Remark 1.24 With respect to the decomposition (1.8) we have

$$\begin{aligned} A_0 &\simeq (\mathbb{Z}/2\mathbb{Z})^b \times (\mathbb{Z}/4\mathbb{Z})^c \\ A^\theta &\simeq (\mathbb{Z}/2\mathbb{Z})^b \times (\mathbb{Z}/2\mathbb{Z})^c \\ A \cap T &\simeq 1 \times (\mathbb{Z}/2\mathbb{Z})^c \end{aligned}$$

where $\mathbb{Z}/4\mathbb{Z} = \{\pm(1,1), \pm(i,-i)\} \subset \mathbb{C}^* \times \mathbb{C}^*$. This makes (1.22) explicit.

Proposition 1.25 *The strong real forms of (G,γ) are are parametrized by elements x of $T\delta$ satisfying $x^2 \in Z$, modulo the action of W_δ.*

It is convenient to mod out by the translations in $T\cap A$; this amounts to replacing T with $H/A \simeq T/T\cap A$. Let

$$\overline{T} = T/T\cap A, \quad \overline{T}^\Gamma = \overline{T} \times \langle\delta\rangle \tag{1.26}$$

Note that W^θ acts on $\overline{T}$. Also every element of $T\cap A$ has order 2, so the condition $x^2 \in Z$ for $x \in \overline{T}$ is well defined. This gives:

Proposition 1.27 *The strong real forms of (G,γ) are are parametrized by elements x of $\overline{T}\delta$ satisfying $x^2 \in Z$, modulo the action of W^θ.*

One advantage of $\overline{T}\delta$ over $T\delta$ is that Z acts naturally on $\overline{T}$, via the isomorphism $\overline{T} \simeq H/A$.

To compute the orbits of W_δ on $\overline{T}\delta$ we pass to the tangent space, in which W_δ becomes an affine Weyl group. See the Appendix for some generalities about affine root systems and Weyl groups.

2 Affine Weyl group and strong real forms

We are interested in computing the orbits of W^θ acting on $\overline{T}\delta$ (Proposition 1.25).

Let $\pi : E \to \overline{T}\delta$ be the tangent space of $\overline{T}\delta$ at δ. We recall a few definitions from the Appendix. The space E is an affine space, with group of translations $\mathfrak{t} = Lie(T)$. The space of affine linear functions $E \to E$ is denoted $\mathrm{Aff}(E,E)$.

Definition 2.1 *Suppose B is a subgroup of $Aut(\overline{T}\delta)$. Let $\widetilde{B}$ be the lift of B to $Aff(E,E)$. That is*

$$\widetilde{B} = \{\phi \in \mathit{Aff}(E,E) \,|\, \phi \textit{ factors to an element of } B\}.$$

From Proposition 1.27 we see:

Lemma 2.2 *Strong real forms of G are parametrized by elements X of E satisying $\pi(X)^2 \in Z$ modulo the action of $\widetilde{W^\theta}$.*

We consider the problem of finding a fundamental domain for the action of $\widetilde{W^\theta}$ on E, and return later to the question of finding the subset of X such that $\pi(X)^2 \in Z$.

We first suppose G is simply connected. From the Appendix (Definitions B.7 and B.9 and Proposition B.12)

$$\widetilde{W^\theta} = W_{\text{aff}} \simeq W^\theta \ltimes L_{sc}$$

(the last isomorphism depending on a choice of $\widetilde{\delta}$ lying over δ). Also W_{aff} is the affine Weyl group of the affine root system D_{Aff}. The underlying finite root system is Δ_θ.

There is a standard choice of a fundamental domain for the action of W_{aff} on E. Choose a set of simple roots $\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_n$ of Δ_{aff}, and let

$$\overline{\mathcal{D}} = \{e \in E \,|\, \widetilde{\alpha}_i(e) \geq 0, i = 0, \dots, n\}.$$

If we choose $\widetilde{\delta}$ then we may identify E with V, and write $\widetilde{\alpha}_i = (\alpha_i, 0)$ ($i = 1, \dots, n$) and $\widetilde{\alpha}_0 = (\alpha_0, c)$. Let $\beta = -\alpha_0$; recall β is the highest long (resp. short) root of Δ if $c = 1$ (respectively $c = 2$). Then

$$\overline{\mathcal{D}} = \{v \in V \,|\, \alpha_i(v) \geq 0 \, (i = 1, \dots, n), \, \beta(v) \leq c\}.$$

If G is not simply connected then $W_{\text{aff}} \subset \widetilde{W^\theta}$, and $\widetilde{W^\theta}$ is an *extended affine Weyl group*. Its fundamental domain will be a quotient of $\mathcal{D}$ by a finite group.

Definition 2.3 *Let*

$$L(G) = X_*(T/T \cap A). \tag{2.4}$$

In particular we have

$$L(G)/X_*(T) \simeq T \cap A \tag{2.5}$$

Lemma 2.6

$$L(G) = \langle \frac{1}{c} \sum_{k=0}^{c-1} \theta^k(\gamma^\vee) \,|\, \gamma \in X_*(H) \rangle$$

If $c=1,2$ we have

$$L=\{\frac{1}{2}(\alpha^\vee+\theta\alpha^\vee)\,|\,\alpha\in X_*(H\}\quad (c=1,2). \tag{2.7}$$

If G is simply connected then $L(G)=L_{sc}$ (Definition B.9).

Lemma 2.8 *Setting $L=L(G)$ we have an exact sequence*

$$1\to L\to \widetilde{W^\theta}\to W^\theta\to 1 \tag{2.9)(a}$$

Given $\widetilde{\delta}$ we obtain a splitting of (2.9)(a)*, so*

$$\widetilde{W^\theta}\simeq W^\theta\ltimes L. \tag{2.9)(b}$$

If G is simply connected then (2.9)(a-b) reduce to (B.13)(a-b).

To find a fundamental domain for $\widetilde{W^\theta}$ we relate it to W_{aff}.

Lemma 2.10 *We have an exact sequence*

$$1\to W_{\mathit{aff}}\to\widetilde{W^\theta}\to L/L_{sc}\to 1 \tag{2.11}$$

Given $\widetilde{\delta}$ we obtain a splitting taking L/L_{sc} to the stabilizer of $\mathcal{D}$. Thus

$$\widetilde{W^\theta}\simeq W_{\mathit{aff}}\rtimes L/L_{sc} \tag{2.12}$$

and L/L_{sc} acts as automorphisms of $\mathcal{D}$.

Recall we are given (Δ,θ), to which we have associated the affine root system Δ_{aff}, with Dynkin diagram D_{Aff}. See the Appendix.

Lemma 2.13 *The stabilizer of $\mathcal{D}$ in the Euclidean group of E is isomorphic to the automorphism group of D_{Aff}.*

Thus we have an action of L/L_{sc} on D_{Aff}. It behooves us to understand L/L_{sc}.

2.1 The group L/L_{sc}

From (B.11) we have

$$L/L_{sc} = \frac{\langle\{\frac{1}{2}(\gamma^\vee + \theta\gamma^\vee) \,|\, \gamma^\vee \in X_*(H)\}\rangle}{\langle\{\frac{1}{2}(\alpha^\vee + \theta\alpha^\vee) \,|\, \gamma^\vee \in R^\vee\}\rangle}$$

Let G_{sc} be the simply connected cover of G, with center $Z_{sc} = Z(G_{sc})$. We have an exact sequence

$$1 \to \pi_1 \to G_{sc} \to G \to 1$$

with $\pi_1 = \pi_1(G) \subset Z_{sc}$. Write $H_{sc} = T_{sc}A_{sc}$ for the Cartan subgroup in G_{sc} with image H.

Lemma 2.14

(2.15) $$L/L_{sc} \simeq \pi_1/\pi_1 \cap A_{sc}$$

Proof. A standard fact is that $\pi_1 \simeq X_*(H)/R^\vee$. The map $\gamma^\vee \to \frac{1}{2}(\gamma^\vee + \theta\gamma^\vee)$ takes $X_*(H)$ onto L and factors to a surjection

$$\pi_1 \twoheadrightarrow L/L_{sc}.$$

The kernel is

$$\{\gamma^\vee \in X_*(H) \,|\, (1+\theta)\gamma^\vee \in (1+\theta)R^\vee\}/R^\vee$$

If $(1+\theta)\gamma^\vee = (1+\theta)\mu^\vee$ for some $\mu^\vee \in R^\vee$ then $(1+\theta)(\gamma^\vee - \mu^\vee) = 0$. So we may replace the numerator with $\{\gamma^\vee \,|\, (1+\theta)\gamma^\vee = 0\}$. This says $\exp(2\pi i\gamma^\vee) \in A_{sc}$, so the kernel is $\pi_1 \cap A_{sc}$. ■

Remark 2.16 Note that

$$(1-\theta)\pi_1 \subset \pi_1 \cap A_{sc} \subset \pi_1^{-\theta}$$

and both inclusions may be proper. If G is adjoint then $\pi_1 = Z_{sc}$ and one can see $Z_{sc} \cap A_{sc} = (1-\theta)Z_{sc}$, which gives

(2.17) $$L_{ad}/L_{sc} = Z_{sc}/(1-\theta)Z_{sc}.$$

However it is not easy to describe $\pi_1 \cap A_{sc}$ in general.

Definition 2.18 *Let*

$$\pi_1^\dagger = \pi_1/\pi_1 \cap A_{sc} \tag{2.19}$$

Let $\tau : \pi_1^\dagger \to Aut(D_{Aff})$ be the action of $\pi_1^\dagger$ on the affine Dynkin diagram via Lemmas 2.10, 2.13 and (2.15)*.*

Here is another description of τ. First take G to be simply connected, so $Z = Z_{sc}$. Note that Z acts by left multiplication on $H\delta$ and therefore on $\overline{T}\delta$. Explicitly $z = ta \in Z$ acts on $\overline{T}\delta$ by multiplication by t. Although t, a are only defined up to $T \cap A$, this action is well defined on $\overline{T}\delta$. Clearly this action factors to $Z/Z \cap A$, lifts to an action on E, and induces actions of $Z/Z \cap A$ on $\mathcal{D}$ and D_{Aff}.

Suppose $z = ta = \exp(2\pi i\gamma^\vee)$ with $\gamma^\vee \in P^\vee$. Then $t = \exp(2\pi i\frac{1}{2}(\gamma^\vee + \theta\gamma^\vee))$, and it follows that under the isomorphism (2.15) L_{ad}/L_{sc} acts by translation on E.

Now drop the assumption that G is simply connected. Then $\pi_1(G) \subset Z_{sc}$ acts on $\mathcal{D}$ and D_{Aff} by the preceding construction, and this action factors to an action of $\pi_1^\dagger(G)$.

Lemma 2.20 *We may parametrize $\overline{D}$ as $\{(a_0, \dots, a_n)\}$ where $a_i \geq 0$ and*

$$\sum_{i=0}^{n} n_i a_i = \frac{1}{c}. \tag{2.21}$$

Here $(a_0, \dots, a_n)$ corresponds to the element X of $\mathcal{D}$ satisfying

$$\alpha_i(X) = a_i \quad (i = 1, \dots, n)$$

Lemma 2.22 *Suppose $(a_0, a_1, \dots, a_n)$ satisfies* (2.21)*, and let $X \in \mathcal{D}$ be the corresponding element. Then $x = \pi(X) \in \overline{T}\delta$ satisfies $x^m \in Z$ if and only if $ma_i \in \mathbb{Z}$ for all $i = 0, \dots, n$.*

Example 2.23 Take $m = 1$. We must take $c = 1$ and each $a_i = 0$ or 1. We conclude from (2.21) that Z is in bijection with the nodes of $\widetilde{D}$ with label 1.

Given m choose integers b_i and let $a_i = b_i/m$ $(0 \leq i \leq n)$. Then $(a_0, \dots, a_n)$ corresponds to an element of $\mathcal{D}$ if

$$c \sum_{i=0}^{n} n_i b_i = m \tag{2.24}$$

To complete our classification of strong real forms we take $m = 1$ or 2.

Definition 2.25 *A Kac diagram for (G, γ) is a subset S of D_{Aff} satisfying $c \sum_{i \in S} n_i \le 2$.*

Clearly $|S| \le 2$ and $n_i \le 2$ for all $i \in S$.

Theorem 2.26 *Fix G and an inner class γ of real forms. Let $c = order(\gamma)$. Let θ be the fundamental real form in the given inner class. Let Δ be the root system of G, Δ_θ the quotient of Δ by θ, and D_{Aff} the affine Dynkin diagram associated to Δ_θ.*

The strong real forms of (G, γ) are parametrized by Kac diagrams for D_{Aff}, modulo the action of $\pi_1^\dagger(G)$ on D_{Aff}.

Suppose S is a Kac diagram corresponding to a real form, with (complexified) maximal compact subgroup K_S. Then the Dynkin diagram of K_s is obtained by by deleting the nodes of S from D_{Aff}.

For the usual classification of real forms see the next section.

For example, a compact group is given by $m = 1$, $c = 1$ and $S = \{i\}$ with $n_i = 1$.

Suppose $m = 2$. If $c = 1$, then $S = \{i\}$ with $n_i = 2$, or $S = \{i, j\}$ with $n_i = n_j = 1$. If $c = 2$ then $S = \{i\}$ with $n_i = 1$.

2.2 The Kac classification of real forms

The Kac classification of real forms of $\mathfrak{g}$ amounts to taking G to be the adjoint group. In this case $\pi_1^\dagger(G) = Z_{sc}/(1-\theta)Z_{sc}$ (2.17). Recall (2.19) acts by τ on D_{Aff} (Definition 2.18).

Theorem 2.27 *Traditional real forms of $(\mathfrak{g}, \gamma)$ are parametrized by subsets S as in Theorem 2.26, modulo the action of $Aut(D_{Aff})$.*

Real forms of $(\mathfrak{g}, \gamma)$ are parametrized by subsets S modulo the action of $Z_{sc}/(1-\theta)Z_{sc}$.

Proof. The second statement is an immediate consequence of Theorem 2.26. The first follows from the following Lemma. ∎

Remark 2.28 This also gives the classification for G either simply connected or adjoint. For general G equivalence will be by the subgroup stabilizing $Z(G)$.

Lemma 2.29 *We have a split exact sequence*

$$1 \to \pi_1^\dagger(G) \to Aut(\mathcal{D}) \to Out(G) \to 1$$

or equivalently

$$1 \to \pi_1^\dagger(G) \to Aut(D_{Aff}) \to Aut(D_\theta) \to 1$$

Here D_θ is the Dynkin diagram of Δ_θ, the underlying finite root system of D_{Aff}. See the Appendix.

Remark 2.30 If $\theta = 1$ and G is simply connected this becomes

$$1 \to Z \to \mathrm{Aut}(D_{\mathrm{Aff}}) \to \mathrm{Aut}(D) \to 1$$

If $\theta \neq 1$ then $\mathrm{Aut}(D_\theta) = 1$ and we have

$$\pi_1^\dagger \simeq \mathrm{Aut}(D_{\mathrm{Aff}})$$

See [6, Exercise 15, page 217]. For an explicit formula for the map $Z \to \mathrm{Aut}(D_{\mathrm{Aff}})$ see [2, Chapter VI, §2.3, Proposition 6].

3 Simplified Kac Diagrams and Vogan Diagrams

If $\gamma \neq 1$ the classification of real forms via the Kac diagram is quite subtle, due to its use of the extended Dynkin diagram of Δ_θ, rather than that of Δ. Here is a version using the extended Dynkin diagram of Δ.

So fix (G, γ) with G simple and $\gamma \neq 1$. Choosing a splitting datum, in particular a Cartan subgroup H we obtain the fundamental automorphism θ of G as in Section 1. Write $H = TA$ as usual.

For simplicity we assume G is adjoint, so strong real forms and real forms coincide. Suppose $\gamma \neq 1$. By Proposition 1.27 the real forms of (G, γ) are parametrized by elements $t \in T$ of order 2 (corresponding to $x = t\delta \in \overline{T}\delta$), modulo $T \cap A$ and conjugation by W^θ.

On the other hand the real forms of $(G, 1)$ are parametrized by elements of H of order 2, modulo conjugation by G. If two elements of t are conjugate by W then they are necessarily conjugate by W^θ. If S is the Kac diagram of a real form of $(G, 1)$, then the corresponding element h is in T if and only if S is θ-invariant. This gives a surjective map from

$$\theta - \text{invariant Kac diagrams for } (G, 1) \to \text{strong real forms of } (G, \gamma)$$

This map is not injective: on the left hand side equivalence is by the action of W^θ, and on the right by W^θ and $A \cap T$. It turns out that if we require that S is pointwise fixed by θ then we get a bijection.

Proposition 3.1 *Given (G, γ) let D_{Aff} be the extended Dynkin diagram of $\Delta = \Delta(G, H)$. Then real forms of (G, γ) are parametrized by Kac diagrams S for which each node of S is fixed by θ, modulo $Aut(D_{Aff})$. That is, S is a set of θ-fixed nodes of D_{Aff}, such that $c\sum_{i \in S} n_i \leq 2$.*

To be honest there is some case-by-case checking here. One point is this. Suppose α is a complex root, and $n_\alpha = 1$. Then $S = \{\alpha, \theta\alpha\}$ defines an element t of T of order 2, and a real form of $(G, 1)$. It also defines a real form of (G, γ), but this one is obtained from another set S which is pointwise fixed.

3.1 Vogan Diagrams

We continue to assume (G, γ) and H have been fixed, and θ is the fundamental real form of G. Let D be the Dykin diagram of G. Suppose θ' is a real form of (G, γ) and B is a θ-stable Borel subgroup of G containing H. Associated to this data is a *Vogan Diagram*: color each of the θ-fixed nodes of D black if the cooresponding imaginary root is non-compact, and white otherwise. See [4, Section VI.8]. Alternatively, let $\mathcal{S}$ be the subset (of black nodes) of the θ-fixed nodes of D. This gives a map from real forms of (G, γ) to Vogan diagrams. This map is not injective: it depends on the choice of B. If we choose B to be the "Borel de Siebenthal" choice [4, Theorem 6.96], i.e. for which at most one simple root is non-compact, then we get a set $\mathcal{S}$ with at most one element.

The is closely related to the simplified Kac diagram. Here is the precise statement.

Proposition 3.2 *Suppose S is a modified Kac diagram of a real form. If S contains a node with label 1 we may assume (via the action of Z_{sc}) this is the affine node. Deleting this node we obtain a subset of the finite Dynkin diagram. This is the Vogan diagram of the real form.*

Conversely suppose S is a Vogan diagram with at most one node, corresponding to a real form of G. Also assume it satisfies the condition in the last line of [4, Thoerem6.96]; equivalently the label on this node is ≤ 2. If S is empty this is the compact form. Suppose $S = \{i\}$. The Kac diagram of this real form is $S \cup \{0\}$ if $n_i = 1$, and S if $n_i = 1$.

Remark 3.3 One of the subtleties of the Vogan diagram is that we do not need a diagram $S = \{i\}$ if $n_i \ge 3$. The fact that such Kac diagram is not needed is explained by (2.24).

Appendix: Affine root systems and Weyl groups

Let V be a real vector space of dimension n and E an affine space with translations V. That is V acts simply transitively on E, written $v, e \to v+e$. A function If E, E' are affine spaces a function $f : E \to E'$ is said to be *affine* if there exists a linear function $df : V \to V'$ such that

$$f(v+e) = df(v) + f(e) \quad \text{for all } v \in V, e \in E. \tag{A.1}$$

In particular if E' is one dimensional we say f is an affine linear functional. In this case $df : V \to \mathbb{R}$, i.e. $df \in V^*$. We say df is the differential of f. The set $\mathrm{Aff}(E)$ of all affine linear functionals is a vector space of dimension $n+1$. The map $f \to df$ is a linear map from $\mathrm{Aff}(E)$ to V^*, and this gives an exact sequence

$$0 \to \mathbb{R} \to \mathrm{Aff}(E) \to V^* \to 0. \tag{A.2}$$

The first inclusion takes $x \in \mathbb{R}$ to the constant function $f_x(e) = x$ for all $e \in E$; this satisfies $df = 0$.

Choose an element $e_0 \in E$. This gives an isomorphism $V \simeq E$ via $v \to v + e_0$. For $\lambda \in V^*$ let $s(\lambda)(v + e_0) = \lambda(v)$. This defines a splitting of (A.2):

Lemma A.3 *Given e_0 we obtain an isomorphism*

$$\mathit{Aff}(E) \simeq V^* \oplus \mathbb{R} \tag{A.4)(a}$$

According to this decomposition we write $f \in \mathit{Aff}(E)$ as

$$f = (\lambda, c). \tag{A.4)(b}$$

We make the isomorphism (A.4)(a) explicit. In one direction $f \in \mathrm{Aff}(E)$ goes to $\lambda = df$ and $c = f(e_0)$. For the other direction (λ, c) goes to $f \in \mathrm{Aff}(E)$ defined by $f(v + e_0) = \lambda(v) + c$.

We now assume V is equipped with a positive definite non-degenerate symmetric form $(,)$, and identify V and V^*. In particular we may identify df with an element of V. Define $(,)$ on $\mathrm{Aff}(V)$ by

$$(f, g) = (df, dg)$$

and for $f \in \mathrm{Aff}(E)$ not a constant function let

$$f^\vee = \frac{2f}{(f, f)}.$$

The affine reflection $s_f : V \to V$ is

$$\begin{aligned} s_f(v) &= v - f^\vee(v)df \\ &= v - f(v)(df)^\vee \\ &= v - \frac{2f(v)}{(f, f)}df \end{aligned}$$

Definition A.5 (Macdonald [5]) *An affine root system on E is a subset S of $\mathit{Aff}(E)$ satisfying*

1. *S spans $\mathit{Aff}(E)$, and the elements of S are non-constant functions,*

2. *$s_\alpha(\beta) \in S$ for all $\alpha, \beta \in S$,*

3. *$\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in S$,*

4. *The Weyl group $W = W(S)$ is the group generated by the reflections $\{s_\alpha \,|\, \alpha \in S\}$. We require that W acts properly on V.*

The Weyl group $W(S)$ is an *affine Weyl group*. The notions of simple roots $\Pi(S)$ and Dynkin diagram $D(S)$ are simlar to those for classical root systems. Also the dual $S^\vee$ of S defined in the obvious way is an affine root system, with Dynkin diagram $D(S^\vee) = D(S)^\vee$. Here the dual of a Dynkin diagram means the same diagram with arrows reversed, as usual.

Choose a base point e_0 in E and write elements of $\mathrm{Aff}(E)$ as (λ, c) as in Lemma A.3.

Suppose $\Delta \subset V$ is a classical (not necessarily reduced) root system. If Δ is simply laced we say each root is long. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots. For each i let $\widetilde{\alpha}_i = (\alpha_i, 0)$, and let $\widetilde{\alpha}_0 = (-\beta, 1)$ where β is the highest root. Note that β is long. Then $\{\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_n\}$ is a set of simple roots of an affine root system denoted $\widetilde{\Delta}$.

Let $D = D(\Delta)$ be the Dynkin diagram of Δ. Let $\widetilde{D}$ be the extended Dynkin diagram of D, i.e. obtained by adjoining $-\beta$ where β is the highest root. Then the Dynkin diagram of $\widetilde{\Delta}$ is the extended Dynkin diagram of Δ, i.e.

$$D(\widetilde{\Delta}) = \widetilde{D(\Delta)}.$$

We will use Δ (resp. S) to denote a typical classical (resp. affine) root system.

Suppose Δ is a classical root system with Dykin diagram $D = D(\Delta)$. Let and $S = \widetilde{\Delta}$, so $D(S) = \widetilde{D}$. Then $S^\vee = (\widetilde{\Delta})^\vee$ is also an affine root system, with Dynkin diagram $D(S^\vee) = (\widetilde{D})^\vee$. If Δ is not simply laced then it is *not* necessarily the case that $(\widetilde{\Delta})^\vee = \widetilde{(\Delta^\vee)}$ or $(\widetilde{D})^\vee = \widetilde{(D^\vee)}$. Note that $\widetilde{D}$ is obtained from D by adding a long root, so $(\widetilde{D})^\vee$ has an extra short root. On the other hand $\widetilde{(D^\vee)}$ is obtained from $D^\vee$ by adding an extra long root.

Theorem A.6 (Macdonald [5]) *Every reduced, irreducible affine root system is equivalent to either $\widetilde{\Delta}$ or $(\widetilde{\Delta})^\vee$ where Δ is a classical (not necessarily reduced) root system.*

Remark A.7 A remarkable fact is that every reduced, irreducible affine root system is also obtained from a classical root system and involution, as discussed in the next section.

Affine root system and Weyl group associated to (Δ, θ)

Let Δ be an irreducible root system, and θ an automorphism of Δ preserving a set of simple roots. Thus θ corresponds to an automorphism of the Dynkin diagram $D = D(\Delta)$ of Δ. Let $c \in \{1, 2, 3\}$ be the order of δ. Associated to (Δ, θ) is an affine root system, which we now describe.

The quotient Δ/θ is naturally a root system [7], which we denote Δ_θ. Here are the possibilities with $\theta \neq 1$. We list the finite root systems Δ, Δ_θ, the names of the affine root system according to [5] and [6], the simply connected group G with root system Δ, the real form of G corresponding to θ, and G^θ.

Δ	Δ_θ	Δ_{aff}	Δ_{aff}	G	$G(\mathbb{R})$	K
A_{2n}	BC_n	$\widetilde{BC_n}$	$A_{2n}^{(2)}$	$SL(2n+1)$	$SL(2n+1, \mathbb{R})$	$SO(2n+1)$
A_{2n-1}	C_n	$\widetilde{B}_n^\vee$	$A_{2n-1}^{(2)}$	$SL(2n)$	$SL(n, \mathbb{H})$	$Sp(n)$
D_n	B_n	$\widetilde{C}_n^\vee$	$D_n^{(2)}$	$Spin(2n)$	$Spin(2n-1, 1)$	$Spin(2n-1)$
E_6R	F_4	$\widetilde{F_4}^\vee$	$E_6^{(2)}$	E_6	$E_6(F_4)$	F_4
$D_4, \quad \theta^3 = 1$	G_2	$\widetilde{G_2}^\vee$	$D_4^{(3)}$	$Spin(8)$		G_2

As in section 1 there is an algebraic group G, and splitting data $(H, B, \{X_\alpha\})$ so that $\Delta = \Delta(G, H)$, and θ may be viewed as an automorphism of G preserving the splitting data. (For these purposes we may as well take G simply connected.) Then $T = H^\theta$ acts on $\mathfrak{g}$, and the set of roots $\Delta(G, T) \subset \mathfrak{t}^*$ is a (possibly reduced) root system.

The following Lemma is more or less immediate.

Lemma B.1 *Restriction from H to T defines isomorphisms*

$$\Delta(G, T) \simeq \Delta_\theta$$

and

$$W^\theta \simeq W(\Delta_\theta).$$

Also $\Delta(K,T)$ is the reduced root system of Δ_θ (obtained by taking only the shorter of two roots $\alpha, 2\alpha$) and $W(K,T) \simeq W(\Delta_\theta)$. See Remark 1.4.

Now T^Γ acts on the complex Lie algebra $\mathfrak{g}$ of G. Let $\Delta(G,T^\Gamma)$ be the set of roots, i.e. we have a root space decomposition

$$\mathfrak{g} = \sum_{\alpha\in\Delta(G,T^\Gamma)} \mathfrak{g}_\alpha.$$

Clearly restriction from T^Γ to T is a surjection $\Delta(G,T^\Gamma) \to \Delta(G,T)$.

If $c = 1$ this is simply $\Delta(G,T)$. For simplicity assume $c = 2$. Then $\Delta(G,T^\Gamma)$ may be thought of as a $\mathbb{Z}/2\mathbb{Z}$-graded root system. That is a character α of T^Γ is a pair (α_0,ϵ) with $\alpha_0 \in \Delta(G,T) \simeq \Delta_\theta$ and $\epsilon = \pm 1$, where $\alpha_0 = \alpha|_T$ and $\epsilon = \alpha(\delta)$. We can define the reflection associated to $\alpha \in \Delta(G,T^\Gamma)$ in the usual way, preserving $\Delta(G,T^\Gamma)$. To be precise, if $\alpha = (\alpha_0,\epsilon)$ and $\beta = (\beta_0,\delta)$ then

$$s_\alpha(\beta) = (s_{\alpha_0}(\beta_0), \epsilon\delta(-1)^{\langle\beta,\alpha^\vee\rangle}). \tag{B.2}$$

Let $\pi : E \to T\delta$ be the universal cover. Then E is an affine space with translations $\mathfrak{t} = \mathrm{Lie}(\mathfrak{t})$.

Suppose λ is a character of $T^\Gamma \to \mathbb{C}^*$. Note that λ is determined by its restriction to $T\delta$. By the property of covering spaces λ lifts to a family of functions $\widetilde{\lambda} : E \to \mathbb{C}$ satisfying

$$\lambda(\pi(X)) = e^{2\pi i\widetilde{\lambda}(X)}$$

i.e. $d\widetilde{\lambda} = d\lambda$, where the left hand side is in the sense of (A.1) and the right is the ordinary differential of λ. We say $\widetilde{\lambda}$ lies over λ. Any two such functions differ by constant.

Definition B.3 *The affine root system Δ_{aff} associated to (Δ,θ) is the set of affine functions in Aff(E) lying over $\Delta(G,T^\Gamma)$.*

Note that the underlying finite root system, i.e. the differentials of all affine roots is $\Delta(G,T) \simeq \Delta_\theta$, i.e.

$$d : \Delta_{\text{aff}} \twoheadrightarrow \Delta_\theta$$

The following Lemma is an immediate consequence of the fact that $\Delta(G,T^\Gamma)$ is a root system in the sense of (B.2).

Lemma B.4 *Δ_{aff} is an affine root system.*

To be explicit, choose $\widetilde{\delta} \in E$ with $\pi(\widetilde{\delta}) = \delta$. Suppose $\alpha \in \widehat{T^\Gamma}$. To avoid excessive notation we write α for the differential of α restricted to T, rather than $d\alpha$. Then in the decomposition of Lemma A.3 we may write the set of $\widetilde{\alpha}$ lying over α as

$$\{(\alpha, c) \,|\, e^{2\pi i c} = \alpha(\delta)\}$$

In particular note that the set of roots lying over α is

$$\{(\alpha, c) \,|\, c \in \mathbb{Z}\} \quad \text{ if } \alpha(\delta) = 1$$

or

$$\{(\alpha, c) \,|\, c \in \mathbb{Z} + \frac{1}{2}\} \quad \text{ if } \alpha(\delta) = -1$$

Similarly if δ has order 3 then $c \in \mathbb{Z} + \frac{1}{3}$ or $\mathbb{Z} + \frac{2}{3}$.

For $\alpha \in \Delta_\theta$ let $c_\alpha = 1$ if α is long, or $\frac{1}{c}$ if α is short, where $c = \text{order}(\theta)$.

Proposition B.5 *Let Δ_{aff} be the affine root system associated to (Δ, θ), and let $c = order(\theta) \in \{1, 2, 3\}$. Then*

$$\Delta_{aff} = \{(\alpha, x) \,|\, x \in c_\alpha \mathbb{Z}\}$$

Proposition B.6 *Fix a set $\alpha_1, \ldots, \alpha_n$ of simple roots of Δ_θ. For each i let $\widetilde{\alpha}_i = (\alpha_i, 0)$. Let β be the highest (long) root of $\Delta = \Delta_\theta$ if $c = 1$ or the highest short root otherwise. Let*

$$\widetilde{\alpha}_0 = (-\beta, \frac{1}{c}).$$

Then $\{\widetilde{\alpha}_0, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n\}$ is a set of simple roots of Δ_{aff}.

Definition B.7 *The affine Weyl group associated to (Δ, θ) is the subgroup of $Aff(E, E)$ generated by the affine reflections $s_{\widetilde{\alpha}}$ for $\widetilde{\alpha} \in \Delta_{aff}$. Alternatively,*

(B.8) $\quad W_{aff} = \{\phi \in Aff(E, E) \,|\, \phi$ *factors to an element of* $W(\Delta_\theta) = W^\theta\}$.

We now describe W_{aff}.

Definition B.9 *Let*

$$L_{sc} = \langle \frac{1}{c} \sum_{k=0}^{c-1} \theta^k(\alpha^\vee) \,|\, \alpha \in \Delta \rangle \tag{B.10}$$

We are primarily interested in $c = 1, 2$, in which case:

(B.11) $$L_{sc} = \{\frac{1}{2}(\alpha^\vee + \theta\alpha^\vee) \,|\, \alpha \in \Delta\}$$

Proposition B.12 *The lattice L_{sc} is the set of translations in W_{aff}. There is an exact sequence*

(B.13)(a) $$0 \to L_{sc} \to W_{\mathit{aff}} \to W^\theta \to 1$$

If we choose an element $\widetilde{\delta} \in E$ lying over δ we obtain a splitting of (1.18)*, taking W^θ to the the stabilizer in Aff(E) of $\widetilde{\delta}$, i.e.*

(B.13)(b) $$W_{\mathit{aff}} \simeq W^\theta \ltimes L_{sc}$$

We give a few details of the map $p : W_{\text{aff}} \to W_\delta$. Suppose $\alpha \in \Delta_\theta$ and $x \in \mathbb{Z}$. Then

$$p(s_{(\alpha,x)}) = s_\alpha.$$

Suppose $c = 2$, $\alpha \in \Delta_\theta$ is a short root and $x \in \mathbb{Z} + \frac{1}{2}$. Then $m_\alpha = \alpha^\vee(-1) \in T \cap A$ and

$$p(s_{(\alpha,x)}) = s_\alpha m_\alpha$$

and

$$p(t_{\frac{1}{2}\alpha^\vee}) = m_\alpha.$$

where $t_{\frac{1}{2}\alpha^\vee} \in W_{\text{aff}}$ is translation by $\frac{1}{2}\alpha^\vee$.

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