

Algebraic methods in the theory of generalized Harish-Chandra modules

Ivan Penkov and Gregg Zuckerman

Abstract

This paper is a review of results on generalized Harish-Chandra modules in the framework of cohomological induction. The main results, obtained during the last 10 years, concern the structure of the fundamental series of $(\mathfrak{g}, \mathfrak{k})$ -modules, where \mathfrak{g} is a semisimple Lie algebra and \mathfrak{k} is an arbitrary algebraic reductive in \mathfrak{g} subalgebra. These results lead to a classification of simple $(\mathfrak{g}, \mathfrak{k})$ -modules of finite type with generic minimal \mathfrak{k} -types, which we state. We establish a new result about the Fernando-Kac subalgebra of a fundamental series module. In addition, we pay special attention to the case when \mathfrak{k} is an eligible r -subalgebra (see the definition in section 4) in which we prove stronger versions of our main results. If \mathfrak{k} is eligible, the fundamental series of $(\mathfrak{g}, \mathfrak{k})$ -modules yields a natural algebraic generalization of Harish-Chandra's discrete series modules.

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Ivan Penkov
Jacobs University Bremen, School of Engineering and Science
Campus Ring 1, 28759, Bremen, Germany
e-mail: i.penkov@jacobs-university.de

Gregg Zuckerman
Yale University, Department of Mathematics
10 Hillhouse Avenue, P.O. Box 208283,
New Haven, CT 06520-8283, USA
e-mail: gregg.zuckerman@yale.edu

Introduction

Generalized Harish-Chandra modules have now been actively studied for more than 10 years. A *generalized Harish-Chandra module* M over a finite-dimensional reductive Lie algebra \mathfrak{g} is a \mathfrak{g} -module M for which there is a reductive in \mathfrak{g} subalgebra \mathfrak{f} such that as a \mathfrak{f} -module, M is the direct sum of finite-dimensional generalized \mathfrak{f} -isotypic components. If M is irreducible, \mathfrak{f} acts necessarily semisimply on M , and in what follows we restrict ourselves to the study of generalized Harish-Chandra modules on which \mathfrak{f} acts semisimply; see [Z] for an introduction to the topic.

In this paper we present a brief review of results obtained in the past 10 years in the framework of algebraic representation theory, more specifically in the framework of cohomological induction, see [KV] and [Z]. In fact, generalized Harish-Chandra modules have been studied also with geometric methods, see for instance [PSZ] and [PS1], [PS2], [PS3], [Pe], but the geometric point of view remains beyond the scope of the current review. In addition, we restrict ourselves to finite-dimensional Lie algebras \mathfrak{g} and do not review the paper [PZ4], which deals with the case of locally finite Lie algebras. We omit the proofs of most results which have already appeared.

The cornerstone of the algebraic theory of generalized Harish-Chandra modules so far is our work [PZ2]. In this work we define the notion of simple generalized Harish-Chandra modules with generic minimal \mathfrak{f} -type and provide a classification of such modules. The result extends in part the Vogan-Zuckerman classification of simple Harish-Chandra modules. It leaves open the questions of existence and classification of simple $(\mathfrak{g}, \mathfrak{f})$ -modules of finite type whose minimal \mathfrak{f} -types are not generic. While the classification of such modules presents the main open problem in the theory of generalized Harish-Chandra modules, in the note [PZ3] we establish the existence of simple $(\mathfrak{g}, \mathfrak{f})$ -modules with arbitrary given minimal \mathfrak{f} -type.

In the paper [PZ5] we establish another general result, namely the fact that each module in the fundamental series of generalized Harish-Chandra modules has finite length. We then consider in detail the case when $\mathfrak{f} = \mathfrak{sl}(2)$. In this case the highest weights of \mathfrak{f} -types are just non-negative integers μ and the genericity condition is the inequality $\mu \geq \Gamma$, Γ being a bound depending on the pair $(\mathfrak{g}, \mathfrak{f})$. In [PZ5] we improve the bound Γ to an, in general, much lower bound Λ . Moreover, we show that in a number of low dimensional examples the bound Λ is sharp in the sense that our classification results do not hold for simple $(\mathfrak{g}, \mathfrak{f})$ -modules with minimal \mathfrak{f} -type $V(\mu)$ for μ lower than Λ . In [PZ5] we also conjecture that the Zuckerman functor establishes an equivalence of a certain subcategory of the thickening of category \mathcal{O} and a subcategory of the category of $(\mathfrak{g}, \mathfrak{f} \simeq \mathfrak{sl}(2))$ -modules.

Sections 2 and 3 of the present paper are devoted to a brief review of the above results. We also establish some new results in terms of the algebra $\tilde{\mathfrak{f}} := \mathfrak{f} + C(\mathfrak{f})$ (where $C(\cdot)$ stands for centralizer in \mathfrak{g}). A notable such result is Corollary 2.10 which gives a sufficient condition on a simple $(\mathfrak{g}, \mathfrak{f})$ -module

M for $\tilde{\mathfrak{k}}$ to be a maximal reductive subalgebra of \mathfrak{g} which acts locally finitely on M .

The idea of bringing $\tilde{\mathfrak{k}}$ into the picture leads naturally to considering a preferred class of reductive subalgebras \mathfrak{k} which we call eligible: they satisfy the condition $C(\mathfrak{t}) = \mathfrak{t} + C(\mathfrak{k})$ where \mathfrak{t} is Cartan subalgebra of \mathfrak{k} . In section 5 we study a natural generalization of Harish-Chandra's discrete series to the case of an eligible subalgebra \mathfrak{k} . A key statement here is that under the assumption of eligibility of \mathfrak{k} , the isotypic component of the minimal \mathfrak{k} -type of a generalized discrete series module is an irreducible $\tilde{\mathfrak{k}}$ -module (Theorem 5.1).

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1 Notation and preliminary results

We start by recalling the setup of [PZ2] and [PZ5].

1.1 Conventions

The ground field is \mathbb{C} , and if not explicitly stated otherwise, all vector spaces and Lie algebras are defined over \mathbb{C} . The sign \otimes denotes tensor product over \mathbb{C} . The superscript $*$ indicates dual space. The sign \in stands for semidirect sum of Lie algebras (if $\mathfrak{l} = \mathfrak{l}' \in \mathfrak{l}''$, then \mathfrak{l}' is an ideal in \mathfrak{l} and $\mathfrak{l}'' \cong \mathfrak{l}/\mathfrak{l}'$). $H^i(\mathfrak{l}, M)$ stands for the cohomology of a Lie algebra \mathfrak{l} with coefficients in an \mathfrak{l} -module M , and $M^{\mathfrak{l}} = H^0(\mathfrak{l}, M)$ stands for space of \mathfrak{l} -invariants of M . By $Z(\mathfrak{l})$ we denote the center of \mathfrak{l} , and by \mathfrak{l}_{ss} we denote the semisimple

part of \mathfrak{l} when \mathfrak{l} is reductive. $\wedge(\cdot)$ and $S(\cdot)$ denote respectively the exterior and symmetric algebra.

If \mathfrak{l} is a Lie algebra, then $U(\mathfrak{l})$ stands for the enveloping algebra of \mathfrak{l} and $Z_{U(\mathfrak{l})}$ denotes the center of $U(\mathfrak{l})$. We identify \mathfrak{l} -modules with $U(\mathfrak{l})$ -modules. It is well known that if \mathfrak{l} is finite dimensional and M is a simple \mathfrak{l} -module (or equivalently a simple $U(\mathfrak{l})$ -module), $Z_{U(\mathfrak{l})}$ acts on M via a $Z_{U(\mathfrak{l})}$ -character, i.e. via an algebra homomorphism $\theta_M : Z_{U(\mathfrak{l})} \rightarrow \mathbb{C}$, see Proposition 2.6.8 in [Dix].

We say that an \mathfrak{l} -module M is *generated* by a subspace $M' \subseteq M$ if $U(\mathfrak{l}) \cdot M' = M$, and we say that M is *cogenerated* by $M' \subseteq M$, if for any non-zero homomorphism $\psi : M \rightarrow \bar{M}$, $M' \cap \ker \psi \neq \{0\}$.

By $\text{Soc}M$ we denote the socle (i.e. the unique maximal semisimple submodule) of an \mathfrak{l} -module M . If $\omega \in \mathfrak{l}^*$, we put $M^\omega := \{m \in M \mid \mathfrak{l} \cdot m = \omega(\mathfrak{l})m \ \forall \mathfrak{l} \in \mathfrak{l}\}$. By $\text{supp}_\mathfrak{l}M$ we denote the set $\{\omega \in \mathfrak{l}^* \mid M^\omega \neq \{0\}\}$.

A finite *multiset* is a function f from a finite set D into \mathbb{N} . A *submultiset* of f is a multiset f' defined on the same domain D such that $f'(d) \leq f(d)$ for any $d \in D$. For any finite multiset f , defined on a subset D of a vector space, we put $\rho_f := \frac{1}{2} \sum_{d \in D} f(d)d$.

If $\dim M < \infty$ and $M = \bigoplus_{\omega \in \mathfrak{l}^*} M^\omega$, then M determines the finite multiset $\text{ch}_\mathfrak{l}M$ which is the function $\omega \mapsto \dim M^\omega$ defined on $\text{supp}_\mathfrak{l}M$.

1.2 Reductive subalgebras, compatible parabolics and generic \mathfrak{k} -types

Let \mathfrak{g} be a finite-dimensional semisimple Lie algebra. By \mathfrak{g} -mod we denote the category of \mathfrak{g} -modules. Let $\mathfrak{k} \subset \mathfrak{g}$ be an algebraic subalgebra which is reductive in \mathfrak{g} . We set $\tilde{\mathfrak{k}} = \mathfrak{k} + C(\mathfrak{k})$ and note that $\tilde{\mathfrak{k}} = \mathfrak{k}_{\text{ss}} \oplus C(\mathfrak{k})$ where $C(\cdot)$ stands for centralizer in \mathfrak{g} . We fix a Cartan subalgebra \mathfrak{t} of $\tilde{\mathfrak{k}}$ and let \mathfrak{h} denote an as yet unspecified Cartan subalgebra of \mathfrak{g} . Everywhere, but in subsection 1.3 below, we assume that $\mathfrak{t} \subseteq \mathfrak{h}$, and hence that $\mathfrak{h} \subseteq C(\mathfrak{t})$. By Δ we denote the set of \mathfrak{h} -roots of \mathfrak{g} , i.e. $\Delta = \{\text{supp}_\mathfrak{h}\mathfrak{g}\} \setminus \{0\}$. Note that, since $\tilde{\mathfrak{k}}$ is reductive in \mathfrak{g} , \mathfrak{g} is a \mathfrak{t} -weight module, i.e. $\mathfrak{g} = \bigoplus_{\eta \in \mathfrak{t}^*} \mathfrak{g}^\eta$. We set $\Delta_\mathfrak{t} := \{\text{supp}_\mathfrak{t}\mathfrak{g}\} \setminus \{0\}$. Note also that the \mathbb{R} -span of the roots of \mathfrak{h} in \mathfrak{g} fixes a real structure on \mathfrak{h}^* , whose projection onto \mathfrak{t}^* is a well-defined real structure on \mathfrak{t}^* . In what follows, we denote by $\text{Re}\eta$ the real part of an element $\eta \in \mathfrak{t}^*$. We fix also a Borel subalgebra $\mathfrak{b}_\mathfrak{k} \subseteq \tilde{\mathfrak{k}}$ with $\mathfrak{b}_\mathfrak{k} \supseteq \mathfrak{t}$. Then $\mathfrak{b}_\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{n}_\mathfrak{k}$, where $\mathfrak{n}_\mathfrak{k}$ is the nilradical of $\mathfrak{b}_\mathfrak{k}$. We set $\rho := \rho_{\text{ch}_\mathfrak{k}\mathfrak{n}_\mathfrak{k}}$. The quartet $\mathfrak{g}, \tilde{\mathfrak{k}}, \mathfrak{b}_\mathfrak{k}, \mathfrak{t}$ will be fixed throughout the paper. By W we denote the Weyl group of \mathfrak{g} .

As usual, we parametrize the characters of $Z_{U(\mathfrak{g})}$ via the Harish-Chandra homomorphism. More precisely, if \mathfrak{b} is a given Borel subalgebra of \mathfrak{g} with $\mathfrak{b} \supset \mathfrak{h}$ (\mathfrak{b} will be specified below), the $Z_{U(\mathfrak{g})}$ -character corresponding to $\zeta \in \mathfrak{h}^*$ via the Harish-Chandra homomorphism defined by \mathfrak{b} is denoted by θ_ζ ($\theta_{\rho_{\text{ch}_\mathfrak{g}\mathfrak{b}}}$ is the trivial $Z_{U(\mathfrak{g})}$ -character). Sometimes we consider a reductive subalgebra

$l \subset \mathfrak{g}$ instead of \mathfrak{g} and apply this convention to the characters of $Z_{U(l)}$. In this case we write θ_ζ^l for $\zeta \in \mathfrak{h}_l^*$, where \mathfrak{h}_l is a Cartan subalgebra of l .

By $\langle \cdot, \cdot \rangle$ we denote the unique \mathfrak{g} -invariant symmetric bilinear form on \mathfrak{g}^* such that $\langle \alpha, \alpha \rangle = 2$ for any long root of a simple component of \mathfrak{g} . The form $\langle \cdot, \cdot \rangle$ enables us to identify \mathfrak{g} with \mathfrak{g}^* . Then \mathfrak{h} is identified with \mathfrak{h}^* , and \mathfrak{k} is identified with \mathfrak{k}^* . We sometimes consider $\langle \cdot, \cdot \rangle$ as a form on \mathfrak{g} . The superscript \perp indicates orthogonal space. Note that there is a canonical \mathfrak{k} -module decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$ and a canonical decomposition $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{t}^\perp$ with $\mathfrak{t}^\perp \subseteq \mathfrak{k}^\perp$. We also set $\|\zeta\|^2 := \langle \zeta, \zeta \rangle$ for any $\zeta \in \mathfrak{h}^*$.

We say that an element $\eta \in \mathfrak{t}^*$ is $(\mathfrak{g}, \mathfrak{k})$ -regular if $\langle \text{Re}\eta, \sigma \rangle \neq 0$ for all $\sigma \in \Delta_{\mathfrak{k}}$. To any $\eta \in \mathfrak{t}^*$ we associate the following parabolic subalgebra \mathfrak{p}_η of \mathfrak{g} :

$$\mathfrak{p}_\eta = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_\eta} \mathfrak{g}^\alpha \right),$$

where $\Delta_\eta := \{\alpha \in \Delta \mid \langle \text{Re}\eta, \alpha \rangle \geq 0\}$. By \mathfrak{m}_η and \mathfrak{n}_η we denote respectively the reductive part of \mathfrak{p} (containing \mathfrak{h}) and the nilradical of \mathfrak{p} . In particular $\mathfrak{p}_\eta = \mathfrak{m}_\eta \rtimes \mathfrak{n}_\eta$, and if η is $\mathfrak{b}_\mathfrak{t}$ -dominant, then $\mathfrak{p}_\eta \cap \mathfrak{k} = \mathfrak{b}_\mathfrak{t}$. We call \mathfrak{p}_η a *t-compatible parabolic subalgebra*. Note that

$$\mathfrak{p}_\eta = C(\mathfrak{t}) \oplus \left(\bigoplus_{\beta \in \Delta_{\mathfrak{t}, \eta}^+} \mathfrak{g}^\beta \right)$$

where $\Delta_{\mathfrak{t}, \eta}^+ := \{\beta \in \Delta_{\mathfrak{t}} \mid \langle \text{Re}\eta, \beta \rangle > 0\}$. Hence, \mathfrak{p}_η depends upon our choice of \mathfrak{t} and η , but not upon the choice of \mathfrak{h} .

A *t-compatible parabolic subalgebra* $\mathfrak{p} = \mathfrak{m} \rtimes \mathfrak{n}$ (i.e. $\mathfrak{p} = \mathfrak{p}_\eta$ for some $\eta \in \mathfrak{t}^*$) is *t-minimal* (or simply *minimal*) if it does not properly contain another *t-compatible parabolic subalgebra*. It is an important observation that if $\mathfrak{p} = \mathfrak{m} \rtimes \mathfrak{n}$ is minimal, then $\mathfrak{t} \subseteq Z(\mathfrak{m})$. In fact, a *t-compatible parabolic subalgebra* \mathfrak{p} is minimal if and only if \mathfrak{m} equals the centralizer $C(\mathfrak{t})$ of \mathfrak{t} in \mathfrak{g} , or equivalently if and only if $\mathfrak{p} = \mathfrak{p}_\eta$ for a $(\mathfrak{g}, \mathfrak{k})$ -regular $\eta \in \mathfrak{t}^*$. In this case $\mathfrak{n} \cap \mathfrak{k} = \mathfrak{n}_\mathfrak{k}$.

Any *t-compatible parabolic subalgebra* $\mathfrak{p} = \mathfrak{p}_\eta$ has a well-defined opposite parabolic subalgebra $\bar{\mathfrak{p}} := \mathfrak{p}_{-\eta}$; clearly \mathfrak{p} is minimal if and only if $\bar{\mathfrak{p}}$ is minimal.

A *k-type* is by definition a simple finite-dimensional \mathfrak{k} -module. By $V(\mu)$ we denote a *k-type* with $\mathfrak{b}_\mathfrak{t}$ -highest weight μ . The weight μ is then \mathfrak{k} -integral (or, equivalently, \mathfrak{k}_{ss} -integral) and $\mathfrak{b}_\mathfrak{t}$ -dominant.

Let $V(\mu)$ be a *k-type* such that $\mu + 2\rho$ is $(\mathfrak{g}, \mathfrak{k})$ -regular, and let $\mathfrak{p} = \mathfrak{m} \rtimes \mathfrak{n}$ be the minimal compatible parabolic subalgebra $\mathfrak{p}_{\mu+2\rho}$. Put $\tilde{\rho}_\mathfrak{n} := \rho_{\text{ch}_\mathfrak{b}, \mathfrak{n}}$ and $\rho_\mathfrak{n} := \rho_{\text{ch}_\mathfrak{t}, \mathfrak{n}}$. Clearly $\rho_\mathfrak{n} = \tilde{\rho}_\mathfrak{n}|_{\mathfrak{t}}$. We define $V(\mu)$ to be *generic* if the following two conditions hold:

1. $\langle \text{Re}\mu + 2\rho - \rho_\mathfrak{n}, \alpha \rangle \geq 0 \forall \alpha \in \text{supp}_{\mathfrak{t}} \mathfrak{n}_\mathfrak{t}$;
2. $\langle \text{Re}\mu + 2\rho - \rho_S, \rho_S \rangle > 0$ for every submultiset S of $\text{ch}_{\mathfrak{t}, \mathfrak{n}}$.

It is easy to show that there exists a positive constant C depending only on $\mathfrak{g}, \mathfrak{f}$ and \mathfrak{p} such that $\langle \operatorname{Re} \mu + 2\rho, \alpha \rangle > C$ for every $\alpha \in \operatorname{supp}_i \mathfrak{n}$ implies $\mathfrak{p}_{\mu+2\rho} = \mathfrak{p}$ and that $V(\mu)$ is generic.

1.3 Generalities on \mathfrak{g} -modules

Suppose M is a \mathfrak{g} -module and \mathfrak{l} is a reductive subalgebra of \mathfrak{g} . M is *locally finite over $Z_{U(\mathfrak{l})}$* if every vector in M generates a finite-dimensional $Z_{U(\mathfrak{l})}$ -module. Denote by $\mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})})$ the full subcategory of \mathfrak{g} -modules which are locally finite over $Z_{U(\mathfrak{l})}$.

Suppose $M \in \mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})})$ and θ is a $Z_{U(\mathfrak{l})}$ -character. Denote by $P(\mathfrak{l}, \theta)(M)$ the generalized θ -eigenspace of the restriction of M to \mathfrak{l} . The $Z_{U(\mathfrak{l})}$ -spectrum of M is the set of characters θ of $Z_{U(\mathfrak{l})}$ such that $P(\mathfrak{l}, \theta)(M) \neq 0$. Denote the $Z_{U(\mathfrak{l})}$ spectrum of M by $\sigma(\mathfrak{l}, M)$. We say that θ is a *central character of \mathfrak{l} in M* if $\theta \in \sigma(\mathfrak{l}, M)$. The following is a standard fact.

Lemma 1.1 *If $M \in \mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})})$, then*

$$M = \bigoplus_{\theta \in \sigma(\mathfrak{l}, M)} P(\mathfrak{l}, \theta)(M).$$

A \mathfrak{g} -module M is *locally Artinian over \mathfrak{l}* if for every vector $v \in M$, $U(\mathfrak{l}) \cdot v$ is an \mathfrak{l} -module of finite length.

Lemma 1.2 *If M is locally Artinian over \mathfrak{l} , then $M \in \mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})})$.*

Proof The statement follows from the fact that $Z_{U(\mathfrak{l})}$ acts via a character on any simple \mathfrak{l} -module. \square

If \mathfrak{p} is a parabolic subalgebra of \mathfrak{g} , by a $(\mathfrak{g}, \mathfrak{p})$ -module M we mean a \mathfrak{g} -module M on which \mathfrak{p} acts locally finitely. By $\mathcal{M}(\mathfrak{g}, \mathfrak{p})$ we denote the full subcategory of \mathfrak{g} -modules which are $(\mathfrak{g}, \mathfrak{p})$ -modules.

In the remainder of this subsection we assume that \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} such that $\mathfrak{h}_\mathfrak{l} := \mathfrak{h} \cap \mathfrak{l}$ is a Cartan subalgebra of \mathfrak{l} , and that \mathfrak{p} is a parabolic subalgebra of \mathfrak{g} such that $\mathfrak{h} \subset \mathfrak{p}$ and $\mathfrak{p} \cap \mathfrak{l}$ is a parabolic subalgebra of \mathfrak{l} . By M we denote a \mathfrak{g} -module from $\mathcal{M}(\mathfrak{g}, \mathfrak{p})$.

Lemma 1.3 *The set $\operatorname{supp}_{\mathfrak{h}} M$ is independent of the choice of $\mathfrak{h} \subseteq \mathfrak{p}$.*

Proof Suppose \mathfrak{h}_1 and \mathfrak{h}_2 are Cartan subalgebras of \mathfrak{g} such that $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq \mathfrak{p}$. Let \mathfrak{m}_j be the maximal reductive subalgebra of \mathfrak{p} such that $\mathfrak{h}_j \subseteq \mathfrak{m}_j$, $j = 1, 2$. There exists an inner automorphism $\Psi(\mathfrak{m}_1) = \mathfrak{m}_2$. Then, $\Psi(\mathfrak{h}_1)$ and \mathfrak{h}_2 are Cartan subalgebras of \mathfrak{m}_2 . There exists an inner automorphism Φ of \mathfrak{m}_2 such that $\Phi(\Psi(\mathfrak{h}_1)) = \mathfrak{h}_2$. Hence, for any finite dimensional \mathfrak{p} -module W , $\operatorname{supp}_{\mathfrak{h}_1} W = \operatorname{supp}_{\mathfrak{h}_2} W$. By assumption M is a union of finite-dimensional \mathfrak{p} -modules. \square

Proposition 1.4 *M is locally Artinian over \mathfrak{l} .*

Proof We apply Proposition 7.6.1 in [Dix] to the pair $(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{p})$. In particular, if $v \in M$, then $U(\mathfrak{l}) \cdot v$ has finite length as an \mathfrak{l} -module. \square

Corollary 1.5 $M \in \mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})})$.

Lemma 1.6 $\sigma(\mathfrak{l}, M) \subseteq \{\theta_{(\eta|_{\mathfrak{h}_1}) + \rho_1}^{\mathfrak{l}} \mid \eta \in \text{supp}_{\mathfrak{g}} M\}$.

Proof The simple \mathfrak{l} -subquotients of M are $(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{p})$ -modules, and our claim follows the well-known relationship between the highest weight of a highest weight module and its central character. \square

Let N be a \mathfrak{g} -module, and let $\mathfrak{g}[N]$ be the set of elements $x \in \mathfrak{g}$ that act locally finitely in N . Then $\mathfrak{g}[N]$ is a Lie subalgebra of \mathfrak{g} , the *Fernando-Kac subalgebra associated to N* . The fact has been proved independently by V. Kac in [K] and by S. Fernando in [F].

Theorem 1.7 *Let M_1 be a non-zero subquotient of M . Assume that $\eta|_{\mathfrak{h}_1}$ is non-integral relative to \mathfrak{l} for all $\eta \in \text{supp}_{\mathfrak{g}} M$. Then $\mathfrak{l} \not\subseteq \mathfrak{g}[M_1]$.*

Proof By Lemma 1.6, no central character of \mathfrak{l} in M_1 is \mathfrak{l} -integral. Therefore, no non-zero \mathfrak{l} -submodule of M_1 is finite dimensional. But $M_1 \neq 0$. Hence, $\mathfrak{l} \not\subseteq \mathfrak{g}[M_1]$. \square

In agreement with [PZ2], we define a \mathfrak{g} -module M to be a $(\mathfrak{g}, \mathfrak{k})$ -module if M is isomorphic as a \mathfrak{k} -module to a direct sum of isotypic components of \mathfrak{k} -types. If M is a $(\mathfrak{g}, \mathfrak{k})$ -module, we write $M[\mu]$ for the $V(\mu)$ -isotypic component of M , and we say that $V(\mu)$ is a \mathfrak{k} -type of M if $M[\mu] \neq 0$. We say that a $(\mathfrak{g}, \mathfrak{k})$ -module M is of *finite type* if $\dim M[\mu] \neq \infty$ for every \mathfrak{k} -type $V(\mu)$ of M . Sometimes, we also refer to $(\mathfrak{g}, \mathfrak{k})$ -modules of finite type as *generalized Harish-Chandra modules*.

Note that for any $(\mathfrak{g}, \mathfrak{k})$ -module of finite type M and any \mathfrak{k} -type $V(\sigma)$ of M , the finite-dimensional \mathfrak{k} -module $M[\sigma]$ is a $\tilde{\mathfrak{k}}$ -module. In particular, M is a $(\mathfrak{g}, \tilde{\mathfrak{k}})$ -module of finite type. We will write $M\langle\delta\rangle$ for the $\tilde{\mathfrak{k}}$ -isotypic components of M where $\delta \in (\mathfrak{h} \cap \tilde{\mathfrak{k}})^*$.

If M is a module of finite length, a \mathfrak{k} -type $V(\mu)$ of M is *minimal* if the function $\mu' \mapsto \|\text{Re}\mu' + 2\rho\|^2$ defined on the set $\{\mu' \in \mathfrak{t}^* \mid M[\mu'] \neq 0\}$ has a minimum at μ . Any non-zero $(\mathfrak{g}, \mathfrak{k})$ -module M of finite length has a minimal \mathfrak{k} -type.

1.4 Generalities on the Zuckerman functor

Recall that the functor of \mathfrak{k} -finite vectors $\Gamma_{\mathfrak{g}, \mathfrak{k}}^{\mathfrak{g}, \mathfrak{k}}$ is a well-defined left-exact functor on the category of $(\mathfrak{g}, \mathfrak{k})$ -modules with values in $(\mathfrak{g}, \mathfrak{k})$ -modules,

$$\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{t}}(M) := \sum_{M' \subset M, \dim M' = 1, \dim U(\mathfrak{t}) \cdot M' < \infty} M'.$$

By $R\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{t}} := \bigoplus_{i \geq 0} R^i \Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{t}}$ we denote as usual the total right derived functor of $\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{t}}$, see [Z] and the references therein.

Proposition 1.8 *If \mathfrak{l} is any reductive subalgebra of \mathfrak{g} containing \mathfrak{k} , then there is a natural isomorphism of \mathfrak{l} -modules*

$$R\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{t}}(N) \cong R\Gamma_{\mathfrak{l},\mathfrak{k}}^{\mathfrak{l},\mathfrak{t}}(N). \quad (1)$$

Proof See Proposition 2.5 in [PZ4]. \square

Proposition 1.9 *If $\tilde{N} \in \mathcal{M}(\mathfrak{l}, \mathfrak{t}, Z_{U(\mathfrak{l})}) := \mathcal{M}(\mathfrak{l}, Z_{U(\mathfrak{l})}) \cap \mathcal{M}(\mathfrak{l}, \mathfrak{t})$, then*

$$R\Gamma_{\mathfrak{l},\mathfrak{k}}^{\mathfrak{l},\mathfrak{t}}(\tilde{N}) \in \mathcal{M}(\mathfrak{l}, \mathfrak{k}, Z_{U(\mathfrak{l})}).$$

Moreover,

$$\sigma(\mathfrak{l}, R\Gamma_{\mathfrak{l},\mathfrak{k}}^{\mathfrak{l},\mathfrak{t}}(\tilde{N})) \subset \sigma(\mathfrak{l}, \tilde{N}).$$

Proof See Proposition 2.12 and Corollary 2.8 in [Z]. \square

Corollary 1.10 *If $N \in \mathcal{M}(\mathfrak{g}, \mathfrak{t}, Z_{U(\mathfrak{l})}) := \mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})}) \cap \mathcal{M}(\mathfrak{g}, \mathfrak{t})$, then*

$$R\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{t}}(N) \in \mathcal{M}(\mathfrak{g}, \mathfrak{k}, Z_{U(\mathfrak{l})}).$$

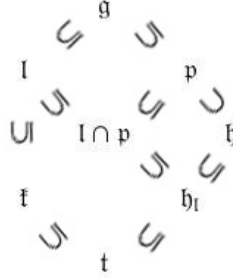
Moreover,

$$\sigma(\mathfrak{l}, R\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{t}}(N)) \subseteq \sigma(\mathfrak{l}, N).$$

Proof Apply Propositions 1.8 and 1.9. \square

Note that the isomorphism (1) enables us to write simply $\Gamma_{\mathfrak{k},\mathfrak{t}}$ instead of $\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{t}}$.

For $\mathfrak{g} \supseteq \mathfrak{l} \supseteq \mathfrak{k} \supseteq \mathfrak{t}$ as above, let \mathfrak{p} be a \mathfrak{t} -compatible parabolic subalgebra of \mathfrak{g} . It follows immediately that $\mathfrak{l} \cap \mathfrak{p}$ is a \mathfrak{t} -compatible parabolic subalgebra of \mathfrak{l} . Let $\mathfrak{h}_{\mathfrak{l}} \subset \mathfrak{l} \cap \mathfrak{p}$ be a Cartan subalgebra of \mathfrak{l} containing \mathfrak{t} , and let $\mathfrak{h} \subset \mathfrak{p}$ be a Cartan subalgebra of \mathfrak{g} such that $\mathfrak{h}_{\mathfrak{l}} = \mathfrak{h} \cap \mathfrak{l}$. We have the following diagram of subalgebras:



In this setup we have the following result.

Theorem 1.11 *Suppose $N \in \mathcal{M}(\mathfrak{g}, \mathfrak{p}) \cap \mathcal{M}(\mathfrak{g}, \mathfrak{t})$, M is a non-zero subquotient of $R\Gamma_{\mathfrak{t}, \mathfrak{t}}(N)$ and $\eta|_{\mathfrak{b}_l}$ is not l -integral for all $\eta \in \text{supp}_{\mathfrak{b}_l} N$. Then $l \not\subseteq \mathfrak{g}[M]$.*

Proof Every central character of l in M is a central character of l in N . This follows from Corollary 2.8 in [Z]. By our assumptions, no central character of l in N is l -integral. Hence, no l -submodule of M is finite dimensional, and thus $l \not\subseteq \mathfrak{g}[M]$. \square

2 The fundamental series: main results

We now introduce one of our main objects of study: the fundamental series of generalized Harish-Chandra modules.

We start by fixing some more notation: if \mathfrak{q} is a subalgebra of \mathfrak{g} and J is a \mathfrak{q} -module, we set $\text{ind}_{\mathfrak{q}}^{\mathfrak{g}} J := U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} J$ and $\text{pro}_{\mathfrak{q}}^{\mathfrak{g}} J := \text{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), J)$. For a finite-dimensional \mathfrak{p} - or $\bar{\mathfrak{p}}$ -module E we set $N_{\mathfrak{p}}(E) := \Gamma_{\mathfrak{t}, 0}(\text{pro}_{\mathfrak{p}}^{\mathfrak{g}}(E \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n})))$, $N_{\bar{\mathfrak{p}}}(E^*) := \Gamma_{\mathfrak{t}, 0}(\text{pro}_{\bar{\mathfrak{p}}}^{\mathfrak{g}}(E^* \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n}^*)))$. One can show that both $N_{\mathfrak{p}}(E)$ and $N_{\bar{\mathfrak{p}}}(E^*)$ have simple socles as long as E itself is simple.

The *fundamental series* of $(\mathfrak{g}, \mathfrak{k})$ -modules of finite type $F(\mathfrak{k}, \mathfrak{p}, E)$ is defined as follows. Let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ be a minimal compatible parabolic subalgebra, E be a simple finite dimensional \mathfrak{p} -module on which \mathfrak{t} acts via the weight $\omega \in \mathfrak{t}^*$, and $\mu := \omega + 2\rho_{\mathfrak{n}}^{\perp}$ where $\rho_{\mathfrak{n}}^{\perp} := \rho_{\mathfrak{n}} - \rho$. Set

$$F(\mathfrak{k}, \mathfrak{p}, E) := R\Gamma_{\mathfrak{t}, \mathfrak{t}}(N_{\mathfrak{p}}(E)).$$

In the rest of the paper we assume that $\mathfrak{h} \cap \mathfrak{k}$ is a Cartan subalgebra of \mathfrak{k} .

Theorem 2.1 *a) $F(\mathfrak{k}, \mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k})$ -module of finite type and $Z_{U(\mathfrak{g})}$ acts on $F(\mathfrak{p}, E)$ via the $Z_{U(\mathfrak{g})}$ -character $\theta_{\nu + \tilde{\rho}}$ where $\tilde{\rho} := \rho_{\text{ch}_{\mathfrak{b}}}$ for some Borel subalgebra \mathfrak{b} of \mathfrak{g} with $\mathfrak{b} \supset \mathfrak{h}$, $\mathfrak{b} \subset \mathfrak{p}$ and $\mathfrak{b} \cap \mathfrak{k} = \mathfrak{b}_{\mathfrak{t}}$, and where ν is the \mathfrak{b} -highest weight of E (note that $\nu|_{\mathfrak{k}} = \omega$).*

b) $F(\mathfrak{k}, \mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k})$ -module of finite length.

c) *There is a canonical isomorphism*

$$F(\mathfrak{k}, \mathfrak{p}, E) \simeq R\Gamma_{\tilde{\mathfrak{k}} \cap \mathfrak{m}}(\Gamma_{\tilde{\mathfrak{k}} \cap \mathfrak{m}, 0}(\mathrm{pro}_{\mathfrak{p}}^{\mathfrak{g}}(E \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n}))). \quad (2)$$

Proof Part a) is a recollection of Theorem 2, a) in [PZ2]. Part b) is a recollection of Theorem 2.5 in [PZ5]. Part c) follows from the comparison principle (Proposition 2.6) in [PZ4]. \square

Corollary 2.2 *$F(\mathfrak{k}, \mathfrak{p}, E)$ is a $(\mathfrak{g}, \tilde{\mathfrak{k}})$ -module of finite type.*

Proof As we observed in subsection 1.3, every $(\mathfrak{g}, \mathfrak{k})$ -module of finite type is a $(\mathfrak{g}, \tilde{\mathfrak{k}})$ -module of finite type. \square

Corollary 2.3 *Let \mathfrak{k}_1 and \mathfrak{k}_2 be two algebraic reductive subalgebras such that $\tilde{\mathfrak{k}}_1 = \tilde{\mathfrak{k}}_2$. Suppose that \mathfrak{p} is a parabolic subalgebra which is both \mathfrak{k}_1 - and \mathfrak{k}_2 -compatible and \mathfrak{k}_1 - and \mathfrak{k}_2 -minimal for some Cartan subalgebras \mathfrak{t}_1 of \mathfrak{k}_1 and \mathfrak{t}_2 of \mathfrak{k}_2 . Then there exists a canonical isomorphism*

$$F(\mathfrak{k}_1, \mathfrak{p}, E) \simeq F(\mathfrak{k}_2, \mathfrak{p}, E).$$

Proof Consider the isomorphism (2) for \mathfrak{k}_1 and \mathfrak{k}_2 , and notice that

$$R\Gamma_{\tilde{\mathfrak{k}} \cap \mathfrak{m}}(\Gamma_{\tilde{\mathfrak{k}} \cap \mathfrak{m}, 0}(\mathrm{pro}_{\mathfrak{p}}^{\mathfrak{g}}(E \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n}))))$$

depends only on $\tilde{\mathfrak{k}}$ and \mathfrak{p} , but not on \mathfrak{k}_1 and \mathfrak{k}_2 . \square

Corollary 2.4 *Let M be any non-zero subquotient of $F(\mathfrak{k}, \mathfrak{p}, E)$. If the \mathfrak{b} -highest weight $\nu \in \mathfrak{h}^*$ of E is non-integral after restriction to $\mathfrak{h} \cap \mathfrak{l}$ for any reductive subalgebra \mathfrak{l} of \mathfrak{g} such that $\mathfrak{l} \supset \tilde{\mathfrak{k}}$, then $\tilde{\mathfrak{k}}$ is a maximal reductive subalgebra of $\mathfrak{g}[M]$.*

Proof Corollary 2.2 shows that $\tilde{\mathfrak{k}} \subseteq \mathfrak{g}[M]$. Theorem 1.11 shows that if \mathfrak{l} is a reductive subalgebra of \mathfrak{g} such that \mathfrak{l} is strictly larger than $\tilde{\mathfrak{k}}$, then $\mathfrak{l} \not\subseteq \mathfrak{g}[M]$. The assumption on ν implies that all weights in $\mathrm{supp}_{\mathfrak{h} \cap \mathfrak{l}}(N_{\mathfrak{p}}(E))$ are non-integral with respect to \mathfrak{l} . \square

Example

Here is an example to Corollary 2.4. Let $\mathfrak{g} = F_4$, $\mathfrak{k} \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(6)$. Then $\mathfrak{k} = \tilde{\mathfrak{k}}$. By inspection, there is only one proper intermediate subalgebra \mathfrak{l} , $\tilde{\mathfrak{k}} \subset \mathfrak{l} \subset \mathfrak{g}$, and \mathfrak{l} is isomorphic to $\mathfrak{so}(9)$. We have $\mathfrak{t} = \mathfrak{h}$, and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ is a standard basis of \mathfrak{h}^* , see [Bou]. A weight $\nu = \sum_{i=1}^4 m_i \varepsilon_i$ is \mathfrak{k} -integral iff $m_1 \in \mathbb{Z}$ or $m_1 \in \mathbb{Z} + \frac{1}{2}$, and $(m_2, m_3, m_4) \in \mathbb{Z}^3$ or $(m_2, m_3, m_4) \in \mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. On the other hand, ν is \mathfrak{l} -integral if $(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4$ or $(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. So if the \mathfrak{b} -highest weight ν of E is not \mathfrak{l} -integral, Corollary 2.4 implies that $\mathfrak{g}[M] = \tilde{\mathfrak{k}}$ for any simple subquotient M of $F(\mathfrak{k}, \mathfrak{p}, E)$.

Remark

- a) In [PZ1] another method, based on the notion of a small subalgebra introduced by Willenbring and Zuckerman in [WZ], for computing maximal reductive subalgebras of simple subquotients of $F(\mathfrak{k}, \mathfrak{p}, E)$ is suggested. Note that the subalgebra $\mathfrak{k} \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(6)$ of F_4 considered in the above example is not small in $\mathfrak{so}(9)$, so the above conclusion that $\mathfrak{g}[M] = \mathfrak{k}$ does not follow from [PZ1]. On the other hand, if one replaces \mathfrak{k} in the example by $\mathfrak{k}' \simeq \mathfrak{so}(5) \oplus \mathfrak{so}(4)$, then a conclusion similar to that of the example can be reached both by the method of [PZ1] and by Corollary 2.4.
- b) There are pairs $(\mathfrak{g}, \mathfrak{k})$ to which neither the method of [PZ1] nor Corollary 2.4 apply. Such an example is a pair $(\mathfrak{g} = F_4, \mathfrak{k} \simeq \mathfrak{so}(8))$. The only proper intermediate subalgebra in this case is $\mathfrak{l} \simeq \mathfrak{so}(9)$; however $\mathfrak{so}(8)$ is not small in $\mathfrak{so}(9)$ and any $\mathfrak{k} = \tilde{\mathfrak{k}}$ -integrable weight is also \mathfrak{l} -integrable.

If M is a $(\mathfrak{g}, \mathfrak{k})$ -module of finite type, then $\Gamma_{\mathfrak{k},0}(M^*)$ is a well-defined $(\mathfrak{g}, \mathfrak{k})$ -module of finite type and $\Gamma_{\mathfrak{k},0}(\cdot)$ is an involution on the category of $(\mathfrak{g}, \mathfrak{k})$ -modules of finite type. We put $\Gamma_{\mathfrak{k},0}(M^*) := M_{\mathfrak{k}}^*$. There is an obvious \mathfrak{g} -invariant non-degenerate pairing $M \times M_{\mathfrak{k}}^* \rightarrow \mathbb{C}$.

The following five statements are recollections of the main results of [PZ2] (Theorem 2 through Corollary 4).

Theorem 2.5 *Assume that $V(\mu)$ is a generic \mathfrak{k} -type and that $\mathfrak{p} = \mathfrak{p}_{\mu+2\rho}$ (μ is necessarily $\mathfrak{b}_{\mathfrak{k}}$ -dominant and \mathfrak{k} -integral).*

- a) $F^i(\mathfrak{k}, \mathfrak{p}, E) = 0$ for $i \neq s := \dim \mathfrak{n}_{\mathfrak{k}}$.
- b) *There is a \mathfrak{k} -module isomorphism*

$$F^s(\mathfrak{k}, \mathfrak{p}, E)[\mu] \cong \mathbb{C}^{\dim E} \otimes V(\mu),$$

and $V(\mu)$ is the unique minimal \mathfrak{k} -type of $F^s(\mathfrak{k}, \mathfrak{p}, E)$.

- c) *Let $\bar{F}^s(\mathfrak{k}, \mathfrak{p}, E)$ be the \mathfrak{g} -submodule of $F^s(\mathfrak{k}, \mathfrak{p}, E)$ generated by $F^s(\mathfrak{k}, \mathfrak{p}, E)[\mu]$. Then $\bar{F}^s(\mathfrak{k}, \mathfrak{p}, E)$ is simple and $\bar{F}^s(\mathfrak{k}, \mathfrak{p}, E) = \text{Soc} F^s(\mathfrak{k}, \mathfrak{p}, E)$. Moreover, $F^s(\mathfrak{k}, \mathfrak{p}, E)$ is cogenerated by $F^s(\mathfrak{k}, \mathfrak{p}, E)[\mu]$. This implies that $F^s(\mathfrak{k}, \mathfrak{p}, E)_{\mathfrak{k}}^*$ is generated by $F^s(\mathfrak{k}, \mathfrak{p}, E)_{\mathfrak{k}}^*[w_m(-\mu)]$, where $w_m \in W_{\mathfrak{k}}$ is the element of maximal length in the Weyl group $W_{\mathfrak{k}}$ of \mathfrak{k} .*
- d) *For any non-zero \mathfrak{g} -submodule M of $F^s(\mathfrak{k}, \mathfrak{p}, E)$ there is an isomorphism of \mathfrak{m} -modules*

$$H^r(\mathfrak{n}, M)^{\omega} \cong E.$$

Theorem 2.6 *Let M be a simple $(\mathfrak{g}, \mathfrak{k})$ -module of finite type with minimal \mathfrak{k} -type $V(\mu)$ which is generic. Then $\mathfrak{p} := \mathfrak{p}_{\mu+2\rho} = \mathfrak{m} \supset \mathfrak{n}$ is a minimal compatible parabolic subalgebra. Let $\omega := \mu - 2\rho_{\mathfrak{n}}^{\perp}$ (recall that $\rho_{\mathfrak{n}}^{\perp} = \rho_{\text{ch}_{\mathfrak{k}}(\mathfrak{n} \cap \mathfrak{k}^{\perp})}$), and let E be the \mathfrak{p} -module $H^r(\mathfrak{n}, M)^{\omega}$ with trivial \mathfrak{n} -action, where $r = \dim(\mathfrak{n} \cap \mathfrak{k}^{\perp})$. Then E is a simple \mathfrak{p} -module, the pair (\mathfrak{p}, E) satisfies the hypotheses of Theorem 2.5, and M is canonically isomorphic to $\bar{F}^s(\mathfrak{p}, E)$ for $s = \dim(\mathfrak{n} \cap \mathfrak{k})$.*

Corollary 2.7 (*Generic version of a theorem of Harish-Chandra*). *There exist at most finitely many simple $(\mathfrak{g}, \mathfrak{k})$ -modules M of finite type with a fixed $Z_{U(\mathfrak{g})}$ -character such that a minimal \mathfrak{k} -type of M is generic. (Moreover, each such M has a unique minimal \mathfrak{k} -type by Theorem 2.5 b).)*

Proof By Theorems 2.1 a) and 2.6, if M is a simple $(\mathfrak{g}, \mathfrak{k})$ -module of finite type with generic minimal \mathfrak{k} -type $V(\mu)$ for some μ , then the $Z_{U(\mathfrak{g})}$ -character of M is $\theta_{\nu+\tilde{\rho}}$. There are finitely many Borel subalgebras \mathfrak{b} as in Theorem 2.1 a); thus, if $\theta_{\nu+\tilde{\rho}}$ is fixed, there are finitely many possibilities for the weight ν (as $\theta_{\nu+\tilde{\rho}}$ determines $\nu + \tilde{\rho}$ up to a finite choice). Therefore, up to isomorphism, there are finitely many possibilities for the \mathfrak{p} -module E , and hence, up to isomorphism, there are finitely many possibilities for M . \square

Theorem 2.8 *Assume that the pair $(\mathfrak{g}, \mathfrak{k})$ is regular, i.e. \mathfrak{k} contains a regular element of \mathfrak{g} . Let M be a simple $(\mathfrak{g}, \mathfrak{k})$ -module (a priori of infinite type) with a minimal \mathfrak{k} -type $V(\mu)$ which is generic. Then M has finite type, and hence by Theorem 2.6, M is canonically isomorphic to $\bar{F}^s(\mathfrak{p}, E)$ (where \mathfrak{p}, E and s are as in Theorem 2.6).*

Corollary 2.9 *Let the pair $(\mathfrak{g}, \mathfrak{k})$ be regular.*

- a) *There exist at most finitely many simple $(\mathfrak{g}, \mathfrak{k})$ -modules M with a fixed $Z_{U(\mathfrak{g})}$ -character, such that a minimal \mathfrak{k} -type of M is generic. All such M are of finite type (and have a unique minimal \mathfrak{k} -type by Theorem 2.5 b)).*
- b) *(Generic version of Harish-Chandra's admissibility theorem). Every simple $(\mathfrak{g}, \mathfrak{k})$ -module with a generic minimal \mathfrak{k} -type has finite type.*

Proof The proof of a) is as the proof of Corollary 2.7 but uses Theorem 2.8 instead of Theorem 2.6, and b) is a direct consequence of Theorem 2.8. \square

The following statement follows from Corollary 2.4 and Theorem 2.6.

Corollary 2.10 *Let M be as in Theorem 2.6. If the \mathfrak{b} -highest weight of E is not \mathfrak{l} -integral for any reductive subalgebra \mathfrak{l} with $\tilde{\mathfrak{k}} \subset \mathfrak{l} \subseteq \mathfrak{g}$, then $\tilde{\mathfrak{k}}$ is a maximal reductive subalgebra of $\mathfrak{g}[M]$.*

Definition 2.11 *Let $\mathfrak{p} \supset \mathfrak{b}_{\mathfrak{k}}$ be a minimal \mathfrak{k} -compatible parabolic subalgebra and let E be a simple finite dimensional \mathfrak{p} -module on which \mathfrak{k} acts by ω . We say that the pair (\mathfrak{p}, E) is allowable if $\mu = \omega + 2\rho_{\mathfrak{n}}^{\perp}$ is dominant integral for \mathfrak{k} , $\mathfrak{p}_{\mu+2\rho} = \mathfrak{p}$, and $V(\mu)$ is generic.*

Theorem 2.6 provides a classification of simple $(\mathfrak{g}, \mathfrak{k})$ -modules with generic minimal \mathfrak{k} -type in terms of allowable pairs. Note that for any minimal \mathfrak{k} -compatible parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}_{\mathfrak{k}}$, there exists a \mathfrak{p} -module E such that (\mathfrak{p}, E) is allowable.

3 The case $\mathfrak{k} \simeq \mathfrak{sl}(2)$

Let $\mathfrak{k} \simeq \mathfrak{sl}(2)$. In this case there is only one minimal \mathfrak{t} -compatible parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ of \mathfrak{g} which contains $\mathfrak{b}_{\mathfrak{t}}$. Furthermore, we can identify the elements of \mathfrak{t}^* with complex numbers, and the $\mathfrak{b}_{\mathfrak{t}}$ -dominant integral weights of \mathfrak{t} in $\mathfrak{n} \cap \mathfrak{t}^{\perp}$ with non-negative integers. It is shown in [PZ2] that in this case the genericity assumption on a \mathfrak{k} -type $V(\mu)$, $\mu \geq 0$, amounts to the condition $\mu \geq \Gamma := \tilde{\rho}(h) - 1$ where $h \in \mathfrak{h}$ is the semisimple element in a standard basis e, h, f of $\mathfrak{k} \simeq \mathfrak{sl}(2)$.

In our work [PZ5] we have proved a different sufficient condition for the main results of [PZ2] to hold when $\mathfrak{k} \simeq \mathfrak{sl}(2)$. Let λ_1 and λ_2 be the maximum and submaximum weights of \mathfrak{t} in $\mathfrak{n} \cap \mathfrak{t}^{\perp}$ (if λ_1 has multiplicity at least two in $\mathfrak{n} \cap \mathfrak{t}^{\perp}$, then $\lambda_2 = \lambda_1$; if $\dim \mathfrak{n} \cap \mathfrak{t}^{\perp} = 1$, then $\lambda_2 = 0$). Set $\Lambda := \frac{\lambda_1 + \lambda_2}{2}$.

Theorem 3.1 *If $\mathfrak{k} \simeq \mathfrak{sl}(2)$, all statements of section 2 from Theorem 2.5 through Corollary 2.9 hold if we replace the assumption that μ is generic by the assumption $\mu \geq \Lambda$. As a consequence, the isomorphism classes of simple $(\mathfrak{g}, \mathfrak{k})$ -modules whose minimal \mathfrak{k} -type is $V(\mu)$ with $\mu \geq \Lambda$ are parameterized by the isomorphism classes of simple \mathfrak{p} -modules E on which \mathfrak{t} acts via $\mu - 2\rho_{\mathfrak{n}}^+$.*

The $\mathfrak{sl}(2)$ -subalgebras of a simple Lie algebra are classified (up to conjugation) by Dynkin in [D]. We will now illustrate the computation of the bound Λ as well as the genericity condition on μ in examples.

We first consider three types of $\mathfrak{sl}(2)$ -subalgebras of a simple Lie algebra: long root- $\mathfrak{sl}(2)$, short root- $\mathfrak{sl}(2)$ and principal $\mathfrak{sl}(2)$ (of course, there are short roots only for the series B, C and for G_2 and F_4). We compare the bounds Λ and Γ in the following table.

	long root	short root	principal
$A_n, n \geq 2$	$\Gamma = n - 1 \geq 1 = \Lambda$	not applicable	$\Gamma = \frac{n(n+1)(n+2)}{6} - 1 \geq 2n - 1 = \Lambda$
$B_n, n \geq 2$	$\Gamma = 2n - 3 \geq 1 = \Lambda$	$\Gamma = 2n - 2 \geq 2 = \Lambda$	$\Gamma = \frac{n(n+1)(4n-1)}{6} - 1 > 4n - 3 = \Lambda$
$C_n, n \geq 3$	$\Gamma = n - 1 > 1 = \Lambda$	$\Gamma = 2n - 2 > 2 = \Lambda$	$\Gamma = \frac{n(n+1)(2n+1)}{3} - 1 > 4n - 3 = \Lambda$
$D_n, n \geq 4$	$\Gamma = 2n - 4 > 1 = \Lambda$	not applicable	$\Gamma = \frac{2(n-1)n(n+1)}{3} - 1 > 4n - 7 = \Lambda$
E_6	$\Gamma = 10 > 1 = \Lambda$	not applicable	$\Gamma = 155 > 21 = \Lambda$
E_7	$\Gamma = 16 > 1 = \Lambda$	not applicable	$\Gamma = 398 > 33 = \Lambda$
E_8	$\Gamma = 28 > 1 = \Lambda$	not applicable	$\Gamma = 1239 > 57 = \Lambda$
F_4	$\Gamma = 7 > 1 = \Lambda$	$\Gamma = 10 > 2 = \Lambda$	$\Gamma = 109 > 21 = \Lambda$
G_2	$\Gamma = 2 > 1 = \Lambda$	$\Gamma = 4 > 3 = \Lambda$	$\Gamma = 15 > 9 = \Lambda$

Table A

Let's discuss the case $\mathfrak{g} = F_4$ in more detail. Recall that the *Dynkin index* of a semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is the quotient of the normalized \mathfrak{g} -invariant symmetric bilinear form on \mathfrak{g} restricted to \mathfrak{s} and the

normalized \mathfrak{s} -invariant symmetric bilinear form on \mathfrak{s} , where for both \mathfrak{g} and \mathfrak{s} the square length of a long root equals 2. According to Dynkin [D], the conjugacy class of an $\mathfrak{sl}(2)$ -subalgebra \mathfrak{k} of F_4 is determined by the Dynkin index of \mathfrak{k} in F_4 . Moreover, for $\mathfrak{g} = F_4$ the following integers are Dynkin indices of $\mathfrak{sl}(2)$ -subalgebras: 1(long root), 2(short root), 3, 4, 6, 8, 9, 10, 11, 12, 28, 35, 36, 60, 156. The bounds Λ and Γ are given in the following table.

Dynkin index	1	2	3
	$\Gamma = 7 > 1 = \Lambda$	$\Gamma = 10 > 2 = \Lambda$	$\Gamma = 14 > 3 = \Lambda$
Dynkin index	4	6	8
	$\Gamma = 15 > 3 = \Lambda$	$\Gamma = 16 > 4 = \Lambda$	$\Gamma = 17 > 4 = \Lambda$
Dynkin index	9	10	11
	$\Gamma = 25 > 5 = \Lambda$	$\Gamma = 26 > 5 = \Lambda$	$\Gamma = 28 > 6 = \Lambda$
Dynkin index	12	28	35
	$\Gamma = 29 > 6 = \Lambda$	$\Gamma = 45 > 9 = \Lambda$	$\Gamma = 50 > 10 = \Lambda$
Dynkin index	36	60	156
	$\Gamma = 51 > 10 = \Lambda$	$\Gamma = 67 > 13 = \Lambda$	$\Gamma = 109 > 21 = \Lambda$

Table B

We conclude this section by recalling a conjecture from [PZ5]. Let $\mathcal{C}_{\bar{\mathfrak{p}}, \mathfrak{t}, n}$ denote the full subcategory of \mathfrak{g} -mod consisting of finite-length modules with simple subquotients which are $\bar{\mathfrak{p}}$ -locally finite $(\mathfrak{g}, \mathfrak{t})$ -modules N whose \mathfrak{t} -weight spaces N^β , $\beta \in \mathbb{Z}$, satisfy $\beta \geq n$. Let $\mathcal{C}_{\mathfrak{t}, n}$ be the full subcategory of \mathfrak{g} -mod consisting of finite length modules whose simple subquotients are $(\mathfrak{g}, \mathfrak{k})$ -modules with minimal $\mathfrak{k} \simeq \mathfrak{sl}(2)$ -type $V(\mu)$ for $\mu \geq n$. We show in [PZ5] that the functor $R^1\Gamma_{\mathfrak{t}, \mathfrak{t}}$ is a well-defined fully faithful functor from $\mathcal{C}_{\bar{\mathfrak{p}}, \mathfrak{t}, n+2}$ to $\mathcal{C}_{\mathfrak{t}, n}$ for $n \geq 0$. Moreover, we make the following conjecture.

Conjecture 3.2 *Let $n \geq \Lambda$. Then $R^1\Gamma_{\mathfrak{t}, \mathfrak{t}}$ is an equivalence between the categories $\mathcal{C}_{\bar{\mathfrak{p}}, \mathfrak{t}, n+2}$ and $\mathcal{C}_{\mathfrak{t}, n}$.*

We have proof of this conjecture for $\mathfrak{g} \simeq \mathfrak{sl}(2)$ and, jointly with V. Serganova, for $\mathfrak{g} \simeq \mathfrak{sl}(3)$.

4 Eligible subalgebras

In what follows we adopt the following terminology. A *root subalgebra* of \mathfrak{g} is a subalgebra which contains a Cartan subalgebra of \mathfrak{g} . An *r-subalgebra* of \mathfrak{g} is a subalgebra \mathfrak{l} whose root spaces (with respect to a Cartan subalgebra of

\mathfrak{l}) are root spaces of \mathfrak{g} . The notion of r -subalgebra goes back to [D]. A root subalgebra is, of course, an r -subalgebra.

We now give the following key definition.

Definition 4.1 *An algebraic reductive in \mathfrak{g} subalgebra \mathfrak{k} is eligible if $C(\mathfrak{t}) = \mathfrak{t} + C(\mathfrak{k})$.*

Note that in the above definition one can replace \mathfrak{t} with any Cartan subalgebra of \mathfrak{k} . Furthermore, if \mathfrak{k} is eligible then $\mathfrak{h} \subset C(\mathfrak{t}) = \mathfrak{t} + C(\mathfrak{k}) \subset \tilde{\mathfrak{k}} = \mathfrak{k} + C(\mathfrak{k})$, i.e. \mathfrak{h} is a Cartan subalgebra of both $\tilde{\mathfrak{k}}$ and \mathfrak{g} . In particular, $\tilde{\mathfrak{k}}$ is a reductive root subalgebra of \mathfrak{g} . As \mathfrak{k} is an ideal in $\tilde{\mathfrak{k}}$, \mathfrak{k} is an r -subalgebra of \mathfrak{g} .

Proposition 4.2 *Assume \mathfrak{k} is an r -subalgebra of \mathfrak{g} . The following three conditions are equivalent:*

- (i) \mathfrak{k} is eligible;
- (ii) $C(\mathfrak{k})_{ss} = C(\mathfrak{t})_{ss}$;
- (iii) $\dim C(\mathfrak{k})_{ss} = \dim C(\mathfrak{t})_{ss}$.

Proof The implications (i) \Rightarrow (ii) \Rightarrow (iii) are obvious. To see that (iii) implies (i), observe that if \mathfrak{k} is an r -subalgebra of \mathfrak{g} , then $\mathfrak{h} \subseteq \mathfrak{t} + C(\mathfrak{k}) \subseteq C(\mathfrak{t})$. Therefore the inclusion $\mathfrak{t} + C(\mathfrak{k}) \subseteq C(\mathfrak{t})$ is proper if and only if $\mathfrak{g}^{\pm\alpha} \in C(\mathfrak{t}) \setminus C(\mathfrak{k})$ for some root $\alpha \in \Delta$, or, equivalently, if the inclusion $C(\mathfrak{k})_{ss} \subseteq C(\mathfrak{t})_{ss}$ is proper. \square

An algebraic, reductive in \mathfrak{g} , r -subalgebra \mathfrak{k} may or may not be eligible. If \mathfrak{k} is a root subalgebra, then \mathfrak{k} is always eligible. If \mathfrak{g} is simple of types A, C, D and \mathfrak{k} is a semisimple r -subalgebra, then \mathfrak{k} is necessarily eligible. In general, a semisimple r -subalgebra is eligible if and only if the roots of \mathfrak{g} which vanish on \mathfrak{t} are strongly orthogonal to the roots of \mathfrak{k} . For example, if \mathfrak{g} is simple of type B and \mathfrak{k} is a simple r -subalgebra of type B of rank less or equal than $\text{rk } \mathfrak{g} - 2$, then $C(\mathfrak{k})_{ss}$ is simple of type D whereas $C(\mathfrak{t})_{ss}$ is simple of type B . Hence in this case \mathfrak{k} is not eligible.

Note, however that any semisimple r -subalgebra \mathfrak{k}' can be extended to an eligible subalgebra \mathfrak{k} just by setting $\mathfrak{k} := \mathfrak{k}' + \mathfrak{h}_{C(\mathfrak{k}')}$ where $\mathfrak{h}_{C(\mathfrak{k}')}$ is a Cartan subalgebra of $C(\mathfrak{k}')$. Finally, note that if x is any algebraic regular semisimple element of $C(\mathfrak{k}')$, then $\mathfrak{k} := \mathfrak{k}' \oplus Z(C(\mathfrak{k}')) + \mathbb{C}x$ is an eligible subalgebra of \mathfrak{g} . Indeed, if $\mathfrak{t}' \subseteq \mathfrak{k}'$ is a Cartan subalgebra of \mathfrak{k}' , and $\mathfrak{h}_{\mathfrak{t}'} := \mathfrak{t}' \oplus Z(C(\mathfrak{k}')) + \mathbb{C}x$ is the corresponding Cartan subalgebra of \mathfrak{k} , then $C(\mathfrak{h}_{\mathfrak{t}'})$ is a Cartan subalgebra of \mathfrak{g} . Hence,

$$C(\mathfrak{h}_{\mathfrak{t}'}) = \mathfrak{h}_{\mathfrak{t}'} + C(\mathfrak{k}) \tag{3}$$

as the right-hand side of (3) necessarily contains a Cartan subalgebra of \mathfrak{g} .

To any eligible subalgebra \mathfrak{k} we assign a unique weight $\varkappa \in \mathfrak{h}^*$ (the "canonical weight associated with \mathfrak{k} "). It is defined by the conditions $\varkappa|_{(\mathfrak{h} \cap \mathfrak{k}_{ss})} = \rho$, $\varkappa|_{(\mathfrak{h} \cap C(\mathfrak{k}))} = 0$.

5 The generalized discrete series

In what follows we assume that \mathfrak{f} is eligible and $\mathfrak{h} \subset \tilde{\mathfrak{f}}$. In this case \mathfrak{h} is a Cartan subalgebra both of $\tilde{\mathfrak{f}}$ and \mathfrak{g} . Let $\lambda \in \mathfrak{h}^*$ and set $\gamma := \lambda|_{\mathfrak{t}}$. Assume that $\mathfrak{m} := \mathfrak{m}_\gamma = C(\mathfrak{t})$. Assume furthermore that λ is \mathfrak{m} -integral and let E_λ be a simple finite-dimensional \mathfrak{m} -module with \mathfrak{b} -highest weight λ . Then

$$D(\tilde{\mathfrak{f}}, \lambda) := F^s(\tilde{\mathfrak{f}}, \mathfrak{p}_\gamma, E_\lambda \otimes \Lambda^{\dim \mathfrak{m}_\gamma}(\mathfrak{n}_\gamma^*))$$

is by definition a *generalized discrete series module*.

Note that since $D(\tilde{\mathfrak{f}}, \lambda)$ is a fundamental series module, Theorem 2.1 applies to $D(\tilde{\mathfrak{f}}, \lambda)$. In the case when \mathfrak{f} is a root subalgebra and λ is regular, we have $\lambda = \gamma$ and \mathfrak{p}_γ is a Borel subalgebra of \mathfrak{g} which we denote by \mathfrak{b}_λ . Then $D(\tilde{\mathfrak{f}}, \lambda) = R^s \Gamma_{\tilde{\mathfrak{f}}, \mathfrak{b}}(\Gamma_{\mathfrak{b}}(\text{pro}_{\mathfrak{b}_\lambda}^{\mathfrak{g}} E_\lambda))$, i.e. $D(\tilde{\mathfrak{f}}, \lambda)$ is cohomologically co-induced from a 1-dimensional \mathfrak{b}_λ -module. If in addition, $\tilde{\mathfrak{f}}$ is a symmetric subalgebra, λ is $\tilde{\mathfrak{f}}$ -integral, and $\lambda - \tilde{\rho}$ is \mathfrak{b}_λ -dominant regular, then $D(\tilde{\mathfrak{f}}, \lambda)$ is a $(\mathfrak{g}, \tilde{\mathfrak{f}})$ -module in Harish-Chandra's discrete series, see [KV], Ch.XI.

Suppose \mathfrak{f} is eligible but \mathfrak{f} is not a root subalgebra. Suppose further that $\tilde{\mathfrak{f}}$ is symmetric. Any simple subquotient M of $D(\tilde{\mathfrak{f}}, \lambda)$ is a $(\mathfrak{g}, \tilde{\mathfrak{f}})$ -module and thus a Harish-Chandra module for $(\mathfrak{g}, \tilde{\mathfrak{f}})$. However, M may or may not be in the discrete series of $(\mathfrak{g}, \tilde{\mathfrak{f}})$ -modules. This becomes clear in Theorem 5.6 below.

Our first result is a sharper version of the main result of [PZ3] for an eligible \mathfrak{f} .

Theorem 5.1 *Let $\mathfrak{f} \subseteq \mathfrak{g}$ be eligible. Assume that $\lambda - 2\chi$ is $\tilde{\mathfrak{f}}$ -integral and dominant. Then, $D(\tilde{\mathfrak{f}}, \lambda) \neq 0$. Moreover, if we set $\mu := (\lambda - 2\chi)|_{\mathfrak{t}}$, then $V(\mu)$ is the unique minimal $\tilde{\mathfrak{f}}$ -type of $D(\tilde{\mathfrak{f}}, \lambda)$. Finally, there are isomorphisms of simple finite-dimensional $\tilde{\mathfrak{f}}$ -modules*

$$D(\tilde{\mathfrak{f}}, \lambda)[\mu] \cong D(\tilde{\mathfrak{f}}, \lambda)\langle \lambda - 2\chi \rangle \simeq V_{\tilde{\mathfrak{f}}}(\lambda - 2\chi).$$

Proof Note that $\mu = \gamma - 2\rho$. By Lemma 2 in [PZ3]

$$\dim \text{Hom}_{\tilde{\mathfrak{f}}}(V(\mu), D(\tilde{\mathfrak{f}}, \lambda)) = \dim E_\lambda,$$

and hence $D(\tilde{\mathfrak{f}}, \lambda) \neq 0$. In addition, $V(\mu)$ is the unique minimal $\tilde{\mathfrak{f}}$ -type of $D(\tilde{\mathfrak{f}}, \lambda)$. By construction, $D(\tilde{\mathfrak{f}}, \lambda)[\mu]$ is a finite-dimensional $\tilde{\mathfrak{f}}$ -module. We will use Theorem 2.1 c) to compute $D(\tilde{\mathfrak{f}}, \lambda)[\mu]$ as a $\tilde{\mathfrak{f}}$ -module. Since \mathfrak{f} is eligible, we have $\mathfrak{m} = \mathfrak{t} + C(\mathfrak{f})$. As $[\mathfrak{t}, C(\mathfrak{f})] = 0$ and \mathfrak{t} is toral, the restriction of E_λ to $C(\mathfrak{f})$ is simple. We have

$$\tilde{\mathfrak{f}} = \mathfrak{f}_{\text{ss}} \oplus C(\mathfrak{f}),$$

and hence there is an isomorphism of $\tilde{\mathfrak{f}}$ -modules

$$V_{\tilde{\mathfrak{f}}}(\lambda - 2\chi) \cong (V(\mu)|_{\mathfrak{f}_{\text{ss}}}) \boxtimes E_\lambda.$$

Consequently, we have isomorphisms of $C(\mathfrak{f})$ -modules

$$\mathrm{Hom}_{\mathfrak{k}}(V(\mu), V_{\mathfrak{k}}(\lambda - 2\kappa)) \cong \mathrm{Hom}_{\mathfrak{k}_{\mathrm{ss}}}((V(\mu)|_{\mathfrak{k}_{\mathrm{ss}}}), V_{\mathfrak{k}}(\lambda - 2\kappa)) \cong E_{\lambda}. \quad (4)$$

Write $\mathfrak{p}_{\gamma} = \mathfrak{p}$ and note that $\mathfrak{k} \cap \mathfrak{m} = \mathfrak{m}$. By Theorem 2.1 c), we have a canonical isomorphism

$$D(\mathfrak{k}, \lambda) \cong R^s \Gamma_{\mathfrak{k}, \mathfrak{m}}(\Gamma_{\mathfrak{m}, 0}(\mathrm{pro}_{\mathfrak{p}}^{\mathfrak{g}} E_{\lambda})).$$

According to the theory of the bottom layer [KV], Ch. V, Sec. 6, $D(\mathfrak{k}, \lambda)$ contains the \mathfrak{k} -module

$$R^s \Gamma_{\mathfrak{k}, \mathfrak{m}}(\Gamma_{\mathfrak{m}, 0}(\mathrm{pro}_{\mathfrak{k} \cap \mathfrak{p}}^{\mathfrak{k}} E_{\lambda}))$$

which is in turn isomorphic to $V_{\mathfrak{k}}(\lambda - 2\kappa)$.

By the above argument, we have a sequence of injections

$$V_{\mathfrak{k}}(\lambda - 2\kappa) \hookrightarrow D(\mathfrak{k}, \lambda)\langle \lambda - 2\kappa \rangle \hookrightarrow D(\mathfrak{k}, \lambda)[\mu].$$

We conclude from (4) that the above sequence of injections is in fact a sequence of isomorphisms of simple \mathfrak{k} -modules. \square

Corollary 5.2 *Under the assumptions of Theorem 5.1, there exists a simple $(\mathfrak{g}, \mathfrak{k})$ -module M of finite type over \mathfrak{k} , such that if $V(\mu_M)$ is a minimal \mathfrak{k} -type of M , then $V(\mu_M)$ is the unique minimal \mathfrak{k} -type of M and there is an isomorphism of finite-dimensional \mathfrak{k} -modules*

$$M[\mu_M] \cong V_{\mathfrak{k}}(\lambda - 2\kappa).$$

In particular, $M[\mu_M]$ is a simple \mathfrak{k} -submodule of M .

Proof First we construct a module M as required. Let $\bar{D}(\mathfrak{k}, \lambda)$ be the $U(\mathfrak{g})$ -submodule of $D(\mathfrak{k}, \lambda)$ generated by the \mathfrak{k} -isotypic component $D(\mathfrak{k}, \lambda)\langle \lambda - 2\kappa \rangle$. Suppose N is a proper \mathfrak{g} -submodule of $\bar{D}(\mathfrak{k}, \lambda)$. Since $D(\mathfrak{k}, \lambda)\langle \lambda - 2\kappa \rangle$ is simple over \mathfrak{k} ,

$$N \cap (D(\mathfrak{k}, \lambda)\langle \lambda - 2\kappa \rangle) = 0.$$

Thus, if $N(\mathfrak{k}, \lambda)$ is the maximum proper submodule of $\bar{D}(\mathfrak{k}, \lambda)$, the quotient module

$$M = \bar{D}(\mathfrak{k}, \lambda)/N(\mathfrak{k}, \lambda)$$

is a simple $(\mathfrak{g}, \mathfrak{k})$ -module, and M has finite type over \mathfrak{k} . Writing $\mu_M = \mu = \gamma - 2\rho$, we see that M has unique minimal \mathfrak{k} -type $V(\mu_M)$. Finally, by Theorem 5.1, we have an isomorphism of finite-dimensional \mathfrak{k} -modules,

$$M[\mu_M] \cong V_{\mathfrak{k}}(\lambda - 2\kappa).$$

\square

If \mathfrak{k} is symmetric (and hence \mathfrak{k} is a root subalgebra due to the eligibility of \mathfrak{k}), Theorem 5.1 and Corollary 5.2 go back to [V] (where they are proven by a different method).

The following two statements are consequences of the main results of section 2 and Theorem 5.1.

Corollary 5.3 *Let \mathfrak{k} be eligible, $\lambda \in \mathfrak{h}^*$ be such that $\lambda - 2\kappa$ is $\tilde{\mathfrak{k}}$ -integral and $V(\mu)$ is generic for $\mu := \lambda|_{\mathfrak{k}} - 2\rho$.*

- a) *Soc $D(\mathfrak{k}, \lambda)$ is a simple $(\mathfrak{g}, \mathfrak{k})$ -module with unique minimal \mathfrak{k} -type $V(\mu)$.*
- b) *There is a canonical isomorphism of $C(\mathfrak{k})$ -modules*

$$\mathrm{Hom}_{\mathfrak{k}}(V(\mu), \mathrm{Soc} D(\mathfrak{k}, \lambda)) \simeq E_{\lambda}.$$

- c) *There is a canonical isomorphism of $\tilde{\mathfrak{k}}$ -modules*

$$V(\mu) \otimes \mathrm{Hom}_{\mathfrak{k}}(V(\mu), \mathrm{Soc} D(\mathfrak{k}, \lambda)) \simeq V_{\tilde{\mathfrak{k}}}(\lambda - 2\kappa),$$

i.e. the $V(\mu)$ -isotypic component of $\mathrm{Soc} D(\mathfrak{k}, \lambda)$ is a simple $\tilde{\mathfrak{k}}$ -module isomorphic to $V_{\tilde{\mathfrak{k}}}(\lambda - 2\kappa)$.

- d) *If $\lambda - 2\kappa$ is not \mathfrak{l} -integral for any reductive subalgebra \mathfrak{l} such that $\tilde{\mathfrak{k}} \subset \mathfrak{l} \subseteq \mathfrak{g}$, then $\tilde{\mathfrak{k}}$ is a maximal reductive subalgebra of $\mathfrak{g}[M]$ for any subquotient M of $D(\mathfrak{k}, \lambda)$, in particular of $\mathrm{Soc} D(\mathfrak{k}, \lambda)$.*

Proof

a) Observe that $\mathfrak{p}_{\gamma} = \mathfrak{p}_{\mu+2\rho}$, and $D(\mathfrak{k}, \lambda) = F^s(\mathfrak{k}, \mathfrak{p}_{\mu+2\rho}, E_{\lambda} \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n}^*))$. So, a) follows from Theorem 2.5 c).

b) By Theorem 2.5 c), $\mathrm{Hom}_{\mathfrak{k}}(V(\mu), \mathrm{Soc} D(\mathfrak{k}, \lambda)) = \mathrm{Hom}_{\mathfrak{k}}(V(\mu), D(\mathfrak{k}, \lambda))$, which in turn is isomorphic to $\mathrm{Hom}_{\mathfrak{k}}(V(\mu), V_{\tilde{\mathfrak{k}}}(\lambda - 2\kappa))$ by Theorem 5.1. The desired isomorphism follows now from (4).

c) This follows from the isomorphism in b) and the isomorphism $V(\mu) \otimes E_{\lambda} \cong V_{\tilde{\mathfrak{k}}}(\lambda - 2\kappa)$ of $\tilde{\mathfrak{k}}$ -modules.

d) Follows from Corollary 2.4. Note that, since \mathfrak{k} is eligible, $\tilde{\mathfrak{k}}$ is a root subalgebra and the condition that $\lambda - 2\kappa$ be not \mathfrak{l} -integral involves only finitely many subalgebras \mathfrak{l} . \square

Corollary 5.4 *Let \mathfrak{k} be eligible and let $V(\mu)$ be a generic \mathfrak{k} -type.*

- a) *Let M be a simple $(\mathfrak{g}, \mathfrak{k})$ -module of finite type with minimal \mathfrak{k} -type $V(\mu)$. Then $M[\mu]$ is a simple finite-dimensional $\tilde{\mathfrak{k}}$ -module isomorphic to $V_{\tilde{\mathfrak{k}}}(\lambda)$ for some weight $\lambda \in \mathfrak{h}^*$ such that $\lambda|_{\mathfrak{k}} = \mu + 2\rho$ and $\mu - 2\kappa$ is $\tilde{\mathfrak{k}}$ -integral. Moreover,*

$$M \cong \mathrm{Soc} D(\mathfrak{k}, \lambda).$$

If in addition λ is not \mathfrak{l} -integral for any reductive subalgebra \mathfrak{l} with $\tilde{\mathfrak{k}} \subset \mathfrak{l} \subseteq \mathfrak{g}$, then $\tilde{\mathfrak{k}}$ is a unique maximal reductive subalgebra of $\mathfrak{g}[M]$.

- b) *If \mathfrak{k} is regular in \mathfrak{g} , then a) holds for any simple $(\mathfrak{g}, \mathfrak{k})$ -module with generic minimal \mathfrak{k} -type $V(\mu)$. In particular M has finite type over $\tilde{\mathfrak{k}}$.*

Proof

a) We apply Theorem 2.6. Since $V(\mu)$ is generic, $\mathfrak{p} = \mathfrak{p}_{\mu+2\rho} = \mathfrak{m}\mathfrak{D}\mathfrak{n}$ is a minimal \mathfrak{t} -compatible parabolic subalgebra. Let $\omega := \mu - 2\rho_{\mathfrak{n}}^{\perp}$ (recall that $\rho_{\mathfrak{n}}^{\perp} = \rho_{\mathfrak{n}} - \rho$) and let Q be the \mathfrak{m} -module $H^r(\mathfrak{n}, M)^{\omega}$ where $r = \dim(\mathfrak{t}^{\perp} \cap \mathfrak{n})$.

Observe that Q is a simple \mathfrak{m} -module and M is canonically isomorphic to $\bar{F}^s(\mathfrak{p}, Q) = \text{Soc } F^s(\mathfrak{p}, Q)$. Let $\lambda \in \mathfrak{h}^*$ be so that $\lambda - 2\tilde{\rho}_{\mathfrak{n}}$ is an extreme weight of \mathfrak{h} in Q . Thus, $F^s(\mathfrak{p}, Q) = F^s(\mathfrak{p}, E_{\lambda} \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n}^*)) = D(\mathfrak{k}, \lambda)$. Finally, $M \cong \text{Soc } D(\mathfrak{k}, \lambda)$, and $\lambda|_{\mathfrak{t}} = \mu + 2\rho$. It follows that $\lambda - 2\chi$ is both \mathfrak{k} -integral and $C(\mathfrak{k})$ -integral. Hence, the weight $\lambda - 2\chi$ is $\tilde{\mathfrak{k}}$ -integral.

b) We apply Theorem 2.8. \square

Corollary 5.5 *If $\mathfrak{k} \simeq \mathfrak{sl}(2)$, the genericity assumption on $V(\mu)$ in Corollaries 5.3 and 5.4 can be replaced by the assumption $\mu \geq \Lambda$.*

Proof The statement follows directly from Theorem 3.1. \square

We conclude this paper by discussing in more detail an example of an eligible $\mathfrak{sl}(2)$ -subalgebra. Note first that if \mathfrak{g} is any simple Lie algebra and $\tilde{\mathfrak{k}}$ is a long root $\mathfrak{sl}(2)$ -subalgebra, then the pair $(\mathfrak{g}, \tilde{\mathfrak{k}})$ is a symmetric pair. This is a well-known fact and it implies in particular that any $(\mathfrak{g}, \tilde{\mathfrak{k}})$ -module of finite type and of finite length is a Harish-Chandra module for the pair $(\mathfrak{g}, \tilde{\mathfrak{k}})$. The latter modules are classified under the assumption of simplicity see [KV], Ch.XI; however, in general, it is an open problem to determine which simple $(\mathfrak{g}, \tilde{\mathfrak{k}})$ -modules have finite type over $\tilde{\mathfrak{k}}$. Without having been explicitly stated, this problem has been discussed in the literature, see [OW] and the references therein. On the other hand, in this case $\Lambda = 1$, hence Corollaries 5.4 and 5.5 provide a classification of simple $(\mathfrak{g}, \tilde{\mathfrak{k}})$ -modules of finite type with minimal $\tilde{\mathfrak{k}}$ -types $V(\mu)$ for $\mu \geq 1$. So the above problem reduces to matching the above two classifications in the case $\mu \geq 1$, and finding all simple $(\mathfrak{g}, \tilde{\mathfrak{k}})$ -modules of finite type whose minimal $\tilde{\mathfrak{k}}$ -type equals $V(0)$ among the simple Harish-Chandra modules for the pair $(\mathfrak{g}, \tilde{\mathfrak{k}})$. We do this here in a special case.

Let $\mathfrak{g} = \mathfrak{sp}(2n+2)$ for $n \geq 2$. By assumption, $\tilde{\mathfrak{k}}$ is a long root $\mathfrak{sl}(2)$ -subalgebra, and $\tilde{\mathfrak{k}} \simeq \mathfrak{sp}(2n) \oplus \mathfrak{k}$. Consider simple $(\mathfrak{g}, \tilde{\mathfrak{k}})$ -modules with $Z_{U(\mathfrak{g})}$ -character equal to the character of a trivial module. According to the Langlands classification, there are precisely $(n+1)^2$ pairwise non-isomorphic such modules, one of which is the trivial module. Following [Co] (see figure 4.5 on page 93) we enumerate them as σ_t for $0 \leq t \leq n$ and σ_{ij} for $0 \leq i \leq n-1, 1 \leq j \leq 2n, i < j, i+j \leq 2n$. The modules σ_t are discrete series modules. The modules σ_{ij} are Langlands quotients of the principal series (all of them are proper quotients in this case).

We announce the following result which we intend to prove elsewhere.

Theorem 5.6 *Let $\mathfrak{g} = \mathfrak{sp}(2n+2)$ for $n \geq 2$ and $\tilde{\mathfrak{k}}$ be a long root $\mathfrak{sl}(2)$ -subalgebra.*

a) *Any simple $(\mathfrak{g}, \tilde{\mathfrak{k}})$ -module of finite type is isomorphic to a subquotient of the generalized discrete series module $D(\tilde{\mathfrak{k}}, \lambda)$ for some $\tilde{\lambda} = \mathfrak{sp}(2n) \oplus \tilde{\mathfrak{k}}$ -integral weight $\lambda - 2\chi$.*

b) The modules σ_0, σ_{0i} for $i = 1, \dots, 2n, \sigma_{12}$ are, up to isomorphism, all of the simple $(\mathfrak{g}, \mathfrak{k})$ -modules of finite type whose $Z_{U(\mathfrak{g})}$ -character equals that of a trivial \mathfrak{g} -module. Moreover, their minimal \mathfrak{k} -types are as follows:

<i>module</i>	<i>minimal \mathfrak{k}-type</i>
σ_0	$V(2n)$
$\sigma_{0j}, n+1 \leq j \leq 2n$	$V(j-1)$
$\sigma_{0j}, 2 \leq j \leq n$	$V(j-2)$
σ_{01} (trivial representation)	$V(0)$
σ_{12}	$V(0)$

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