

# THE GENUINE OMEGA-REGULAR UNITARY DUAL OF THE METAPLECTIC GROUP

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ABSTRACT. We classify all genuine unitary representations of the metaplectic group whose infinitesimal character is real and at least as regular as that of the oscillator representation. In a previous paper we exhibited a certain family of representations satisfying these conditions, obtained by cohomological induction from the tensor product of a one-dimensional representation and an oscillator representation. Our main theorem asserts that this family exhausts the genuine omega-regular unitary dual of the metaplectic group.

## 1. INTRODUCTION

In [4], we formulated a conjecture that provides a classification of all the genuine unitary *omega-regular* representations of the metaplectic group. (Roughly, a representation is called “omega-regular” if its infinitesimal character is real and at least as regular as that one of the oscillator representation. See Definition 2.1.) In this paper, we prove that conjecture. In particular, we show that all such representations are obtained by cohomological parabolic induction from the tensor product of a one-dimensional representation and an oscillator representation. The reader may think of the notion of omega-regular representations as a generalization, in the context of genuine representations of the metaplectic group, of the idea of *strongly regular* representations. (Recall that an infinitesimal character is called “strongly regular” if it is at least as regular as the infinitesimal character of the trivial representation.) Then our classification appears, for the case of the metaplectic group, as a generalization of the main result of [10], which asserts that, for real reductive Lie groups, every irreducible unitary representation with strongly regular infinitesimal character is cohomologically induced from a unitary character. For double covers of other linear groups one can define a similar notion of “omega-regular” for which every genuine unitary omega-regular representation should be obtained by cohomological induction from one of a small set of “basic” unitary representations. We make this more explicit after Theorem 1.4 below.

All representations considered in this paper have real infinitesimal character. Let  $Mp(2n)$  be the metaplectic group of rank  $n$ , *i. e.*, the two-fold connected cover of the symplectic group  $Sp(2n, \mathbb{R})$ . We recall the construction of the  $A_{\mathfrak{q}}(\Omega)$  representations of  $Mp(2n)$ . Choose a theta stable parabolic subalgebra  $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$  of  $\mathfrak{sp}(2n, \mathbb{C})$ , and let  $L$  be the Levi subgroup of  $Mp(2n)$  corresponding to  $\mathfrak{l}$ . We may

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identify  $L$  with a quotient of

$$\bar{L} = \left[ \prod_{i=1}^u \tilde{U}(p_i, q_i) \right] \times Mp(2m).$$

Here, for  $i = 1 \dots u$ ,  $\tilde{U}(p_i, q_i)$  is a connected two-fold cover of  $U(p_i, q_i)$ . However, we will abuse notation and say  $L \simeq \bar{L}$ . (See Section 2 for details.)

If  $\mathbb{C}_\lambda$  is a genuine (unitary) character of  $\prod_{i=1}^u \tilde{U}(p_i, q_i)$ , and  $\omega$  is an irreducible summand of one of the two oscillator representations of  $Mp(2m)$ , we set

$$\Omega = \mathbb{C}_\lambda \otimes \omega,$$

considered as a representation of  $L$ . Let

$$A_{\mathfrak{q}}(\Omega) := \mathcal{R}_{\mathfrak{q}}(\Omega)$$

(see Section 2 for notation).

**Remark 1.1.** Our (new) definition of  $A_{\mathfrak{q}}(\Omega)$  is slightly different from the one given in [4], because we do not require  $\Omega$  to be in the good range for  $\mathfrak{q}$ .

**Definition 1.2.** A representation  $Y$  of  $L$  is in the good range for  $\mathfrak{q}$  if its infinitesimal character  $\gamma^Y$  satisfies

$$\langle \gamma^Y + \rho(u), \alpha \rangle > 0, \quad \forall \alpha \in \Delta(\mathfrak{u}).$$

Here  $\rho(u)$  denotes one half the sum of the roots in  $\Delta(\mathfrak{u})$ .

For brevity of notation, an omega-regular representation will be called “ $\omega$ -regular”. In [4], we proved the following result.

**Proposition 1.3** (Proposition 3 of [4]). *If the representation  $\Omega$  of  $L$  is in the good range for  $\mathfrak{q}$ , then the representation  $A_{\mathfrak{q}}(\Omega)$  of  $Mp(2n)$  is nonzero, irreducible, genuine,  $\omega$ -regular and unitary.*

Our main result is the following converse of this statement.

**Theorem 1.4.** *Let  $X$  be an irreducible genuine  $\omega$ -regular and unitary representation of  $Mp(2n)$ . Then  $X \cong A_{\mathfrak{q}}(\Omega)$ , for some theta stable parabolic subalgebra  $\mathfrak{q}$  of  $\mathfrak{g}$  and some representation  $\Omega = \mathbb{C}_\lambda \otimes \omega$ , in the good range for  $\mathfrak{q}$ , of the Levi subgroup corresponding to  $\mathfrak{q}$ .*

Theorem 1.4 suggests that the four oscillator representations form a “basic” set of building blocks for all genuine omega-regular unitary representations of the metaplectic group, via cohomological parabolic induction. Using the results of [15] and [2], one can verify that similar statements hold true for the simply connected split group of type  $G_2$  and the nonlinear double cover of  $GL(n, \mathbb{R})$ . Here, “omega-regularity” might be defined in terms of  $\frac{1}{2}\rho$ , and the basic representations are the pseudospherical representations which correspond to the trivial representation of the linear group under the Shimura correspondence of [1]. This suggests the following generalization of “omega-regular” for a double cover  $G$  of a *split* linear group. Fix a choice of positive roots with respect to the split Cartan, and consider “metaplectic” roots as in [1], Definition 4.4. Define  $\gamma^\omega$  to be one half the sum of the non-metaplectic positive roots, plus one fourth the sum of the metaplectic positive roots, and call a representation of  $G$  “omega-regular” if its infinitesimal character

is at least as regular as  $\gamma^\omega$  (as in Definition 2.1). This definition agrees with that for  $Mp(2n)$  and the examples considered above, and reduces to the strongly regular definition if  $G$  is linear. If  $G$  is a nonlinear double cover of a split real group as in [1], then the set of basic representations for  $G$  should include the (conjecturally unitary) pseudospherical representations at infinitesimal character  $\gamma^\omega$ , i. e., the Shimura lifts of the trivial representation of the corresponding linear group (e. g., the even oscillator representations of  $Mp(2n)$ ). This collection of pseudospherical representations at infinitesimal character  $\gamma^\omega$  may or may not exhaust the set of basic representations for  $G$ .

We conjecture that a similar notion of “omega-regularity”, and a similar small collection  $\Pi_\omega(G)$  (finite if  $G$  is semisimple) of “basic” genuine, unitary, and omega-regular representations can be defined for any double cover  $G$  of a reductive linear real Lie group.

**Conjecture 1.5.** *For each double cover  $H$  of a real reductive linear Lie group there is a set  $\Pi_\omega(H)$  of basic representations as above, with the following property. Suppose that  $G$  is a double cover of a linear reductive real Lie group. If  $X$  is the Harish-Chandra module of a genuine irreducible omega-regular unitary representation of  $G$  then there is a Levi subgroup  $L$  of  $G$  and a representation  $Y \in \Pi_\omega(L)$  such that  $X$  is obtained from  $Y$  by cohomological parabolic induction.*

**Remark 1.6.** For linear double covers, such as the trivial double cover, or the square root of the determinant cover of  $U(p, q)$ , the basic representations are the (genuine) unitary one-dimensional representations. For these groups, the notion of omega-regularity should coincide with strong regularity, and the conjecture recovers Salamanca-Riba’s result [10].

In [4], we proved Theorem 1.4 for a metaplectic group of rank 2. The proof was based on a case by case calculation. For each (genuine)  $\tilde{U}(2)$ -type  $\mu$  that is the lowest  $K$ -type of an  $A_q(\Omega)$  representation, we showed that there exists a unique unitary and  $\omega$ -regular representation of  $Mp(4)$  with lowest  $K$ -type  $\mu$ . For each genuine  $\tilde{U}(2)$ -type  $\mu$  that is not the lowest  $K$ -type of an  $A_q(\Omega)$  representation, we showed that every  $\omega$ -regular representation of  $Mp(4)$  with lowest  $\tilde{U}(2)$ -type  $\mu$  must be non-unitary. The main tool in the proof of both claims was Parthasarathy’s Dirac Operator Inequality (cf. [6]). This scheme worked for all genuine  $\tilde{U}(2)$ -types, except for the (unique) fine  $\tilde{U}(2)$ -type which occurs in the genuine non-pseudospherical principal series. In this case, we explicitly computed the intertwining operator that gives the invariant Hermitian form on the representation space, and showed that its signature is indefinite.

The case-by-case calculation we used to prove Theorem 1.4 for the case  $n = 2$  is not suitable for a generalization to arbitrary  $n$ . For the general case, we apply a reduction argument similar to the one used in [9], but we also need some of the non-unitarity results and non-unitarity certificates obtained in [5]. (The question of the unitarity of the  $\omega$ -regular principal series of  $Mp(2n)$  was the motivation for [5].)

We sketch the proof of Theorem 1.4 (for arbitrary  $n$ ). Let  $X$  be a genuine admissible irreducible unitary representation of  $Mp(2n)$ . The first step is to realize  $X$  as the lowest  $K$ -type constituent of a module of the form

$$\mathcal{R}_q(X_1 \otimes X_0),$$

where  $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$  is a theta stable subalgebra of  $\mathfrak{sp}(2n, \mathbb{C})$  with Levi subgroup  $L \simeq \tilde{U}(r, s) \times Mp(2d)$ ,  $X_1$  is a genuine strongly regular irreducible representation of  $\tilde{U}(r, s)$ , and  $X_0$  is the irreducible Langlands quotient of a genuine  $\omega$ -regular principal series representation of  $Mp(2d)$ .

If  $X_1 \otimes X_0$  is unitary, then the results of [9] imply that  $X_1 \simeq A_{\mathfrak{q}_2}(\lambda_2)$  for some theta stable parabolic subalgebra  $\mathfrak{q}_2$  and some parameter  $\lambda_2$ , and the results of [5] tell us that  $X_0$  must be the even half of an oscillator representation of  $Mp(2d)$ . In this case, by a version of induction by stages,  $X$  must be of the desired form.

If  $X_1 \otimes X_0$  is not unitary, then there must be an  $(L \cap K)$ -type  $\mu^L$  which detects non-unitarity (in the sense that the invariant Hermitian form on  $\mu^L$  is not positive definite). The results of [9], which rely heavily on Parthasarathy's Dirac Operator Inequality, together with the calculations in [5] (see Lemma 5.2) give us specific information about what  $\mu^L$  could look like. Recall that, associated to the cohomological induction functor  $\mathcal{R}_{\mathfrak{q}}$ , there is the bottom layer map, which takes  $(L \cap K)$ -types (in  $(X_1 \otimes X_0)$ ) to  $K$ -types (in  $\mathcal{R}_{\mathfrak{q}}(X_1 \otimes X_0)$ ). Let  $\mu$  be the image of  $\mu^L$  under this map. If  $\mu$  were nonzero, then by a theorem of Vogan,  $\mu$  would occur in the lowest  $K$ -type constituent  $X$  of  $\mathcal{R}_{\mathfrak{q}}(X_1 \otimes X_0)$  and would carry the same signature as  $\mu^L$ ; hence  $X$  would be not unitary. Because we are assuming that  $X$  is unitary, we deduce that  $\mu^L$  must be mapped to 0. It turns out that, in this case, there exists a different theta stable parabolic subalgebra  $\mathfrak{q}'$  of  $\mathfrak{sp}(2n, \mathbb{C})$  with Levi subgroup  $L' \simeq \tilde{U}(r', s') \times Mp(2d+2)$ , such that  $X$  is the lowest  $K$ -type constituent of  $\mathcal{R}_{\mathfrak{q}'}(X'_1 \otimes X'_0)$ . Here  $X'_1$  is an  $A_{\mathfrak{q}_3}(\lambda_3)$  module of  $\tilde{U}(r', s')$ , and  $X'_0$  is the odd half of an oscillator representation of  $Mp(2d+2)$ . As in the previous case, using induction by stages, we get the desired result.

In [4], we also classified the non-genuine unitary  $\omega$ -regular representations of  $Mp(4)$ , *i. e.*, the  $\omega$ -regular part of the unitary dual of  $Sp(4, \mathbb{R})$ . We plan to address the generalization of this result to metaplectic groups of arbitrary rank in a future paper.

The paper is organized as follows. In Section 2, we set up the notation and recall some properties of the cohomological induction construction. We outline the proof of our main theorem in Section 3. The argument is essentially reduced to two main propositions, which we prove in Sections 6 and 7, and a series of technical lemmas that are presented in Section 4 (the casual reader may want to skip this section). Additional results needed for the proof are included in Sections 5 and 8.

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## 2. DEFINITIONS AND PRELIMINARY RESULTS

We begin with some notation. For the metaplectic group of rank  $r$ , we denote by  $\omega$  an irreducible summand of an oscillator representation and we write  $\omega^r$  for the corresponding infinitesimal character. Recall that there are four such summands, namely, the odd and even halves of the holomorphic and antiholomorphic oscillator representations, respectively. For any compact connected group, we often identify irreducible representations of the group with their highest weight.

For  $G = Mp(2n)$ , set  $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$  and  $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ , and let  $\mathfrak{t}_0$  and  $\mathfrak{t}$  be the real and complexified Lie algebra of a compact Cartan subgroup  $T$  of  $G$ . Let  $\langle \cdot, \cdot \rangle$  denote a fixed non-degenerate  $G$ -invariant  $\theta$ -invariant symmetric bilinear

form on  $\mathfrak{g}_0$ , negative definite on  $\mathfrak{k}_0$  and positive definite on  $\mathfrak{p}_0$ ; use the same notation for its complexification, and its various restrictions and dualizations.

**Definition 2.1.** Let  $\gamma \in i\mathfrak{t}_0^*$ . Choose a positive system  $\Delta^+(\gamma) \subseteq \Delta(\mathfrak{g}, \mathfrak{t})$  such that  $\langle \alpha, \gamma \rangle \geq 0$  for all  $\alpha \in \Delta^+(\gamma)$ , and let  $\omega^n$  be the representative of the infinitesimal character of the oscillator representation of  $G$  which is dominant with respect to  $\Delta^+(\gamma)$ . We call  $\gamma$  “ $\omega$ -regular” if the following regularity condition is satisfied:

$$\langle \alpha, \gamma - \omega^n \rangle \geq 0, \quad \forall \alpha \in \Delta^+(\gamma).$$

We say that a representation of  $G$  is  $\omega$ -regular if its infinitesimal character is.

Given a  $\theta$ -stable parabolic subalgebra  $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$  of  $\mathfrak{g}$ , write  $L$  for the corresponding subgroup of  $Mp(2n)$ . Then  $L$  is a double cover of a Levi subgroup  $L_d$  of  $Sp(2n, \mathbb{R})$  of the form

$$(2.1) \quad L_d \cong \prod_{i=1}^u U(p_i, q_i) \times Sp(2m, \mathbb{R}).$$

For each  $i = 1 \dots u$ , let  $\tilde{U}(p_i, q_i)$  be the inverse image of  $U(p_i, q_i)$  (in (2.1)) under the covering map

$$p : Mp(2n) \longrightarrow Sp(2n, \mathbb{R}),$$

and similarly for  $\tilde{Sp}(2m, \mathbb{R})$ . Then  $\tilde{U}(p_i, q_i)$  is the (connected) “square root of the determinant cover” of  $U(p_i, q_i)$ , and  $\tilde{Sp}(2m, \mathbb{R}) \cong Mp(2m)$ . The groups  $\tilde{U}(p_i, q_i)$  and  $\tilde{Sp}(2m, \mathbb{R})$  intersect in the kernel of the covering map  $p$ , and there is a surjective map

$$(2.2) \quad \bar{L} = \left[ \prod_{i=1}^u \tilde{U}(p_i, q_i) \right] \times Mp(2m) \longrightarrow L$$

given by multiplication inside  $Mp(2n)$ . Since the factors in (2.2) commute, we have that genuine irreducible representations of  $L$  are in correspondence with tensor products of genuine irreducible representations of the factors of  $\bar{L}$ . In order to keep our notation simpler, we will identify  $\bar{L}$  with  $L$ , and just write

$$(2.3) \quad L \cong \left[ \prod_{i=1}^u \tilde{U}(p_i, q_i) \right] \times Mp(2m).$$

Now let  $\left( \mathcal{R}_{\mathfrak{q}}^{(\mathfrak{g}, K)} \right)^i$  be the functors of cohomological parabolic induction carrying  $(\mathfrak{l}, L \cap K)$ -modules to  $(\mathfrak{g}, K)$ -modules (cf. [V], Def. 6.3.1). We will occasionally apply these functors to different settings. When the group and the Lie algebra are clear from the context, we will omit the superscript  $(\mathfrak{g}, K)$  and use the more standard notation  $\mathcal{R}_{\mathfrak{q}}^i$ . In our situation, we only use the degree  $i = \dim(\mathfrak{u} \cap \mathfrak{k})$  (usually denoted by  $S$ ), hence we may omit the superscript  $i$  as well.

**Definition 2.2.** An  $A_{\mathfrak{q}}(\Omega)$  representation is a genuine representation of  $G$  of the following form. Let  $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$  be a theta stable parabolic subalgebra of  $\mathfrak{g}$ , with corresponding Levi subgroup  $L$  (as in Equation 1). Let  $\mathbb{C}_{\lambda}$  be a genuine one-dimensional representation of  $\left[ \prod_{i=1}^u \tilde{U}(p_i, q_i) \right]$  and let  $\omega$  be an irreducible summand of an oscillator representation of  $Mp(2m)$ . We define

$$A_{\mathfrak{q}}(\Omega) := \mathcal{R}_{\mathfrak{q}}(\Omega).$$

**Remark 2.3.** In Definition 2.2, the rank  $m$  of the metaplectic factor of  $L$  is allowed to be equal to 0 or  $n$ ; in these cases, the representation  $A_{\mathfrak{q}}(\Omega)$  of  $Mp(2n)$  is an  $A_{\mathfrak{q}}(\lambda)$  representation or an oscillator representation, respectively.

We will prove Theorem 1.4 in Section 3. Now we recall some properties of the functors of cohomological induction.

Let  $X$  be an irreducible admissible  $(\mathfrak{g}, K)$ -module. Let

$$(2.4) \quad (\lambda_a, \mathfrak{q}_a, L_a, X^{L_a})$$

be the “ $\theta$ -stable parameters” associated to  $X$  via Vogan’s classification of admissible representations, so that  $X$  is the unique lowest  $K$ -type constituent of  $\mathcal{R}_{\mathfrak{q}_a}^{S_a}(X^{L_a})$ , with  $S_a = \dim(\mathfrak{u}_a \cap \mathfrak{k})$  and  $X^{L_a}$  a minimal principal series of  $L_a$  (cf. [13]). Recall that the parameter  $\lambda_a \in \mathfrak{it}_0^*$  determines the theta stable parabolic subalgebra  $\mathfrak{q}_a = \mathfrak{l}_a + \mathfrak{u}_a$ . In particular, the set of roots in  $\mathfrak{u}_a$  is given by

$$\Delta(\mathfrak{u}_a) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \mid \langle \lambda_a, \alpha \rangle > 0\}.$$

Choose and fix the positive system of compact roots

$$(2.5) \quad \Delta_c^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\}$$

so that dominant weights, and therefore highest weights of  $K$ -types, are given by strings of weakly decreasing half-integers. The parameter  $\lambda_a$  is weakly dominant for  $\Delta_c^+$  as well. If

$$(2.6) \quad \lambda_a = \left( \underbrace{g_1, \dots, g_1}_{r_1}, \dots, \underbrace{g_t, \dots, g_t}_{r_t} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{-g_t, \dots, -g_t}_{s_t}, \dots, \underbrace{-g_1, \dots, -g_1}_{s_1} \right)$$

with  $g_1 > \dots > g_t > 0$ , then the centralizer of  $\lambda_a$  on  $G$  is of the form

$$L_a = \text{Centr}_G(\lambda_a) \simeq \left[ \prod_{i=1}^t \tilde{U}(r_i, s_i) \right] \times Mp(2d),$$

and each factor is quasisplit.

Note that the parameter

$$(2.7) \quad \xi = \left( \underbrace{1, 1, \dots, 1}_r \mid \underbrace{0, 0, \dots, 0}_d \mid \underbrace{-1, -1, \dots, -1}_s \right)$$

is a singularization of  $\lambda_a$  (cf. [11]). Here  $r = \sum_{i=1}^t r_i$  and  $s = \sum_{i=1}^t s_i$ . Set

$$L = \text{Cent}_G(\xi) \simeq \tilde{U}(r, s) \times Mp(2d)$$

and  $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ . Then  $L \supseteq L_a$ ,  $\mathfrak{l}_a \subseteq \mathfrak{l}$ ,  $\mathfrak{u}_a = \mathfrak{u} + (\mathfrak{l} \cap \mathfrak{u}_a)$  and  $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \supseteq \mathfrak{q}_a$ .

Proposition 2.4 below lists a few results regarding the functors of cohomological induction and their restriction to  $K$  (see [13] and [3] for proofs). Most of these results are gathered together in [11], but are stated there for the functors  $\mathcal{L}_{\mathfrak{q}}^S$  instead of  $\mathcal{R}_{\mathfrak{q}}^S$ . Note that in our context, the two constructions are isomorphic by a result due to Enright and Wallach (cf. Theorem 5.3 of [14]).

**Proposition 2.4.** *Suppose that  $X$  is an irreducible admissible  $(\mathfrak{g}, K)$ -module of  $Mp(2n)$ . Define  $L$  and  $\mathfrak{q}$  as above, and let  $S = \dim(\mathfrak{u} \cap \mathfrak{k})$ . Then*

- (i) There is a unique irreducible  $(\mathfrak{l}, L \cap K)$ -module  $X^L$  associated to  $X$  so that  $X$  can be realized as the unique lowest  $K$ -type constituent of

$$\mathcal{R}_{\mathfrak{q}}^S(X^L).$$

- (ii) If  $\mu^L$  is (the highest weight of) an irreducible representation  $V_{\mu^L}$  of  $L \cap K$  and  $\mu = \mu^L + 2\rho(\mathfrak{u} \cap \mathfrak{p})$  is  $\Delta^+(\mathfrak{k}, \mathfrak{t})$ -dominant, then every irreducible constituent of  $\left(\mathcal{R}_{\mathfrak{q} \cap \mathfrak{k}}^{(K, \mathfrak{k})}\right)^S(V_{\mu^L})$  has highest weight  $\mu$ . If  $\mu$  is not dominant for  $\Delta^+(\mathfrak{k}, \mathfrak{t})$ , then  $\left(\mathcal{R}_{\mathfrak{q} \cap \mathfrak{k}}^{(K, \mathfrak{k})}\right)^S(V_{\mu^L}) = 0$ .

- (iii) There is a natural injective map, the bottom layer map, of  $K$ -representations

$$\mathcal{B}_{X^L} : \left(\mathcal{R}_{\mathfrak{q} \cap \mathfrak{k}}^{(K, \mathfrak{k})}\right)^S(X^L) \rightarrow \mathcal{R}_{\mathfrak{q}}^S(X^L).$$

Moreover, there is a one-to-one correspondence (with multiplicities) between the lowest  $K$ -types of  $X$  and the lowest  $(L \cap K)$ -types of  $X^L$ .

- (iv) The module  $X$  is endowed with a nonzero invariant Hermitian form  $\langle, \rangle^G$  if and only if the module  $X^L$  is endowed with a nonzero invariant Hermitian form  $\langle, \rangle^L$ .
- (v) The bottom layer map is unitary. That is, for  $X$  Hermitian, on each  $K$ -type in the bottom layer of  $X$  (cf. Remark 2.6), the signature of  $\langle, \rangle^G$  matches the signature of  $\langle, \rangle^L$  on the corresponding  $(L \cap K)$ -type of  $X^L$ .
- (vi) If  $\gamma^{X^L} \in \mathfrak{it}_0^*$  is a representative of the infinitesimal character of  $X^L$ , then

$$\gamma^X = \gamma^{X^L} + \rho(\mathfrak{u})$$

is a representative of the infinitesimal character of  $X$ .

- (vii) If  $\lambda_a^L$  is the Vogan classification parameter associated to  $X^L$ , then

$$\lambda_a = \lambda_a^L + \rho(\mathfrak{u}).$$

*Proof.* Because  $\xi$  in (2.7) is a singularization of  $\lambda_a$ , these facts follow from Lemma 2.7 and Theorem 2.13 in [11]. More precisely, part (i) and (vii) follow from Theorem 2.13 (b); (ii) from Lemma 2.7; (iii) and (v) follow from Theorem 2.13 (d); (iv) is proved in more generality in Proposition 5.2 and Corollary 5.3 of [9], but it is also Theorem 2.13 (c) of [11]. Part (vi) is Proposition 6.3.11 in [13].  $\square$

**Remark 2.5.** If  $X$  has real infinitesimal character, then  $X^L$  and  $X$  are Hermitian by Lemma 6.5 of [9]; the argument given there is easily seen to extend to the case of the metaplectic group. Consequently, we have that in our setting, the forms  $\langle, \rangle^L$  and  $\langle, \rangle^G$  of Proposition 2.4 (iv) always exist.

**Remark 2.6.** The image of the bottom layer map  $\mathcal{B}_{X^L}$  (as in part (iii) of Proposition 2.4) is called “the bottom layer of  $X$ ”. We say that an  $(L \cap K)$ -type  $\mu^L$  survives in the bottom layer if  $\mu = \mu^L + 2\rho(\mathfrak{u} \cap \mathfrak{p})$  is  $\Delta^+(\mathfrak{k}, \mathfrak{t})$ -dominant (as in part (ii) of Proposition 2.4). Note that, in this case,  $\mu$  is the highest weight of a  $K$ -type in the bottom layer of  $X$ .

**Lemma 2.7.** *Retain the notation of Proposition 2.4. Set  $L = L_1 \times L_0$  with  $L_0 = Mp(2d)$  and  $L_1 = \tilde{U}(r, s)$ , and write  $X^L \simeq X_1 \otimes X_0$ , with  $X_i$  an irreducible  $(\mathfrak{l}_i, L_i \cap K)$ -module ( $i = 0, 1$ ). If  $X$  is  $\omega$ -regular, then  $X_1$  is strongly regular for  $L_1$  and  $X_0$  is  $\omega$ -regular for  $L_0$ .*

*Proof.* Let  $\gamma^{X^L} = (\gamma^{X_1}, \gamma^{X_0})$  be (a representative of) the infinitesimal character of  $X^L$ , and write

$$\gamma^X = \gamma^{X^L} + \rho(\mathfrak{u})$$

as in part (vi) of Proposition 2.4. Choose a positive system  $\Delta^+ \subset \Delta(\mathfrak{g}, \mathfrak{t})$  of roots so that  $\gamma^X$  is dominant with respect to  $\Delta^+$ , and choose the representative of the infinitesimal character  $\omega^n$  which is dominant with respect to  $\Delta^+$ . Because  $X$  is  $\omega$ -regular, we have

$$\langle \gamma^X, \alpha \rangle \geq \langle \omega^n, \alpha \rangle, \quad \forall \alpha \in \Delta^+.$$

If we normalize our form so that  $\langle \alpha, \alpha \rangle = 2$  for all short roots  $\alpha$ , then this is equivalent to saying that

$$\langle \gamma^X, \alpha \rangle \geq 1, \quad \forall \alpha \in \Delta^+.$$

Now let

$$\Delta^+(\mathfrak{l}) = \Delta^+ \cap \Delta(\mathfrak{l}, \mathfrak{t}) = [\Delta^+ \cap \Delta(\mathfrak{l}_1, \mathfrak{t})] \cup [\Delta^+ \cap \Delta(\mathfrak{l}_0, \mathfrak{t})].$$

Because  $\rho(\mathfrak{u})$  is orthogonal to the roots of  $\mathfrak{l}$ , we have

$$\langle \gamma^{X^L}, \alpha \rangle \geq 1, \quad \forall \alpha \in \Delta^+(\mathfrak{l}).$$

Note that the roots of  $\mathfrak{l}_1$  are orthogonal to  $\gamma^{X_0}$ , hence

$$\langle \gamma^{X_0}, \alpha \rangle \geq 1, \quad \forall \alpha \in \Delta^+ \cap \Delta(\mathfrak{l}_0, \mathfrak{t})$$

and  $X_0$  is  $\omega$ -regular for  $L_0$ . Similarly, the roots of  $\mathfrak{l}_0$  are orthogonal to  $\gamma^{X_1}$ , hence

$$\langle \gamma^{X_1}, \alpha \rangle \geq 1, \quad \forall \alpha \in \Delta^+(\mathfrak{l}) \cap \Delta(\mathfrak{l}_1, \mathfrak{t}).$$

This implies that  $X_1$  is strongly regular for  $L_1$ .  $\square$

### 3. PROOF OF THEOREM 1.4

The proof of Theorem 1.4 relies on a series of auxiliary lemmas and propositions. We will state these results as needed along the way and postpone the longer proofs to later sections.

Fix  $X$  as in Theorem 1.4, *i. e.*, let  $X$  be a genuine irreducible  $\omega$ -regular unitary representation of  $Mp(2n)$ . By virtue of Proposition 2.4 and Lemma 2.7, we can assume that our genuine irreducible  $\omega$ -regular unitary representation  $X$  is (the unique lowest  $K$ -type constituent of) a representation of the form

$$\mathcal{R}_q^S(X_1 \otimes X_0)$$

with  $X_1$  an irreducible genuine strongly regular  $(\mathfrak{l}_1, L_1 \cap K)$ -module for  $L_1 = \tilde{U}(r, s)$ , and  $X_0$  an irreducible genuine  $\omega$ -regular  $(\mathfrak{l}_0, L_0 \cap K)$ -module for  $L_0 = Mp(2d)$ . Note that, because  $L_0$  is a factor of  $L_a$  (and  $X^{L_a}$  is a minimal principal series representation of  $L_a$ ),  $X_0$  must be a minimal principal series representation of  $L_0$ . The following proposition asserts that, in this setting,  $X_1$  is a *good*  $A_{q_2}(\lambda_2)$  module.



**Proposition 3.1.** *Let  $X$  be an irreducible unitary  $(\mathfrak{g}, K)$ -module of  $Mp(2n)$ . Assume that  $X$  is genuine and  $\omega$ -regular; realize  $X$  as the unique lowest  $K$ -type constituent of  $\mathcal{R}_{\mathfrak{q}}^S(X^L)$ , as in Proposition 2.4, and write*

$$L = L_1 \times L_0 = \tilde{U}(r, s) \times Mp(2d) \quad X^L \simeq X_1 \otimes X_0,$$

as in Lemma 2.7. Suppose that  $r + s \neq 0$ . Then there exist a theta stable parabolic subalgebra  $\mathfrak{q}_2$  and a representation  $\mathbb{C}_{\lambda_2}$  of the Levi factor corresponding to  $\mathfrak{q}_2$ , with  $\lambda_2$  in the good range for  $\mathfrak{q}_2$ , such that  $X_1 \simeq A_{\mathfrak{q}_2}(\lambda_2)$ .

Using Proposition 3.1, we will prove the following result.

**Proposition 3.2.** *In the setting of Proposition 3.1, assume  $d > 0$  and allow  $r + s$  to be possibly equal to zero. Then one of the following (mutually exclusive) options occurs:*

(i)  $X_0$  is an even oscillator representation with lowest  $K$ -type

$$\mu_0 = \pm(1/2, 1/2, \dots, 1/2).$$

(ii) There exist a subgroup  $L' = L'_1 \times L'_0 = \tilde{U}(r', s') \times Mp(2(d+1)) \subset G$  also containing  $L_a$ , a theta stable subalgebra  $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \supseteq \mathfrak{q}$  and representations  $X'_i$  of  $L'_i$  ( $i = 0, 1$ ), such that

(a)  $\tilde{U}(r', s') \subset \tilde{U}(r, s)$ , with either  $r = r' + 1$  or  $s = s' + 1$ ,

(b)  $X'_1$  is a good  $A_{\mathfrak{q}_3}(\lambda_3)$  (or  $r' + s' = 0$ ),

(c)  $X'_0$  is an odd oscillator representation with lowest  $K$ -type

$$\mu'_0 = (3/2, 1/2, \dots, 1/2) \text{ or } (-1/2, -1/2, \dots, -1/2, -3/2),$$

and

(d)  $X$  can be realized as the unique lowest  $K$ -type constituent of  $\mathcal{R}_{\mathfrak{q}'}^{S'}(X'_1 \otimes X'_0)$ .

Putting all these results together, we obtain the following corollary.

**Corollary 3.3.** *Every genuine irreducible  $\omega$ -regular unitary representation  $X$  of  $Mp(2n)$  satisfies one of the following two properties:*

(i)  $X$  is (the unique lowest  $K$ -type constituent of) a representation of the form

$$\mathcal{R}_{\mathfrak{q}}^S(X^L)$$

with  $L = \tilde{U}(r, s) \times Mp(2d) = L_1 \times L_0$  and  $X^L = X_1 \otimes X_0$ . Here

(a)  $X_1$  is a good  $A_{\mathfrak{q}_2}(\lambda_2)$  module for  $\tilde{U}(r, s)$ , unless  $r + s = 0$ , and

(b)  $X_0$  is an even oscillator representation of  $Mp(2d)$  with lowest  $K$ -type

$$\mu_0 = \pm(1/2, 1/2, \dots, 1/2),$$

unless  $d = 0$ .

(ii)  $X$  is (the unique lowest  $K$ -type constituent of) a representation of the form

$$\mathcal{R}_{\mathfrak{q}'}^{S'}(X^{L'})$$

with  $L' = \tilde{U}(r', s') \times Mp(2(d+1)) = L'_1 \times L'_0$  and  $X^{L'} = X'_1 \otimes X'_0$ . Here

(a)  $X'_1$  is a good  $A_{\mathfrak{q}_3}(\lambda_3)$  module for  $\tilde{U}(r', s')$ , unless  $r' + s' = 0$ , and

(b)  $X'_0$  is an odd oscillator representation of  $Mp(2(d+1))$  with lowest  $K$ -type

$$\mu'_0 = (3/2, 1/2, \dots, 1/2) \text{ or } (-1/2, -1/2, \dots, -1/2, -3/2).$$

In order to conclude the proof of Theorem 1.4, we must show that (in both cases) our genuine irreducible  $\omega$ -regular unitary representation  $X$  of  $Mp(2n)$  can also be realized as an  $A_{\mathfrak{q}}(\Omega)$  representation, for some theta stable parabolic subalgebra  $\mathfrak{q}$  and some representation  $\Omega = \mathbb{C}_{\lambda} \otimes \omega$  in the good range for  $\mathfrak{q}$ . For this, we need two more results.

**Proposition 3.4.** *Let  $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$  be a  $\theta$ -stable parabolic subalgebra. Assume that  $L = N_G(\mathfrak{q})$  is a direct product of two reductive subgroups  $L = L_1 \times L_0$ . (Here  $L$  can be of the form  $\tilde{U}(r, s) \times Mp(2d)$ , as in Proposition 2.4, or  $\tilde{U}(r', s') \times Mp(2(d+1))$ , as in Proposition 3.2.) Suppose further that we have a representation*

$$X \simeq \mathcal{R}_{\mathfrak{q}}(A_{\mathfrak{q}'}(\lambda') \otimes \omega)$$

where  $\omega$  is an irreducible summand of an oscillator representation,  $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \subseteq \mathfrak{l}_1$ , and  $A_{\mathfrak{q}'}(\lambda')$  is good for  $\mathfrak{q}'$ . Then there is a theta stable parabolic subalgebra  $\mathfrak{q}_{\omega} = \mathfrak{l}_{\omega} + \mathfrak{u}_{\omega}$  of  $\mathfrak{g}$  such that

$$X \simeq \mathcal{R}_{\mathfrak{q}_{\omega}}(\mathbb{C}_{\lambda'} \otimes \omega) = A_{\mathfrak{q}_{\omega}}(\Omega)$$

with  $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$ .

*Proof.* Assume that  $X \simeq \mathcal{R}_{\mathfrak{q}}(A_{\mathfrak{q}'}(\lambda') \otimes \omega)$ , where  $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \subseteq \mathfrak{l}_1$ . Set

$$\mathfrak{q}_b = \mathfrak{q}' \oplus \mathfrak{l}_0 = (\mathfrak{l}' + \mathfrak{u}') \oplus \mathfrak{l}_0.$$

Then  $\mathfrak{q}_b$  is a theta stable parabolic subalgebra of  $\mathfrak{l}$ , and by Lemma 3.5 (which we will state below), we have that

$$A_{\mathfrak{q}'}(\lambda') \otimes \omega \simeq \mathcal{R}_{\mathfrak{q}'}^{(\mathfrak{l}_1, L_1 \cap K)}(\mathbb{C}_{\lambda'}) \otimes \mathcal{R}_{\mathfrak{l}_0}^{(\mathfrak{l}_0, L_0 \cap K)}(\omega) \simeq \mathcal{R}_{\mathfrak{q}_b}^{(\mathfrak{l}, L \cap K)}(\mathbb{C}_{\lambda'} \otimes \omega).$$

Therefore,

$$X \simeq \mathcal{R}_{\mathfrak{q}}\left(\mathcal{R}_{\mathfrak{q}_b}^{(\mathfrak{l}, L \cap K)}(\mathbb{C}_{\lambda'} \otimes \omega)\right).$$

Now set

$$(3.1) \quad \mathfrak{q}_{\omega} = \mathfrak{q}_b + \mathfrak{u} = \underbrace{(\mathfrak{l}' \oplus \mathfrak{l}_0)}_{\mathfrak{l}_{\omega}} + \underbrace{(\mathfrak{u}' + \mathfrak{u})}_{\mathfrak{u}_{\omega}}.$$

Note that  $\mathfrak{q}_{\omega}$  is a parabolic subalgebra of  $\mathfrak{g}$ . Since  $X_1 \cong A_{\mathfrak{q}'}(\lambda')$ , by Proposition 3.5 in [10], we know that  $\mathfrak{l}' \supseteq \mathfrak{l}_a \cap \mathfrak{l}_1$  and  $\mathfrak{u}' \subseteq \mathfrak{u}_a \cap \mathfrak{l}_1$ . We have

$$(3.2) \quad \mathfrak{l}_{\omega} = \mathfrak{l}' \oplus \mathfrak{l}_0 \subseteq \mathfrak{l}, \quad \mathfrak{u} \subseteq \mathfrak{u}' + \mathfrak{u} = \mathfrak{u}_{\omega} \quad \text{and} \quad \mathfrak{q}_{\omega} \subseteq \mathfrak{q}.$$

Hence, by induction in stages (cf. [13], Proposition 6.3.6), we find that

$$\mathcal{R}_{\mathfrak{q}}\left(\mathcal{R}_{\mathfrak{q}_b}^{(\mathfrak{l}, L \cap K)}(\mathbb{C}_{\lambda'} \otimes \omega)\right) \simeq \mathcal{R}_{\mathfrak{q}_{\omega}}(\mathbb{C}_{\lambda'} \otimes \omega) = A_{\mathfrak{q}_{\omega}}(\Omega),$$

and Proposition 3.4 is proved.  $\square$

**Lemma 3.5.** *For  $i = 1, 2$ , let  $G_i$  be a reductive Lie group with maximal compact subgroup  $K_i$ , complexified Lie algebra  $\mathfrak{g}_i$ , and a theta stable parabolic subalgebra  $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i$ . Moreover, let  $X_i$  be an  $(\mathfrak{l}_i, L_i \cap K_i)$ -module, and let  $S_i = \dim(\mathfrak{u}_i \cap \mathfrak{k}_i)$ . Then*

$$\mathcal{R}_{\mathfrak{q}_1}^{S_1}(X_1) \otimes \mathcal{R}_{\mathfrak{q}_2}^{S_2}(X_2) \simeq \mathcal{R}_{\mathfrak{q}_1 \oplus \mathfrak{q}_2}^{S_1 + S_2}(X_1 \otimes X_2).$$

*Proof.* This follows by tracing through the definitions of the various cohomological induction functors, using standard techniques and identities in homological algebra, including an appropriate Künneth formula.  $\square$

Finally, we need to show that our representation  $\Omega$  is indeed in the good range for  $\mathfrak{q}_\omega$ .

**Proposition 3.6.** *In the setting of Proposition 3.4, let  $X$  be the irreducible (lowest  $K$ -type constituent of the) representation  $\mathcal{R}_{\mathfrak{q}_\omega}(\mathbb{C}_{\lambda'} \otimes \omega)$ . Assume, moreover, that  $\lambda'$  is in the good range for  $\mathfrak{q}'$ . Then  $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$  is in the good range for  $\mathfrak{q}_\omega$ .*

This concludes the proof of Theorem 1.4.

The proofs of Propositions 3.1, 3.2 and 3.6 will be given in Sections 6, 7 and 8, respectively. Before we turn to those proofs, we need to prove several technical lemmas concerning lowest  $K$ -types and to recall some results about genuine minimal principal series representations of  $Mp(2d)$ . We address these issues in the next two sections.

#### 4. TECHNICAL LEMMAS

The purpose of this section is to describe lowest  $K$ -types of irreducible, genuine and  $\omega$ -regular representations of  $Mp(2n)$ , and to present some results that will be needed for the proofs of Propositions 3.1, 3.2 and 3.6.

We begin, in great generality, with irreducible admissible  $(\mathfrak{g}, K)$ -modules  $Mp(2n)$ .

**Lemma 4.1.** *Let  $X$  be an irreducible admissible  $(\mathfrak{g}, K)$ -module of  $Mp(2n)$ , and let  $\mu$  be a lowest  $\tilde{U}(n)$ -type of  $X$ . Let  $\lambda_a$  and  $\mathfrak{q}_a = \mathfrak{l}_a + \mathfrak{u}_a$  be the Vogan classification parameter and the theta stable parabolic subalgebra associated to  $X$ , respectively, as in (2.4). Then  $\mu$  is of the form*

$$(4.1) \quad \mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \delta^{L_a},$$

where  $\delta^{L_a}$  is a fine  $(L_a \cap K)$ -type.

*Proof.* This is a known result due to Vogan. It follows from Proposition 2.4 (ii)-(iii) (when  $\mathfrak{q} = \mathfrak{q}_a$ ) and from the fact that (the highest weight of) the representation  $\mu^{L_a}$  in that Proposition is of the form  $\lambda_a - \rho(\mathfrak{u}_a) + \delta^{L_a}$ , with  $\delta^{L_a}$  a fine  $(L_a \cap K)$ -type. See Proposition 3.2.7 of [7] for a detailed proof in the case of  $G = U(p, q)$ .  $\square$

Next, we restrict our attention to irreducible admissible  $(\mathfrak{g}, K)$ -modules of  $Mp(2n)$  that are genuine and  $\omega$ -regular.

**Lemma 4.2.** *Let  $X$  be an irreducible admissible  $(\mathfrak{g}, K)$ -module of  $Mp(2n)$ . Assume that  $X$  is genuine and  $\omega$ -regular. Realize  $X$  as the unique lowest  $K$ -type constituent of a cohomologically induced representation  $\mathcal{R}_{\mathfrak{q}}^S(X^L)$ , with*

$$L = L_1 \times L_0 = \tilde{U}(r, s) \times Mp(2d) \quad \text{and} \quad X^L \simeq X_1 \otimes X_0,$$

as in Proposition 2.4. Let

$$\lambda_a = \left( \underbrace{g_1, \dots, g_1}_{r_1}, \dots, \underbrace{g_t, \dots, g_t}_{r_t} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{-g_t, \dots, -g_t}_{s_t}, \dots, \underbrace{-g_1, \dots, -g_1}_{s_1} \right)$$

be the Vogan classification parameter of  $X$ , as in Equation 2.6.

- (i) If  $X_1$  is an  $A_{\mathfrak{q}'}(\lambda')$  module of  $\tilde{U}(r, s)$ , then the representation  $X$  has a unique lowest  $\tilde{U}(n)$ -type  $\mu$ .
- (ii) For  $i = 1, \dots, t$ , if  $r_i \neq s_i$ , then  $g_i \in \mathbb{Z} + \frac{1}{2}$ . In this case, the infinitesimal character of  $X$  contains an entry  $\pm g_i$ , and the infinitesimal character of  $X_0$  does not.

- (iii) Suppose that  $X$  has a unique lowest  $\tilde{U}(n)$ -type  $\mu$ .  
 (a)  $\mu$  is of the form

$$(4.2) \quad \mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \mu_0,$$

where  $\mu_0$  is the lowest  $\tilde{U}(d)$ -type of  $X_0$ .

- (b) Moreover, if  $r_i = s_i$ , then  $g_i \in \mathbb{Z}$ .

*Proof.* Recall that  $X_0$  is a genuine minimal principal series of  $Mp(2d)$ , therefore it has a unique lowest  $K$ -type (cf. [5]). If  $X_1$  is an  $A_{q'}(\lambda')$  module of  $\tilde{U}(r, s)$ , then  $X_1$  has a unique lowest  $K$ -type, as well. The uniqueness of the lowest  $K$ -type of  $X$  then follows from part (iii) of Proposition 2.4.

For part (ii), recall from Section 3.1 of [8] the relationship between the parameter  $\lambda_a$ , the infinitesimal character, the lowest  $K$ -types, and the Langlands parameters of a representation. There, the theory is laid out in detail for representations of the symplectic group. The corresponding statements for genuine representations of  $Mp(2n)$  can be obtained by making slight modifications. In the case of  $\omega$ -regular (hence non-singular) representations, all limits of discrete series representations are discrete series, and all the lowest  $K$ -types of a standard module appear in the same irreducible representation. Because  $X$  is a genuine representation with associated parameter  $\lambda_a$  as in (2.6), we can realize  $X$  as the Langlands subquotient of an induced representation from a cuspidal parabolic subgroup with Levi component  $MA$  isomorphic to (a quotient of)  $Mp(2a) \times \widetilde{GL}(2, \mathbb{R})^b \times \widetilde{GL}(1, \mathbb{R})^d$ , where  $a = \sum_{i=1}^t |r_i - s_i|$  and  $b = \sum_{i=1}^t \min\{r_i, s_i\}$ . If  $r_i = s_i + 1$  for some  $i$ , then the entry  $g_i$  coincides with an entry of the Harish-Chandra parameter of a genuine discrete series representation of  $Mp(2a)$ . Similarly for  $-g_i$ , if  $r_i = s_i - 1$ . Therefore,  $g_i \in \mathbb{Z} + \frac{1}{2}$  whenever  $r_i \neq s_i$ . In this case, we also find that  $g_i$  is an entry of the infinitesimal character of  $X$ . Then the fact that  $X_0$  does not contain an entry  $\pm g_i$  follows from the  $\omega$ -regular condition.

For part (iii), assume that our representation has a unique lowest  $\tilde{U}(n)$ -type  $\mu$ . Write

$$\mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \delta^{L_a}$$

(as in Lemma 4.1). By Proposition 2.4, this equals

$$\mu_1 + \mu_0 + 2\rho(\mathfrak{u} \cap \mathfrak{p}).$$

Now look at the restriction to  $L_0 \cap K$ :

$$\begin{aligned} \lambda_a |_{\mathfrak{l}_0 \cap \mathfrak{k}} &= 0 \\ \mu_1 |_{\mathfrak{l}_0 \cap \mathfrak{k}} &= 0 \\ 2\rho(\mathfrak{u} \cap \mathfrak{p}) |_{\mathfrak{l}_0 \cap \mathfrak{k}} &= [\rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k})] |_{\mathfrak{l}_0 \cap \mathfrak{k}}. \end{aligned}$$

Hence

$$\delta^{L_a} |_{\mathfrak{l}_0 \cap \mathfrak{k}} = \mu_0.$$

Next, we show that  $\delta^{L_a} |_{\mathfrak{l}_1 \cap \mathfrak{k}} = 0$ . Because  $X$  has a unique lowest  $\tilde{U}(n)$ -type, the representation  $X_1$  must have the same property. In order to have a representation of  $L_1$  with a unique lowest  $\tilde{U}(r) \times \tilde{U}(s)$ -type, every factor  $\tilde{U}(r_i, s_i)$  of  $L_a$  with  $r_i = s_i$  should carry the trivial fine  $\tilde{U}(r_i) \times \tilde{U}(s_i)$ -type, because non-trivial fine

$\tilde{U}(r_i) \times \tilde{U}(s_i)$ -types on such factors come in pairs. This shows that the restriction of  $\delta^{L_a}$  to  $L_1 \cap K$  is zero, hence  $\delta^{L_a} = \mu_0$  and

$$\mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \mu_0,$$

completing the proof of part (a). For part (b), write

$$r_i = s_i + \varepsilon_i$$

for all  $i = 1, \dots, t$ . Note that  $\varepsilon_i = 0$  or  $\pm 1$ , because each factor of

$$L_a = \left[ \prod_{i=1}^t \tilde{U}(r_i, s_i) \right] \times Mp(2d)$$

is quasisplit. Then

$$(4.3) \quad \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) = \left( \dots, \underbrace{f_i, \dots, f_i}_{r_i}, \dots \left| \underbrace{c, \dots, c}_d \right| \dots, \underbrace{h_i, \dots, h_i}_{s_i}, \dots \right),$$

with

$$(4.4) \quad \begin{aligned} f_i &= \sum_{j < i} (r_j - s_j) + \frac{\varepsilon_i + 1}{2} = \sum_{j < i} \varepsilon_j + \frac{\varepsilon_i + 1}{2} \\ h_i &= \sum_{j < i} (r_j - s_j) + \frac{\varepsilon_i - 1}{2} = \sum_{j < i} \varepsilon_j + \frac{\varepsilon_i - 1}{2} \\ c &= r - s. \end{aligned}$$

If  $r_i = s_i$  for some  $i = 1, \dots, t$ , then  $\varepsilon_i = 0$ . The corresponding entry  $f_i$  of  $\rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k})$  is then in  $\mathbb{Z} + \frac{1}{2}$ . Because genuine  $\tilde{U}(n)$ -types have half-integral entries, this implies that  $g_i \in \mathbb{Z}$ . The proof of Lemma 4.2 is now complete.  $\square$

Remain in the setting of Lemma 4.2. Let  $\mu$  be a lowest  $K$ -type of  $X$ . Recall that  $\mu$  satisfies Equation 4.1:

$$\mu = \lambda_a + \rho(\mathfrak{u}_a \cap \mathfrak{p}) - \rho(\mathfrak{u}_a \cap \mathfrak{k}) + \delta^{L_a}.$$

Write

$$(4.5) \quad \mu = \left( \underbrace{a_1, \dots, a_1}_{r_1}, \dots, \underbrace{a_t, \dots, a_t}_{r_t} \left| \underbrace{c_1, \dots, c_d}_d \right| \underbrace{b_t, \dots, b_t}_{s_t}, \dots, \underbrace{b_1, \dots, b_1}_{s_1} \right),$$

according to how the coordinates break into the factors of the subgroup  $L_a$ . Then the fine  $(L \cap K)$ -type  $\delta^{L_a}$  is of the form

$$(4.6) \quad \delta^{L_a} = \left( \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_t, \dots, y_t}_{r_t} \left| \underbrace{z_1, \dots, z_d}_d \right| \underbrace{y_t, \dots, y_t}_{s_t}, \dots, \underbrace{y_1, \dots, y_1}_{s_1} \right),$$

with  $y_i = 0$  or  $\pm \frac{1}{2}$ , and  $z_i = \pm \frac{1}{2}$  (*cf.* Proposition 6 of [8]). Note that the  $\{z_i\}$  are weakly decreasing.

**Remark 4.3.** If  $r_i = 0$ , then  $a_i$  does not occur as a coordinate of  $\mu$ , but it is still convenient to define

$$(4.7) \quad a_i = g_i + f_i + y_i,$$

with  $f_i$  as in Equation 4.4. (Because  $s_i > 0$  in this case, the quantities  $y_i$  and  $(- )g_i$  can be determined by the coordinates of  $\delta^{L_a}$  and  $\lambda_a$ , respectively.)

Similarly, if  $s_i = 0$  (and  $r_i > 0$ ), we define  $b_i$  by

$$(4.8) \quad b_i = -g_i + h_i + y_i.$$

We obtain sequences  $\{a_i\}$  and  $\{b_i\}$  of half-integers satisfying

$$(4.9) \quad a_1 \geq a_2 \geq \dots \geq a_t \quad \text{and} \quad b_t \geq b_{t-1} \geq \dots \geq b_1.$$

**Lemma 4.4.** *Retain all the previous notation. Then*

(i)  $a_t - b_t \geq 2$ .

(ii) Assume that the  $K$ -type  $\mu$  satisfies the additional condition

$$(4.10) \quad a_t = c_1.$$

(a)  $\varepsilon_t \geq 0$  and  $g_t = \frac{1}{2}$ .

(b) If  $\varepsilon_t = 0$ , then  $\delta^{L_a}$  (cf. (4.1)) must be of the form

$$(4.11) \quad \delta^{L_a} = \left( \dots, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{s_t} \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_q \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{s_t}, \dots \right)$$

for some  $p > 0$  and  $q = d - p$ .

(c) If  $\varepsilon_t = 1$ , then  $\delta^{L_a}$  (cf. (4.1)) must be of the form

$$\delta^{L_a} = \left( \dots, \underbrace{0, \dots, 0}_{s_t+1} \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_q \middle| \underbrace{0, \dots, 0}_{s_t}, \dots \right)$$

for some  $p > 0$  and  $q = d - p$ .

(d)  $c_d > b_t$ .

*Proof.* For part (i), observe that

$$a_t = g_t + (r - s) - (r_t - s_t) + \frac{\varepsilon_t + 1}{2} + y_t = g_t + (r - s) + \frac{1}{2}(1 - \varepsilon_t) + y_t,$$

$$c_1 = (r - s) + z_1,$$

$$c_d = (r - s) + z_d, \text{ and}$$

$$b_t = -g_t + (r - s) - (r_t - s_t) + \frac{\varepsilon_t - 1}{2} + y_t = -g_t + (r - s) + \frac{1}{2}(-1 - \varepsilon_t) + y_t.$$

Hence, we obtain

$$(4.12) \quad a_t - c_1 = g_t + \frac{1}{2}(1 - \varepsilon_t) + y_t - z_1,$$

$$(4.13) \quad c_d - b_t = g_t + \frac{1}{2}(1 + \varepsilon_t) + z_d - y_t,$$

and

$$a_t - b_t = 2g_t + 1 \geq 2,$$

because  $g_t \geq \frac{1}{2}$ . This proves (i).

Now assume that  $\mu$  satisfies Equation 4.10. The coordinate  $g_t$  of  $\lambda_a$  is either an integer or a half-integer. We consider the two cases separately.

First assume that  $g_t \in \mathbb{Z}$ . By Lemma 4.2,  $\varepsilon_t = 0$ , so

$$a_t \in \mathbb{Z} + \frac{1}{2} + y_t \quad \text{and} \quad c_1 \in \mathbb{Z} + z_1.$$

In order for  $\mu$  to be genuine, the coordinates of the fine  $(L_a \cap K)$ -type  $\delta^{L_a}$  must satisfy

$$\begin{cases} y_t \in \mathbb{Z} \\ z_1 \in \mathbb{Z} + \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} y_t = 0 \\ z_1 = \pm \frac{1}{2}. \end{cases}$$

This says that the representation  $\delta^{L_a}$  of  $(L_a \cap K)$  is trivial on the  $\tilde{U}(s_t, s_t)$ -factor of  $L_a$ , and non-trivial on the  $Mp(2d)$ -factor:

$$\delta^{L_a} = \left( \dots, \underbrace{0, \dots, 0}_{s_t} \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \middle| \underbrace{0, \dots, 0}_{s_t}, \dots \right).$$

Equation 4.12 then gives

$$a_t - c_1 = \begin{cases} g_t & \text{if } p > 0, \\ g_t + 1 & \text{if } p = 0. \end{cases}$$

Because  $g_t > 0$ , this contradicts the fact that  $\mu$  satisfies condition (4.10). Hence  $g_t$  cannot be an integer.

Next, assume that  $g_t \in \mathbb{Z} + \frac{1}{2}$ . We need to show that  $g_t = \frac{1}{2}$ .

Recall that  $\varepsilon_t$  is either 0 or  $\pm 1$ . We consider the two cases separately. If  $\varepsilon_t = 0$ , the fine representation  $\delta^{L_a}$  of  $(L_a \cap K)$  must be non-trivial on both the  $\tilde{U}(s_t, s_t)$ -factor and the  $Mp(2d)$ -factor of  $L_a$ . Then either

$$(4.14) \quad \delta^{L_a} = \left( \dots, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t} \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \middle| \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t}, \dots \right)$$

or

$$(4.15) \quad \delta^{L_a} = \left( \dots, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{s_t} \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{s_t}, \dots \right)$$

for some  $p, q \geq 0$  such that  $p + q = d$ . Equation 4.15 contradicts condition (4.10), because the difference  $a_t - c_1$  is always positive; hence Equation 4.14 must hold. We find

$$a_t - c_1 = 0 \iff g_t = \frac{1}{2} \quad \text{and} \quad p > 0.$$

Note that, in this case,  $c_d - b_t > 0$  by Equation 4.13. This proves (b) (and also (d) for the case  $\varepsilon_t = 0$ ).

If  $\varepsilon_t \neq 0$ , the fine representation  $\delta^{L_a}$  of  $(L_a \cap K)$  must be trivial on the  $\tilde{U}(s_t + \varepsilon_t, s_t)$ -factor of  $L_a$  and non-trivial on the  $Mp(2d)$ -factor of  $L_a$ . Hence

$$\delta^{L_a} = \left( \dots, \underbrace{0, \dots, 0}_{s_t} \middle| \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \middle| \underbrace{0, \dots, 0}_{s_t}, \dots \right).$$

Just as above, we get that

$$a_t - c_1 = 0 \iff g_t = \frac{1}{2}, \varepsilon_t = 1 \text{ and } p > 0.$$

Moreover,  $c_d - b_t > 0$  always in this case. This concludes the proofs of (a), (c) and (d). The proof of Lemma 4.4 is now complete.  $\square$

Finally, we look at the case in which the irreducible, admissible, genuine and  $\omega$ -regular  $(\mathfrak{g}, K)$ -module  $X$  of  $Mp(2n)$  is also unitary. Realize  $X$  as the unique lowest  $K$ -type constituent of a representation  $\mathcal{R}_{\mathfrak{q}}^S(X^L)$ , with

$$L = L_1 \times L_0 = \tilde{U}(r, s) \times Mp(2d) \quad \text{and} \quad X^L \simeq X_1 \otimes X_0,$$

as in Proposition 2.4. Proposition 3.1 then implies that  $X_1$  a good  $A_{\mathfrak{q}_2}(\lambda_2)$  module. The next Lemma describes the coordinates of the lowest  $K$ -type of  $X$ , under some technical assumptions that are needed for the proof of Claim (A) in Proposition 3.2.

**Lemma 4.5.** *Let  $X$  be an irreducible, admissible, genuine and  $\omega$ -regular  $(\mathfrak{g}, K)$ -module of  $Mp(2n)$ . Assume that  $X_1$  (as above) is a good  $A_{\mathfrak{q}_2}(\lambda_2)$  module. Let  $\mu$  be the unique lowest  $K$ -type of  $X$  and let  $\lambda_a$  be its Vogan classification parameter. Write the coordinates of  $\mu$  and  $\lambda_a$  as in Equations 4.5 and 2.6, respectively. Set*

$$\begin{aligned} x &= \max \{i < t \mid r_i > 0\} \\ y &= \max \{i < t \mid s_i > 0\}. \end{aligned}$$

If  $g_t = \frac{1}{2}$ ,  $\varepsilon_t = 1$  and  $s_t > 0$ , then

- (i)  $b_t - b_y \geq 2$ .
- (ii) If  $g_{t-1} \in \mathbb{Z}$  or if  $g_{t-1} \in \mathbb{Z} + \frac{1}{2}$  and  $g_{t-1} \geq \frac{7}{2}$ , then  $a_x - a_t \geq 2$ .

*Proof.* Recall that, by Lemma 4.2,  $X$  has a unique lowest  $\tilde{U}(n)$ -type  $\mu$  (of the form (4.2)). Equations 4.4 give

$$\begin{aligned} (4.16) \quad b_t - b_{t-1} &= -g_t + h_t - (-g_{t-1} + h_{t-1}) \\ &= -g_t + g_{t-1} + \frac{\varepsilon_{t-1} + \varepsilon_t}{2} = g_{t-1} + \frac{\varepsilon_{t-1}}{2} \end{aligned}$$

(because  $g_t = 1/2$  and  $\varepsilon_t = 1$ ). Note that  $g_{t-1}$  is a half-integer greater than  $g_t = 1/2$ , and  $\varepsilon_{t-1}$  is 0 or  $\pm 1$ . Then  $b_t - b_{t-1} \geq 1$ . (This is obvious if  $g_{t-1} \in \mathbb{Z} + \frac{1}{2}$ ; when  $g_{t-1}$  is an integer, it follows from the fact that  $\varepsilon_{t-1} = 0$  because  $b_t - b_{t-1}$  must be an integer.) Since the entries  $\{b_i\}_{i=1}^t$  are weakly increasing, we also find that  $b_t - b_y \geq 1$ .

In order to show that the difference  $b_t - b_y$  is, in fact, at least 2, we consider the lowest  $(K \cap L_1)$ -type  $\mu_1$  of  $X_1$ . This is a  $(\tilde{U}(r) \times \tilde{U}(s))$ -type of the form

$$(4.17) \quad \mu_1 = (\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p}))|_{\tilde{U}(r) \times \tilde{U}(s)}.$$

Note that  $2\rho(\mathfrak{u} \cap \mathfrak{p})$  is constant on  $\tilde{U}(r)$  and  $\tilde{U}(s)$  so, to understand the difference  $b_t - b_y$ , it is sufficient to look at the (difference among) coordinates of  $\mu_1$ .

Write

$$(4.18) \quad L_2 = \prod_{i=1}^u \tilde{U}(p_i, q_i).$$



Section 8 of [9] gives a number of properties of lowest  $K$ -types of (good)  $A_{\mathfrak{q}}(\lambda)$  modules. Assuming we have chosen  $\mathfrak{q}_2$  so that  $L_2$  is maximal (there is a unique such choice),  $\mu_1$  must be of the form

$$(4.19) \quad \mu_1 = \left( \underbrace{n_1, \dots, n_1}_{p_1}, \dots, \underbrace{n_u, \dots, n_u}_{p_u} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{m_u, \dots, m_u}_{q_u}, \dots, \underbrace{m_1, \dots, m_1}_{q_1} \right)$$

with  $n_j > n_k$  and  $m_k > m_j$  for all  $1 \leq j < k \leq u$ . Moreover, Proposition 8.6 of [9] implies that

$$(4.20) \quad n_j - n_k \geq q_i + q_{i+1} \quad \text{and} \quad m_k - m_j \geq p_i + p_{i+1}$$

whenever  $j \leq i$  and  $k \geq i + 1$ . Let

$$z = \max \{i < u \mid q_i > 0\},$$

so that  $m_z$  exists. Then we get

$$b_t - b_y = m_u - m_z \geq p_u + p_{u-1}$$

(by (4.20) for  $i = u - 1$ ).

By Proposition 2.4 (vii), and Proposition 3.5 of [10],  $L_a \cap L_1 \subset L_2$ , therefore  $\mathfrak{u}(r_t, s_t) \subset \mathfrak{u}(p_u, q_u)$ , so that

$$r_t \leq p_u \quad \text{and} \quad s_t \leq q_u.$$

In fact,  $s_t = q_u$ , because  $2\rho(\mathfrak{u} \cap \mathfrak{p})$  is constant on  $\tilde{U}(r)$  and  $\tilde{U}(s)$ , and  $b_t - b_{t-1} > 0$ . Hence we can write

$$b_t - b_y \geq p_u + p_{u-1} \geq r_t = s_t + \varepsilon_t \geq 2.$$

This proves (i).

For the proof of (ii), we distinguish two cases.

If  $g_{t-1}$  is an integer, then Lemma 4.2 implies that  $\varepsilon_{t-1} = 0$ , hence  $s_{t-1} = r_{t-1} > 0$  and  $x = y = t - 1$ . Furthermore, because  $L_a \cap L_1 \subset L_2$ , we have  $\mathfrak{u}(r_{t-1}, s_{t-1}) \subset \mathfrak{u}(p_{u-1}, q_{u-1})$ . This implies

$$(p_u, q_u) = (r_t, s_t) \quad \text{and} \quad q_{u-1} \geq s_{t-1} \geq 1.$$

Then

$$a_{t-1} - a_t = n_{u-1} - n_u \geq q_u + q_{u-1} \geq 1 + 1 = 2,$$

proving (ii).

If  $g_{t-1} \geq \frac{7}{2}$ , then using Equations 4.4, we get

$$\begin{aligned} a_{t-1} - a_t &= g_{t-1} - g_t + f_{t-1} - f_t = g_{t-1} - g_t - \frac{\varepsilon_{t-1} + \varepsilon_t}{2} \\ &= g_{t-1} - \frac{\varepsilon_{t-1}}{2} - 1 \geq 2. \end{aligned}$$

This concludes the proof of the lemma.  $\square$

5. GENUINE PRINCIPAL SERIES OF  $Mp(2d)$ 

Genuine minimal principal series of  $Mp(2d)$  were studied in details in [5]. In this section, we summarize the non-unitarity results that are needed for this paper. We refer the reader to [5] for more details.

**Lemma 5.1** (cf. [5]). *(i) Every genuine minimal principal series representation of  $Mp(2d)$  has a unique lowest  $\tilde{U}(d)$ -type of the form*

$$\mu_{\delta_{p,q}} = \left( \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q \right)$$

with  $p + q = d$ . Here  $\delta_{p,q}$  is the character of the finite subgroup  $M$  of  $Mp(2d)$  used to construct the induced representation.

*(ii) For every pair of non-negative integers  $p$  and  $q$  with  $p+q = d$ , the Langlands quotients of genuine minimal principal series of  $Mp(2d)$  with lowest  $\tilde{U}(d)$ -type  $\mu_{\delta_{p,q}}$  are parametrized by  $d$ -tuples of real numbers*

$$\nu = (\nu_1, \dots, \nu_d) := (\nu^p | \nu^q),$$

where  $\nu_1 \geq \nu_2 \geq \dots \geq \nu_p \geq 0$  and  $\nu_{p+1} \geq \nu_{p+2} \geq \dots \geq \nu_{p+q} \geq 0$ .

The infinitesimal character of the corresponding representation is equal to  $\nu$ .

*(iii) Irreducible genuine pseudospherical representations of  $Mp(2d)$ , i. e., those with lowest  $\tilde{U}(d)$ -type  $\mu_{\delta_{d,0}}$  or  $\mu_{\delta_{0,d}}$ , are uniquely determined by their infinitesimal character.*

**Lemma 5.2.** *Let  $J(\delta_{p,q}, \nu)$  be the Langlands quotient of a genuine minimal principal series representation of  $Mp(2d)$ , with lowest  $\tilde{U}(d)$ -type  $\mu_{\delta_{p,q}}$  and parameter  $\nu = (\nu^p | \nu^q)$  as in Lemma 5.1.*

*(i) If  $\nu_p > \frac{1}{2}$  or if  $\nu_i - \nu_{i+1} > 1$  for some  $1 \leq i \leq p-1$ , then  $J(\delta_{p,q}, \nu)$  is not unitary, and the  $\tilde{U}(d)$ -type*

$$\delta_1 = \left( \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p-1}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q, -\frac{3}{2} \right)$$

*detects non-unitarity.*

*(ii) If  $\nu_{p+q} > \frac{1}{2}$  or if  $\nu_i - \nu_{i+1} > 1$  for some  $p+1 \leq i \leq p+q-1$ , then  $J(\delta_{p,q}, \nu)$  is not unitary, and the  $\tilde{U}(d)$ -type*

$$\delta_2 = \left( \frac{3}{2}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q-1} \right)$$

*detects non-unitarity.*

*(iii) If  $J(\delta_{p,q}, \nu)$  is  $\omega$ -regular,  $pq \neq 0$ ,  $\nu^p \neq \omega^p$  and  $\nu^q \neq \omega^q$ , then both  $\delta_1$  and  $\delta_2$  detect non-unitarity. As usual,  $\omega^r$  denotes the infinitesimal character of the oscillator representation of  $Mp(2r)$ .*

*Proof.* The main ingredients for the proof of Lemma 5.2 are certain non-unitarity results contained in [1] and [5]. Note that the assumptions in part (i) of Lemma 5.2 are precisely conditions (1) and (3) of Proposition 5 of [5]; similarly, the assumptions in part (ii) coincide with conditions (2) and (4) of that proposition. Therefore, in both cases, the non-unitarity of the Langlands quotient  $J(\delta_{p,q}, \nu)$  of  $Mp(2d)$  follows directly from Proposition 5 of [5].

We are left with the problem of identifying a  $\tilde{U}(d)$ -type on which the intertwining operator changes sign.

Recall from §7 of [5] that each  $\tilde{U}(d)$ -type  $\mu$  in  $J(\delta_{p,q}, \nu)$  carries a representation  $\psi_\mu$  of the stabilizer

$$W^{\delta_{p,q}} = W(C_p) \times W(C_q)$$

of  $\delta_{p,q}$ . This group is isomorphic to the Weyl group of

$$G^{\delta_{p,q}} = SO(p+1, p) \times SO(q+1, q);$$

hence, associated with  $\psi_\mu$ , there is an intertwining operator for spherical Langlands quotients of  $G^{\delta_{p,q}}$ . If  $\mu$  is petite, the  $Mp(2d)$ -intertwining operator on  $\mu$  with parameters  $\{\delta = \delta_{p,q}, \nu = (\nu^p|\nu^q)\}$  coincides with the  $G^{\delta_{p,q}}$ -intertwining operator on  $\psi_\mu$  with parameters  $\{\delta = \text{triv}, \nu = (\nu^p|\nu^q)\}$ .

If  $\nu_p > \frac{1}{2}$  or if  $\nu_i - \nu_{i+1} > 1$  for some  $1 \leq i \leq p-1$ , the spherical Langlands quotient of  $SO(p+1, p)$  with parameter  $\nu^p$  is not unitary; the reflection representation

$$\sigma_R = (p-1) \times (1)$$

of  $W(C_p)$  detects non-unitarity (see Lemma 14.6 of [1]). For all choices of  $\nu^q$ , the spherical Langlands quotient of  $SO(p+1, p) \times SO(q+1, 1)$  with parameter  $(\nu^p|\nu^q)$  is also not unitary, and the representation

$$\sigma_R \times \text{triv}$$

of  $W^{\delta_{p,q}}$  detects non-unitarity. The computations in §10.1 of [5] show that the  $\tilde{U}(d)$ -type

$$\delta_1 = \left( \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p-1}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q, -\frac{3}{2} \right)$$

carries the representation  $\sigma_R \times \text{triv}$  of  $W^{\delta_{p,q}}$ . Because  $\delta_1$  is petite, the  $Mp(2d)$ -intertwining operator on  $\delta_1$  with parameters  $\{\delta_{p,q}, (\nu^p|\nu^q)\}$  matches the  $G^{\delta_{p,q}}$ -spherical operator with parameter  $(\nu^p|\nu^q)$ , hence it is not positive semi-definite. This shows that  $J(\delta_{p,q}, (\nu^p|\nu^q))$  is not unitary, and concludes the proof of part (i) of the lemma.

The proof of part (ii) is analogous: the  $\tilde{U}(d)$ -type  $\delta_2$  carries the representation  $\text{triv} \otimes \sigma_R$  of  $W^{\delta_{p,q}}$ , hence the  $Mp(2d)$ -intertwining operator on  $\delta_2$  with parameters  $\{\delta_{p,q}, (\nu^p|\nu^q)\}$  is not positive semi-definite and the Langlands quotient  $J(\delta_{p,q}, (\nu^p|\nu^q))$  of  $Mp(2d)$  is not unitary.

For part (iii) of the lemma, note that if  $\nu = (\nu^p|\nu^q)$  is  $\omega$ -regular and  $\nu^p \neq \omega^p$ , then

$$\nu_p > \frac{1}{2} \text{ or } \nu_i - \nu_{i+1} > 1 \text{ for some } 1 \leq i \leq p-1.$$

Similarly, if  $\nu = (\nu^p|\nu^q)$  is  $\omega$ -regular and  $\nu^q \neq \omega^q$ , then

$$\nu_{p+q} > \frac{1}{2} \text{ or } \nu_i - \nu_{i+1} > 1 \text{ for some } p+1 \leq i \leq p+q-1.$$

Therefore, the assumptions of both part (i) and part (ii) of the lemma must hold, and both  $\delta_1$  and  $\delta_2$  detect non-unitarity.  $\square$

## 6. PROOF OF PROPOSITION 3.1.

In this section, we give the proof of Proposition 3.1. For convenience, we begin by restating the result.

**Proposition.** *Let  $X$  be an irreducible unitary  $(\mathfrak{g}, K)$ -module of  $Mp(2n)$ . Assume that  $X$  is genuine and  $\omega$ -regular. Realize  $X$  as the unique lowest  $K$ -type constituent of  $\mathcal{R}_{\mathfrak{q}}^S(X^L)$ , with*

$$L = L_1 \times L_0 = \tilde{U}(r, s) \times Mp(2d) \quad \text{and} \quad X^L \simeq X_1 \otimes X_0,$$

as in Proposition 2.4. Suppose that  $r + s \neq 0$ . Then there exist a theta stable parabolic subalgebra  $\mathfrak{q}_2$  and a representation  $\mathbb{C}_{\lambda_2}$  of the Levi factor corresponding to  $\mathfrak{q}_2$ , with  $\lambda_2$  in the good range for  $\mathfrak{q}_2$ , such that  $X_1 \simeq A_{\mathfrak{q}_2}(\lambda_2)$ .

*Proof.* Note that, by Proposition 2.4, the  $(\mathfrak{l}, L \cap K)$ -module  $X^L$  is irreducible, and so are the Harish-Chandra modules  $X_i$  of  $L_i$  ( $i = 0, 1$ ). Let  $\mu$  be a lowest  $K$ -type of  $X$ , and let  $\mu^L$  be the  $(L \cap K)$ -type of  $X^L$  corresponding to  $\mu$  via the bottom layer map (cf. Proposition 2.4 (ii)-(iii)); then

$$(6.1) \quad \mu^L = \mu_1 + \mu_0$$

with  $\mu_i = \mu^{L_i}$  a lowest  $(L_i \cap K)$ -type of  $X_i$ . By Lemma 2.7, the representation  $X_1$  of  $L_1 = \tilde{U}(r, s)$  is strongly regular (and irreducible).

Assume, by way of contradiction, that  $X_1$  is *not* a good  $A_{\mathfrak{q}}(\lambda)$  module. We will show that  $X$  must be non-unitary, reaching a contradiction.

By Theorem 1.2 in [9] for the case  $G = SU(p, q)$ , if  $X_1$  is not a good  $A_{\mathfrak{q}}(\lambda)$  module, then  $X_1$  not unitary, and there exists an  $(L_1 \cap K)$ -type  $\eta_1$  such that

$$\eta_1 = \mu_1 + \beta$$

for some  $\beta \in \Delta(\mathfrak{l}_1 \cap \mathfrak{p})$ , and the form

$$\langle \cdot, \cdot \rangle^{L_1} \Big|_{V(\mu_1) \oplus V(\eta_1)}$$

is indefinite. (In [9], the infinitesimal character of  $X_1$  is assumed to be integral; however, the proof of Theorem 1.2 only uses the fact that it is real and strongly regular.) If

$$\eta = \mu + \beta = (\mu_1 + \beta) + \mu_0 + 2\rho(\mathfrak{u} \cap \mathfrak{p})$$

is also dominant, then, by the bottom layer argument,  $\eta$  occurs in  $X$  and the Hermitian form on  $V(\mu) \oplus V(\eta)$  is indefinite (cf. Proposition 2.4 (iii) and (v)). This implies that  $X$  is *not* unitary, reaching a contradiction.

So we may assume that  $\eta = \mu + \beta$  is not dominant. Moreover, we can assume that  $\mu$  is weakly dominant with respect to our fixed choice of  $\Delta_c^+$  in (2.5). Because  $\beta$  is a short non-compact root, if  $\eta_1 = \mu_1 + \beta$  is dominant but  $\eta = \mu + \beta$  is *not* dominant, then one of the following two options must occur:

$$(6.2) \quad d = 0 \quad \text{and} \quad a_t = b_t,$$

or

$$(6.3) \quad a_t = c_1 \quad \text{or} \quad c_d = b_t.$$

**Remark 6.1.** Lemma 4.4 shows that (6.2) is not possible, and that the two identities in (6.3) can not hold simultaneously.

Without loss of generality, we may then assume that  $d > 0$  and  $a_t = c_1$ . We first look at the case in Lemma 4.4 when  $\varepsilon_t = 1$  and  $g_t = \frac{1}{2}$ . By Lemma 4.2 (ii), the infinitesimal character  $\gamma^X$  of  $X$  contains an entry  $\frac{1}{2}$ , but  $\gamma^{X_0}$  does not. Recall that  $X_0$  is a genuine principal series representation of  $L_0 = Mp(2d)$ . We claim that in this case,  $X_0$  is not unitary, and that a  $\tilde{U}(d)$ -type detecting non-unitarity survives under the bottom layer map, leading to a contradiction. We make use of the non-unitarity results for genuine minimal principal series of  $Mp(2d)$  contained in Section 5.

Write  $X_0 \cong J(\delta_{p,q}, \nu)$  as in Lemma 5.2; note that  $p > 0$ , by Lemma 4.4. Then, by  $\omega$ -regularity,  $\nu_p$  is strictly greater than  $\frac{1}{2}$ . By part (i) of Lemma 5.2, we know that the representation  $X_0$  is not unitary, and the signature of the Hermitian form changes on  $\delta_1$ . If  $\mu_1$  is as in Equation 6.1, the weight

$$\eta = \mu_1 + \delta_1 + 2\rho(\mathfrak{u} \cap \mathfrak{p})$$

is  $K$ -dominant. By Proposition 2.4 (iii) and (v), our original representation  $X$  is not unitary, which contradicts our assumption.

Now assume that  $\varepsilon_t = 0$ . Recall that the fine  $K$ -type  $\delta^{L_a}$  is of the form (4.11); in particular, its restriction to the subgroup  $\tilde{U}(s_t, s_t)$  of  $L_a$  is given by

$$\delta^{L_a} |_{\tilde{U}(s_t, s_t)} = \left( 0, \dots, 0, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{s_t}, 0, \dots, 0 \right).$$

Hence, the representation  $\mu_1 |_{\tilde{U}(s_t, s_t)}$ , inside  $Mp(2n)$ , is of the form

$$\left( 0, \dots, 0, \underbrace{a, \dots, a}_{s_t} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{-a-1, \dots, -a-1}_{s_t}, 0, \dots, 0 \right)$$

for some  $a \in \frac{1}{2}\mathbb{Z}$ .

By Lemma 8.8 and Lemma 6.3 (b) in [9], if  $\beta = (1, 0 \dots 0; 0, \dots 0, -1)$  is the root  $\epsilon_1 - \epsilon_{2s_t}$  in  $U(s_t, s_t)$ , then  $\mu_1 |_{U(s_t, s_t)} + \beta$  detects the signature change and survives in the bottom layer. Note that the root in  $Mp(2n, \mathbb{R})$  corresponding to  $\beta$  is

$$\left( 0, \dots, 0, \underbrace{1, 0, \dots, 0}_{s_t} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{1, 0, \dots, 0}_{s_t}, 0, \dots, 0 \right).$$

Therefore,  $X$  is not unitary, once again a contradiction. The proof of Proposition 3.1 is now complete.  $\square$

## 7. PROOF OF PROPOSITION 3.2

We now restate and prove Proposition 3.2.

**Proposition.** *Let  $X$  be an irreducible unitary  $(\mathfrak{g}, K)$ -module of  $Mp(2n)$ . Assume that  $X$  is genuine and  $\omega$ -regular; realize  $X$  as the unique lowest  $K$ -type constituent of  $\mathcal{R}_q^S(X^L)$ , with*

$$L = L_1 \times L_0 = \tilde{U}(r, s) \times Mp(2d) \quad \text{and} \quad X^L \simeq X_1 \otimes X_0,$$

as in Proposition 2.4. Suppose that  $d > 0$ . Then one of the following (mutually exclusive) options occurs:

(i)  $X_0$  is an even oscillator representation with lowest  $\tilde{U}(d)$ -type

$$\mu_0 = \pm(1/2, 1/2, \dots, 1/2).$$

(ii) There exist a subgroup  $L' = L'_1 \times L'_0 = \tilde{U}(r', s') \times Mp(2(d+1)) \subset G$  also containing  $L_a$ , a theta stable subalgebra  $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \supseteq \mathfrak{q}$  and representations  $X'_i$  of  $L'_i$  ( $i = 0, 1$ ), such that

(a)  $\tilde{U}(r', s') \subset \tilde{U}(r, s)$ , with either  $r = r' + 1$  or  $s = s' + 1$ ,

(b)  $X'_1$  is a good  $A_{\mathfrak{q}_3}(\lambda_3)$  (or  $r' + s' = 0$ ),

(c)  $X'_0$  is an odd oscillator representation with lowest  $K$ -type

$$\mu'_0 = (3/2, 1/2, \dots, 1/2) \text{ or } (-1/2, -1/2, \dots, -1/2, -3/2),$$

and

(d)  $X$  can be realized as the unique lowest  $K$ -type constituent of  $\mathcal{R}_{\mathfrak{q}'}^{S'}(X'_1 \otimes X'_0)$ .

*Proof.* Let  $\lambda_a$  be the Vogan classification parameter of  $X$ . Recall that, by Proposition 3.1,  $X_1$  is a good  $A_{\mathfrak{q}_2}(\lambda_2)$  module; then, by Lemma 4.2,  $X$  has a unique lowest  $\tilde{U}(n)$ -type  $\mu$ .

To prove Proposition 3.2, we first show that the representation  $X_0$  of  $Mp(2d)$  is pseudospherical, *i. e.*, that  $X_0 \simeq J(\delta_{p,q}, \nu)$  with  $pq = 0$ . Assume, by way of contradiction, that  $X_0 \simeq J(\delta_{p,q}, \nu)$  with  $p > 0$  and  $q > 0$ , and set  $\nu = (\nu^p | \nu^q)$  (as in Lemma 5.1). Write the lowest  $K$ -type  $\mu$  of  $X$  as in Equation 4.5 and the Vogan's classification parameter  $\lambda_a$  as in (2.6).

Recall that the half integers  $a_i$  and  $b_i$  are defined even if  $r_i = 0$  or  $s_i = 0$ , and that they satisfy (4.9). By an argument similar to the proof of Lemma 4.4, (at least) one of the following conditions must hold:

$$a_t > c_1 \quad \text{or} \quad c_d > b_t.$$

Notice that, since  $p > 0$  and  $q > 0$ , the  $\tilde{U}(d)$ -types

$$\delta_1 = \left( \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p-1}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_q, -\frac{3}{2} \right) \quad \text{and} \quad \delta_2 = \left( \frac{3}{2}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_p, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{q-1} \right)$$

(of Lemma 5.2) are obtained from the lowest  $\tilde{U}(d)$ -type  $\mu_0$  of  $X_0$  by adding or subtracting a short root. One can check that if  $a_t > c_1$ , then  $\delta_2$  survives in the bottom layer; if  $c_d > b_t$ , then  $\delta_1$  survives in the bottom layer.

We distinguish two cases.

First, suppose that  $\nu^p \neq \omega^p$  and  $\nu^q \neq \omega^q$ ; then (by Case 3 of Lemma 5.2)  $X_0$  is not unitary and *both*  $\delta_1$  and  $\delta_2$  detect non-unitarity. Because (at least) one of the two  $\tilde{U}(d)$ -types survives in the bottom layer, we conclude that  $X$  is not unitary and we reach a contradiction.

Next, assume that  $\nu^p = \omega^p$  or  $\nu^q = \omega^q$ . (Note that the two options can not hold at the same time, because  $X_0$  is  $\omega$ -regular.) If  $\nu^p = \omega^p$ , then the entries of  $\nu^q$  are all greater than or equal to  $p + \frac{1}{2}$ ; in particular,  $\nu_{p+q} > \frac{1}{2}$ . Hence, we are in Case (ii) of Lemma 5.2:  $X_0$  is not unitary and  $\delta_2$  detects non-unitarity. Similarly, if  $\nu^q = \omega^q$ , then  $X_0$  is not unitary and  $\delta_1$  detects non-unitarity. In either case, the

infinitesimal character of  $X_0$  contains an entry  $\frac{1}{2}$ . Then, by Lemma 4.2,  $g_t \geq 1$ . Because  $g_t \neq \frac{1}{2}$ , Lemma 4.4 (ii) (a) implies that  $a_t > c_1$ ; a similar argument shows that  $c_d > b_t$  (because if  $c_d = b_t$  then  $-g_t = -\frac{1}{2}$ ). We conclude that the coordinates of  $\mu$  satisfy *both* conditions

$$a_t > c_1 \quad \text{and} \quad c_d > b_t$$

and that *both*  $\tilde{U}(d)$ -types  $\delta_1$  and  $\delta_2$  survive in the bottom layer. Hence  $X$  is not unitary, contradicting our assumption.

Now we know that  $X_0$  is pseudospherical. Without loss of generality, we may assume that  $X_0$  has lowest  $\tilde{U}(d)$ -type  $\mu_0 = (-\frac{1}{2}, \dots, -\frac{1}{2})$ . If  $X_0$  is the even anti-holomorphic oscillator representation (*i. e.*, if  $X_0$  has infinitesimal character  $\omega^d$ ), then we are in case (i) of Proposition 3.2, and we are done. So suppose not. Recall that, by Lemma 2.7,  $X_0$  is  $\omega$ -regular. Therefore, if the infinitesimal character of  $X_0$  is not equal to  $\omega^d$ , then it must satisfy one of the conditions of Lemma 5.2 (ii). We conclude that  $X_0$  is not unitary, and the  $\tilde{U}(d)$ -type  $\delta_2$  detects non-unitarity. Note that the  $\tilde{U}(d)$ -type  $\delta_2$  may or may not survive in the bottom layer. If  $\delta_2$  survives in the bottom layer, then  $X$  is not unitary, and we reach a contradiction. Hence, we may assume that  $\delta_2$  does not survive in the bottom layer. In this case, the lowest  $K$ -type  $\mu$  satisfies the condition  $a_t - c_1 \leq 1$ . By Equation 4.12,

$$a_t - c_1 = g_t + \frac{1}{2}(1 - \varepsilon_t) + y_t - z_1.$$

Here  $z_1 = -\frac{1}{2}$  and  $y_t = 0$  because, by Equation 4.2, the fine  $K$ -type  $\delta^{L_a}$  in (4.1) is trivial except on  $Mp(2d)$ . Therefore, we obtain

$$a_t - c_1 = g_t + \frac{1}{2}(1 - \varepsilon_t) + \frac{1}{2} \geq \frac{1}{2} + \frac{1}{2} = 1$$

(because  $g_t \geq \frac{1}{2}$ ). This forces  $a_t - c_1 = 1$ , so  $g_t = \frac{1}{2}$ ,  $\varepsilon_t = 1$  and  $r_t - s_t = 1$ .

Use the (new) singularization

$$\xi' = \left( \underbrace{1, 1, \dots, 1}_{r-r_t} \mid \underbrace{0, 0, \dots, 0}_{d+r_t+s_t} \mid \underbrace{-1, -1, \dots, -1}_{s-s_t} \right)$$

of  $\lambda_a$  to construct a (new) parabolic subalgebra  $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}'$ , just as at the beginning of Section 3, and let

$$L' = L'_1 \times L'_0 \cong \tilde{U}(r - r_t, s - s_t) \times Mp(2(d + r_t + s_t))$$

be the subgroup corresponding to the Levi factor of  $\mathfrak{q}'$ . Then  $X$  may be realized as the unique lowest  $K$ -type constituent of a representation

$$\mathcal{R}_{\mathfrak{q}'}^{S'}(X^{L'}),$$

as in Proposition 2.4. Using notation analogous to the one in Lemma 2.7, write  $X^{L'} = X'_1 \otimes X'_0$ . Note that  $X'_0$  is still  $\omega$ -regular, and  $X'_1$  is still strongly regular, as representations of  $L'_0$  and  $L'_1$ , respectively.

To finish the proof of our proposition, we need to show that, if  $X$  is unitary, then:

- (A)  $s_t = 0$  (hence  $r_t = 1$ );

(B)  $X'_0$  is an odd oscillator representation with lowest  $\tilde{U}(d+1)$ -type

$$\mu'_0 = \left( \frac{3}{2}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_d \right);$$

and

(C)  $X'_1$  is a good  $A_{\mathfrak{q}_3}(\lambda_3)$  module.

We begin with the proof of Claim (A). Assume, to the contrary, that  $c := s_t > 0$ , and let  $\mu'_0$  be the lowest  $\tilde{U}(d+2c+1)$ -type of  $X'_0$ . By Proposition 2.4 (ii), the lowest  $(L' \cap \tilde{U}(n))$ -type  $\mu'$  of  $X^{L'}$  satisfies

$$\mu' = \mu - 2\rho(\mathfrak{u}' \cap \mathfrak{p}).$$

Therefore,  $\mu'_0$  must be of the form

$$\mu'_0 = \left( \underbrace{a_t - r' + s', \dots, a_t - r' + s'}_{c+1}, \right. \\ \left. \underbrace{c_1 - r' + s', \dots, c_d - r' + s'}_d, \underbrace{b_t - r' + s', \dots, b_t - r' + s'}_c \right)$$

with  $r' = r - r_t$  and  $s' = s - s_t$ . Using  $g_t = \frac{1}{2}$  and  $\varepsilon_t = 1$ , we obtain

$$\mu'_0 = \left( \underbrace{\frac{3}{2}, \dots, \frac{3}{2}}_{c+1}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_d, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_c \right).$$

We will show, using Parthasarathy's Dirac Operator Inequality, that  $X'_0$  is not unitary, and that it contains a  $\tilde{U}(d+2c+1)$ -type  $\eta'_0$  on which the signature of the Hermitian form changes. Because  $\eta'_0$  survives under the bottom layer map, this will imply that  $X$  is not unitary, reaching a contradiction and thus completing the proof of Claim (A).

We recall some results from [6], and from Lemmas 6.1 and 6.3 of [9]. Let  $\Delta^+(\mathfrak{l}'_0)$  be a choice of positive roots for  $Mp(2(d+2c+1))$  which is compatible with our fixed system of positive compact roots, and let  $\rho_n$  be the corresponding half sum of non-compact positive roots. Choose a Weyl group element  $w$  such that  $w^{-1}(\mu'_0 - \rho_n)$  is  $K$ -dominant, and denote by  $\gamma^{X'_0}$  the infinitesimal character of  $X'_0$ . If

$$(7.1) \quad \langle \mu'_0 - \rho_n + w\rho_c, \mu'_0 - \rho_n + w\rho_c \rangle < \langle \gamma^{X'_0}, \gamma^{X'_0} \rangle,$$

then  $X'_0$  is not unitary. In this case, there is a non-compact root  $\beta \in \Delta(\mathfrak{l}'_0)$  such that  $\eta'_0 = \mu'_0 - \beta$  occurs in  $X'_0$  and detects non-unitarity (in the sense that the signature of the Hermitian form on  $X'_0$ , restricted to  $\mu'_0 \oplus \eta'_0$ , is indefinite). If the conditions of Lemma 6.3 (b) of [9] are satisfied, then the root  $\beta$  may be chosen from  $\Delta^+(\mathfrak{l}'_0)$ .

We will prove that, in our setting, we can always choose  $\Delta^+(\mathfrak{l}'_0)$  so that Equation 7.1 holds and the  $\tilde{U}(d+2c+1)$ -type  $\eta'_0 = \mu'_0 - \beta$  survives under the bottom layer map. The choice of  $\Delta^+(\mathfrak{l}'_0)$  will depend on the Vogan classification parameter  $\lambda_a$



of  $X$ ; in particular, we will distinguish two cases, according to the possible values of the entry  $g_{t-1}$ .

Recall that  $g_{t-1}$  is a half integer (strictly) greater than  $g_t = \frac{1}{2}$ . First, assume that

$$g_{t-1} \in \mathbb{Z} \quad \text{or} \quad g_{t-1} \in \mathbb{Z} + \frac{1}{2} \quad \text{and} \quad g_{t-1} \geq \frac{7}{2}.$$

In this case, we let  $\Delta^+(\iota'_0)^{(1)}$  be such that

$$\rho(\iota'_0)^{(1)} = (2c + d + 1, 2c + d, \dots, c + 1, -1, -2, \dots, -c)$$

and

$$\rho_n^{(1)} = \left( \underbrace{\frac{d}{2} + c + 1, \dots, \frac{d}{2} + c + 1}_{c+d+1}, \underbrace{\frac{d}{2}, \dots, \frac{d}{2}}_c \right).$$

Then

$$\begin{aligned} \mu'_0 - \rho_n^{(1)} = & \left( \underbrace{-\frac{d}{2} - c + \frac{1}{2}, \dots, -\frac{d}{2} - c + \frac{1}{2}}_{c+1}, \right. \\ & \left. \underbrace{-\frac{d}{2} - c - \frac{1}{2}, \dots, -\frac{d}{2} - c - \frac{1}{2}}_d, \underbrace{-\frac{d}{2} - \frac{1}{2}, \dots, -\frac{d}{2} - \frac{1}{2}}_c \right). \end{aligned}$$

The element  $w^{(1)}\rho_c$  can be chosen to be

$$\left( \underbrace{\frac{d}{2}, \frac{d}{2} - 1, \dots, \frac{d}{2} - c}_{c+1}, \underbrace{\frac{d}{2} - c - 1, \frac{d}{2} - c - 2, \dots, -c - \frac{d}{2}}_d, \underbrace{c + \frac{d}{2}, c + \frac{d}{2} - 1, \dots, \frac{d}{2} + 1}_c \right).$$

Hence, we obtain

$$(7.2) \quad \begin{aligned} \mu'_0 - \rho_n^{(1)} + w^{(1)}\rho_c = & \left( \underbrace{-c + \frac{1}{2}, -c - \frac{1}{2}, \dots, -2c + \frac{1}{2}}_{c+1}, \right. \\ & \left. \underbrace{-2c - \frac{3}{2}, -2c - \frac{5}{2}, \dots, -2c - d - \frac{1}{2}}_d, \underbrace{c - \frac{1}{2}, c - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}}_c \right). \end{aligned}$$

Because  $X'_0$  is  $\omega$ -regular, its infinitesimal character  $\gamma^{X'_0}$  must satisfy the condition

$$(7.3) \quad \langle \gamma^{X'_0}, \gamma^{X'_0} \rangle \geq \langle \omega^{d+2c+1}, \omega^{d+2c+1} \rangle.$$

Writing  $\omega^{d+2c+1}$  in coordinates, we get

$$(7.4) \quad \omega^{d+2c+1} = \left( d + 2c + \frac{1}{2}, d + 2c - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \right).$$

Rearranging the entries of  $\mu'_0 - \rho_n^{(1)} + w^{(1)}\rho_c$  in Equation 7.2 and comparing them with the entries of  $\omega^{d+2c+1}$  in Equation 7.4, we can see that

$$\begin{aligned} \langle \omega^{d+2c+1}, \omega^{d+2c+1} \rangle - \langle \mu'_0 - \rho_n^{(1)} + w^{(1)}\rho_c, \mu'_0 - \rho_n^{(1)} + w^{(1)}\rho_c \rangle &= \\ &= \left(2c + \frac{1}{2}\right)^2 - \left(c - \frac{1}{2}\right)^2 = 3c^2 + 3c. \end{aligned}$$

This quantity is strictly positive if  $c > 0$ . Hence, by (7.3), we find

$$(7.5) \quad \langle \gamma^{X'_0}, \gamma^{X'_0} \rangle - \langle \mu'_0 - \rho_n^{(1)} + w^{(1)}\rho_c, \mu'_0 - \rho_n^{(1)} + w^{(1)}\rho_c \rangle > 0.$$

We conclude that  $X'_0$  is not unitary and there is a non-compact root  $\beta \in \Delta(\mathfrak{l}'_0)$  such that the  $\tilde{U}(d+2c+1)$ -type  $\eta'_0 = \mu'_0 - \beta$  (is dominant and) detects non-unitarity. Set

$$(7.6) \quad \begin{aligned} x &= \max \{i < t \mid r_i > 0\} \\ y &= \max \{i < t \mid s_i > 0\}. \end{aligned}$$

By Lemma 4.5, the coordinates of  $\mu$  (in Equation 4.5) satisfy the conditions

$$(7.7) \quad a_x - a_t \geq 2 \quad \text{and} \quad b_t - b_y \geq 2.$$

Then the  $\tilde{U}(n)$ -type  $\mu - \beta$  is dominant for  $\Delta_c^+$ , and the  $\tilde{U}(d+2c+1)$ -type  $\eta'_0 = \mu'_0 - \beta$  survives in the bottom layer. This implies that  $X$  is not unitary, and gives a contradiction. (If  $x$  or  $y$  do not exist, then the corresponding condition in (7.7) is empty.)

Next, we consider the case in which  $g_{t-1}$  is either  $\frac{3}{2}$  or  $\frac{5}{2}$ , and we choose  $\Delta^+(\mathfrak{l}'_0)^{(2)}$  be such that

$$\rho(\mathfrak{l}'_0)^{(2)} = (2c + d + 1, 2c + d, \dots, 2, 1)$$

and

$$\rho_n^{(2)} = \left( \underbrace{\frac{d}{2} + c + 1, \dots, \frac{d}{2} + c + 1}_{2c+d+1} \right).$$

Note that this choice of positive roots for  $Mp(2(d+2c+1))$  satisfies the conditions of Lemma 6.3 (b) of [9].

Because  $\mu'_0 - \rho_n^{(2)}$  is dominant for our fixed set of positive compact roots, we can choose  $w^{(2)} = 1$ . Then we obtain

$$\begin{aligned} \mu'_0 - \rho_n^{(2)} + \rho_c &= \left( \underbrace{\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \dots, -c + \frac{1}{2}}_{c+1}, \right. \\ &\quad \left. \underbrace{-c - \frac{3}{2}, -c - \frac{5}{2}, \dots, -c - d - \frac{1}{2}}_d, \underbrace{-c - d - \frac{5}{2}, -c - d - \frac{7}{2}, \dots, -2c - d - \frac{3}{2}}_c \right). \end{aligned}$$

Let

$$\tilde{\gamma} := \left( d + 2c + \frac{3}{2}, d + 2c + \frac{1}{2}, \dots, \frac{9}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2} \right).$$

The quantity

$$\begin{aligned} \langle \tilde{\gamma}, \tilde{\gamma} \rangle - \langle \mu'_0 - \rho_n^{(2)} + \rho_c, \mu'_0 - \rho_n^{(2)} + \rho_c \rangle &= \\ &= \left( c + \frac{1}{2} \right)^2 + \left( c + d + \frac{3}{2} \right)^2 - \left( \frac{1}{2} \right)^2 - \left( \frac{5}{2} \right)^2 \end{aligned}$$

is strictly positive if  $c > 0$ . The assumptions that  $g_{t-1} = \frac{3}{2}$  (or  $\frac{5}{2}$ ), along with Lemma 4.2, imply that the infinitesimal character  $\gamma^{X'_0}$  of  $X'_0$  does not contain an entry  $\frac{3}{2}$  (or  $\frac{5}{2}$ ). Then, by  $\omega$ -regularity, we obtain that

$$(7.8) \quad \langle \gamma^{X'_0}, \gamma^{X'_0} \rangle \geq \langle \tilde{\gamma}, \tilde{\gamma} \rangle$$

and

$$\langle \gamma^{X'_0}, \gamma^{X'_0} \rangle - \langle \mu'_0 - \rho_n^{(2)} + \rho_c, \mu'_0 - \rho_n^{(2)} + \rho_c \rangle > 0.$$

We conclude that  $X'_0$  is not unitary and there is a non-compact root  $\beta \in \Delta^+(\mathfrak{l}'_0)^{(2)}$  such that  $\eta'_0 = \mu'_0 - \beta$  (is dominant and) detects non-unitarity. Note that  $\beta$  must be of the form  $\epsilon_i + \epsilon_j$  or  $2\epsilon_j$  for some  $1 \leq i < j \leq 2c + d + 1$ . Because

$$(7.9) \quad b_t - b_y \geq 2$$

(by Lemma 4.5), the  $\tilde{U}(d + 2c + 1)$ -type  $\eta'_0 = \mu'_0 - \beta$  survives in the bottom layer, leading to a contradiction.

This concludes the proof of Claim (A).

Now we know that  $c = s_t = 0$ , and that  $X'_0$  is an irreducible genuine representation of  $Mp(2(d + 1))$  with lowest  $\tilde{U}(d + 1)$ -type

$$\mu'_0 = \left( \frac{3}{2}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_d \right).$$

Let us prove Claim (B). We apply the same argument used for the proof of Claim (A); note that, because  $c = 0$  in this case, our two choices of positive roots  $\Delta^+(\mathfrak{l}'_0)^{(i)}$  coincide (hence the conditions of Lemma 6.3 (b) of [9] are met).

If the inequality (7.3) is strict, *i. e.*, if

$$\langle \gamma^{X'_0}, \gamma^{X'_0} \rangle > \langle \omega^{d+1}, \omega^{d+1} \rangle,$$

then Equation 7.5 still holds:

$$(7.10) \quad \langle \gamma^{X'_0}, \gamma^{X'_0} \rangle - \langle \mu'_0 - \rho_n + w\rho_c, \mu'_0 - \rho_n + w\rho_c \rangle > 0.$$

Again, we conclude that  $X'_0$  is not unitary and there is a non-compact root  $\beta \in \Delta^+(\mathfrak{l}'_0)$  such that  $\eta'_0 = \mu'_0 - \beta$  (is dominant and) detects non-unitarity. The choices for  $\beta$  are  $\epsilon_1 + \epsilon_{d+1}$ ,  $\epsilon_d + \epsilon_{d+1}$ , and  $2\epsilon_{d+1}$ . In each case,  $\eta'_0 = \mu'_0 - \beta$  survives in the bottom layer because

$$(7.11) \quad c_d - b_{t-1} = \frac{1}{2} + g_{t-1} + \frac{\varepsilon_{t-1} + 1}{2} \geq 2.$$

This follows from the fact that  $g_{t-1} \geq 1$ , and from a calculation similar to the one in (4.16). (Note that if  $g_{t-1} = 1$ , then  $\varepsilon_{t-1} = 0$  by Lemma 4.2.)

Next, assume that the infinitesimal character  $\gamma^{X'_0}$  satisfies

$$\langle \gamma^{X'_0}, \gamma^{X'_0} \rangle = \langle \omega^{d+1}, \omega^{d+1} \rangle,$$

so that

$$(7.12) \quad \gamma^{X'_0} = \omega^{d+1}.$$

We want to prove that  $X'_0$  is an odd oscillator representation. It is sufficient to show that  $X'_0$  is uniquely determined by its infinitesimal character. Because the Vogan classification parameter of  $X'_0$  is given by

$$\lambda'_a = \left( \frac{1}{2}, 0, \dots, 0 \right),$$

we can rewrite the infinitesimal character of  $X'_0$  in the form

$$\gamma^{X'_0} = \left( \frac{1}{2} \middle| \nu \right),$$

where  $\nu$  is the continuous parameter on the pseudospherical principal series of  $Mp(2d)$  (cf. Lemma 5.1). Then Equation 7.12 gives

$$(7.13) \quad \nu = \left( d + \frac{1}{2}, d - \frac{1}{2}, \dots, \frac{3}{2} \right).$$

This implies that  $X'_0$  is indeed the appropriate odd oscillator representation, proving Claim (B).

Finally, we turn to the proof of Claim (C), and show that  $X'_1$  is a good  $A_{q_3}(\lambda_3)$  module. As usual, write the coordinates of  $\mu$  as in Equation 4.5; this time,  $r_t = 1$ ,  $s_t = 0$ ,  $a_t - c_1 = 1$ , and  $c_d - b_{t-1} \geq 2$ . Note that it suffices to show that  $a_{t-1} - a_t \geq 1$ . If this is the case, then our Claim (C) follows by the same kind of argument used in the first part of the proof of Proposition 3.1: if  $X'_1$  is not a good  $A_{q_3}(\lambda_3)$  module, then it is not unitary, and we can use Parthasarathy's Dirac Operator Inequality and a bottom layer argument to show that  $X$  is also non-unitary, reaching a contradiction.

We now show that  $a_{t-1} - a_t \geq 1$ . By contradiction, suppose that  $a_{t-1} = a_t$ . Then, by Lemma 4.2 and some calculations similar to those in the proof of Lemma 4.4, we must have  $g_{t-1} = 1$  and  $r_{t-1} = s_{t-1} > 0$ . Roughly, a jump of  $\frac{3}{2}$  or more in the entries of  $\lambda_a$  results in a corresponding jump of 1 or more in the coordinates of  $\mu$ . Note that, by part (iii) of Lemma 4.2, we cannot have  $g_{t-1} = \frac{3}{2}$  because the infinitesimal character of  $X_0$ , given by (7.13), contains an entry  $\frac{3}{2}$ .

As in the proof of Lemma 4.5, consider the lowest  $(\tilde{U}(r) \times \tilde{U}(s))$ -type  $\mu_1$  of the  $A_{q_2}(\lambda_2)$  module  $X_1$ . Here  $L_2$  is given in (4.18), and  $\mu_1$  has the form

$$\mu_1 = (\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p}))|_{\tilde{U}(r) \times \tilde{U}(s)}$$

with

$$(7.14) \quad 2\rho(\mathfrak{u} \cap \mathfrak{p}) = \left( \underbrace{r+d+1, \dots, r+d+1}_r \middle| \underbrace{r-s, \dots, r-s}_d \middle| \underbrace{-s-d-1, \dots, -s-d-1}_s \right).$$

The shape of  $\mu_1$  forces the last factor  $\tilde{U}(p_u, q_u)$  of  $L_2$  to satisfy  $p_u \geq 2$ . Because  $L_a \subseteq L_2 \times L_0$  (hence  $\tilde{U}(r_{t-1}, s_{t-1}) \times \tilde{U}(r_t, s_t) \subseteq L_2$ ), this implies that  $q_u \geq 1$ .

Now look at the infinitesimal character of  $X_1$ :

$$\gamma^{X_1} = \lambda_2 + \rho(\mathfrak{l}_2) + \rho(\mathfrak{u}_2).$$

Recall that  $\lambda_2 = \mu_1 - 2\rho(\mathfrak{u}_2 \cap \mathfrak{p})$ . Write the highest weight of  $\mu_1$  as in Equation (4.19). Because  $a_t = r - s + \frac{1}{2}$  and  $b_{t-1} = r - s - \frac{5}{2}$  (by Equation 7.11 with  $g_{t-1} = 1$  and  $\varepsilon_{t-1} = 0$ ), we find

$$\mu_1 = \left( \underbrace{\dots, -s - d - \frac{1}{2}, \dots, -s - d - \frac{1}{2}}_{p_u} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{r + d - \frac{3}{2}, \dots, r + d - \frac{3}{2}, \dots}_{q_u} \right)$$

and

$$(7.15) \quad 2\rho(\mathfrak{u}_2 \cap \mathfrak{p}) = \left( \underbrace{\dots, -s + q_u, \dots, -s + q_u}_{p_u} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{r - p_u, \dots, r - p_u, \dots}_{q_u} \right).$$

Hence

$$\begin{aligned} \lambda_2 &= \mu_1 - 2\rho(\mathfrak{u}_2 \cap \mathfrak{p}) \\ &= \left( \underbrace{\dots, -d - q_u - \frac{1}{2}, \dots, -d - q_u - \frac{1}{2}}_{p_u} \middle| \underbrace{0, \dots, 0}_d \middle| \underbrace{d + p_u - \frac{3}{2}, \dots, d + p_u - \frac{3}{2}, \dots}_{q_u} \right). \end{aligned}$$

Because  $\lambda_2$  must be constant on  $\tilde{U}(p_u, q_u)$  (with twist because of the embedding into  $\mathfrak{g}$ ), we can conclude that  $p_u = q_u + 2$ . Recall that here  $q_u > 0$ .

Write

$$\begin{aligned} \rho(\mathfrak{u}_2) &= \left( \underbrace{\dots, \frac{-r - s + p_u + q_u}{2}, \dots, \frac{-r - s + p_u + q_u}{2}}_{p_u} \middle| \underbrace{0, \dots, 0}_d \middle| \right. \\ &\quad \left. \underbrace{\frac{r + s - p_u - q_u}{2}, \dots, \frac{r + s - p_u - q_u}{2}}_{q_u}, \dots \right), \end{aligned}$$

and choose

$$(7.16) \quad \rho(\mathfrak{l}_2) = \left( \underbrace{\dots, \frac{p_u + q_u - 1}{2}, \frac{p_u + q_u - 3}{2}, \dots, \frac{q_u - p_u + 1}{2}}_{p_u} \middle| \underbrace{0, \dots, 0}_d \middle| \right. \\ \left. \underbrace{\frac{p_u + q_u - 1}{2}, \frac{p_u + q_u - 3}{2}, \dots, \frac{p_u - q_u + 1}{2}}_{q_u}, \dots \right).$$

Then

$$\begin{aligned} \gamma^{X_1} &= \lambda_2 + \rho(\mathbf{u}_2) + \rho(\mathfrak{l}_2) = \\ &= \left( \underbrace{\dots, -d - \frac{r+s}{2} + p_u - 1, -d - \frac{r+s}{2} + p_u - 2, \dots, -d - \frac{r+s}{2}}_{p_u} \middle| \right. \\ &\quad \left. \underbrace{0, \dots, 0}_d \middle| \underbrace{d + \frac{r+s}{2} + p_u - 2, d + \frac{r+s}{2} + p_u - 3, \dots, d + \frac{r+s}{2} + p_u - q_u - 1, \dots}_{q_u} \right). \end{aligned}$$

Finally, consider the infinitesimal character of  $X$ . By Proposition 2.4 (vi), we have

$$\gamma^X = \gamma^{X^L} + \rho(u) = \gamma^{X_1} + \gamma^{X_0} + \rho(u).$$

Here the infinitesimal characters of  $X_1$  and  $X_0$  are known, and  $\rho(u)$  is given by

$$\begin{aligned} \rho(u) &= \left( \underbrace{d + \frac{r+s+1}{2}, \dots, d + \frac{r+s+1}{2}}_r \middle| \underbrace{0, \dots, 0}_d \right. \\ &\quad \left. \underbrace{-d - \frac{r+s+1}{2}, \dots, -d - \frac{r+s+1}{2}}_s \right). \end{aligned}$$

We obtain

$$\begin{aligned} \gamma^X = \gamma^{X_1} + \gamma^{X_0} + \rho(u) &= \left( \dots, \underbrace{p_u - \frac{1}{2}, p_u - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}}_{p_u} \middle| \right. \\ &\quad \left. d + \frac{1}{2}, d - \frac{1}{2}, \dots, \frac{5}{2}, \frac{3}{2} \middle| \underbrace{p_u - \frac{5}{2}, p_u - \frac{7}{2}, \dots, p_u - q_u - \frac{3}{2}, \dots}_{q_u} \right). \end{aligned}$$

This contradicts our assumption that  $\gamma^X$  is nonsingular, and concludes the proof of Proposition 3.2.  $\square$

## 8. PROOF OF PROPOSITION 3.6

Finally, we give the proof of Proposition 3.6. For convenience, we restate the result.

**Proposition.** *In the setting of Proposition 3.4, let  $X$  be the irreducible (lowest  $K$ -type constituent of the) representation  $\mathcal{R}_{\mathfrak{q}_\omega}(\mathbb{C}_{\lambda'} \otimes \omega)$ . Assume, moreover, that  $\lambda'$  is in the good range for  $\mathfrak{q}'$ . Then  $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$  is in the good range for  $\mathfrak{q}_\omega$ .*

*Proof.* Recall from Equation 3.2 that  $\mathfrak{q}_\omega$  has Levi component  $\mathfrak{l}_\omega = \mathfrak{l}' + \mathfrak{l}_0$  and nilpotent part  $\mathfrak{u}_\omega = \mathfrak{u}' + \mathfrak{u}$ . Let  $\gamma^\Omega$  be (a representative of) the infinitesimal character of  $\Omega = \mathbb{C}_{\lambda'} \otimes \omega$ . We need to show that  $\Omega$  is in the good range for  $\mathfrak{q}_\omega$ , *i. e.*, that

$$\langle \gamma^\Omega + \rho(\mathfrak{u}_\omega), \alpha \rangle > 0$$

for all  $\alpha$  in  $\Delta(\mathfrak{u}_\omega)$ . Note that

$$\rho(\mathfrak{u}_\omega) = \rho(\mathfrak{u}') + \rho(\mathfrak{u}).$$

Let  $d_0 (= d \text{ or } d + 1)$  be the rank of  $L_0$ , so that the infinitesimal character of the oscillator representation of  $L_0$  is  $\omega^{d_0}$ . Then  $\gamma^\Omega = \lambda' + \rho(\mathfrak{l}') + \omega^{d_0}$ , and

$$\gamma^\Omega + \rho(\mathfrak{u}_\omega) = \lambda' + \rho(\mathfrak{l}') + \omega^{d_0} + \rho(\mathfrak{u}') + \rho(\mathfrak{u}).$$

If  $\alpha \in \Delta(\mathfrak{u}')$ , then  $\alpha$  is orthogonal to  $\omega^{d_0}$  and  $\rho(\mathfrak{u})$ , and

$$\langle \gamma^\Omega + \rho(\mathfrak{u}_\omega), \alpha \rangle = \langle \lambda' + \rho(\mathfrak{l}') + \rho(\mathfrak{u}'), \alpha \rangle > 0$$

because  $\lambda'$  is assumed to be in the good range for  $\mathfrak{q}'$ .

It remains to consider the case  $\alpha \in \Delta(\mathfrak{u})$ . In order to keep the notation simple and familiar, we will compute  $\gamma^\Omega + \rho(\mathfrak{u}_\omega)$  in our usual coordinates, for the case that  $X_0$  is an even oscillator representation. If  $X_0$  is an odd oscillator representation, the result will follow in exactly the same way.

Let  $\lambda_a, \mu$  and  $L = L_1 \times L_0$  be as in the proof of Proposition 3.1, and write  $L' = \prod_{i=1}^u \tilde{U}(p_i, q_i)$  as in (4.18) (it was called  $L_2$  there). In these coordinates, the roots of  $\mathfrak{u}$  are those that are positive on

$$\xi = \left( \underbrace{a, a, \dots, a}_r \mid \underbrace{0, 0, \dots, 0}_d \mid \underbrace{-a, -a, \dots, -a}_s \right)$$

for  $a > 0$ . The roots of  $\mathfrak{u}_\omega$  are those that are positive on

$$\xi_\omega = \left( \underbrace{a_1, \dots, a_1}_{p_1}, \dots, \underbrace{a_u, \dots, a_u}_{p_u} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{-a_u, \dots, -a_u}_{q_u}, \dots, \underbrace{-a_1, \dots, -a_1}_{q_u} \right),$$

for  $a_1 > a_2 > \dots > a_u > 0$ . From [9], we know that the parameter  $\lambda'$  is of the form

$$\lambda' = \left( \underbrace{\lambda_1, \dots, \lambda_1}_{p_1}, \dots, \underbrace{\lambda_u, \dots, \lambda_u}_{p_u} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{-\lambda_u, \dots, -\lambda_u}_{q_u}, \dots, \underbrace{-\lambda_1, \dots, -\lambda_1}_{q_u} \right)$$

with  $\lambda_i \in \mathbb{Z} + \frac{1}{2}$ , and  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_u$ . Note that, if we choose a system of positive roots  $\Delta_\omega^+$  for  $\mathfrak{l}_\omega = \mathfrak{l}' + \mathfrak{l}_0$ , then  $\Delta^+ = \Delta_\omega^+ \cup \Delta(\mathfrak{u}') \cup \Delta(\mathfrak{u})$  is a system of positive roots for  $\mathfrak{g}$ . Let  $\Pi \subset \Delta^+$  be the corresponding set of simple roots. If  $\omega^d$  is the representative of the infinitesimal character of the oscillator representation determined by  $\Delta_\omega^+$ , then

$$\gamma^\Omega + \rho(\mathfrak{u}_\omega) = \lambda' + \rho(\mathfrak{l}') + \omega^{d_0} + \rho(\mathfrak{u}') + \rho(\mathfrak{u})$$

is automatically dominant for all roots in  $\Delta_\omega^+ \cup \Delta(\mathfrak{u}')$ . Hence, we only need to show dominance for the *simple* roots  $\Pi \cap \Delta(\mathfrak{u})$  in  $\Delta(\mathfrak{u})$ ; this set will turn out to be a single root.

Without loss of generality, we may assume that  $q_u > 0$ . Set

$$\omega^d = (0, \dots, 0 \mid d - 1/2, d - 3/2, \dots, 1/2 \mid 0, \dots, 0)$$

and choose the restriction of  $\rho(l')$  on each factor in a standard way; the choice is given explicitly (for the last factor  $\tilde{U}(p_u, q_u)$  of  $L'$  only) in (7.16). Then

$$\Pi \cap \Delta(\mathbf{u}) = \{-\epsilon_{r+1} - \epsilon_{r+d+1}\}.$$

If we write

$$\gamma^\Omega + \rho(\mathbf{u}_\omega) = (\gamma_1, \gamma_2, \dots, \gamma_r | d - 1/2, d - 3/2, \dots, 1/2 | \gamma_{r+d+1}, \dots, \gamma_{r+d+s}),$$

then the dominance condition reduces to

$$\gamma_{r+d+1} + d - 1/2 < 0.$$

Note that, given the  $\omega$ -regularity condition and the shape of  $\omega^d$ , it suffices to prove that

$$(8.1) \quad \gamma_{r+d+1} \leq d - \frac{1}{2}.$$

We consider the entries of  $\gamma^\Omega + \rho(\mathbf{u}_\omega)$  corresponding to the last factor  $\tilde{U}(p_u, q_u)$  of  $L'$ . These entries were essentially computed in the proof of (Claim (C) in) Proposition 3.2, with the same choice of positive roots, but with different assumptions on the entries of  $\mu$  and a different notation for the parabolic subalgebra of  $L_1$  (prime algebras were replaced by algebras with subscript 2).

By Equation 4.2 (in Lemma 4.2), the  $r^{\text{th}}$  coordinate of  $\mu$  is the same as the  $r^{\text{th}}$  coordinate of  $\lambda_a + \rho(\mathbf{u}_a \cap \mathfrak{p}) - \rho(\mathbf{u}_a \cap \mathfrak{k})$ , which, in turn, is of the form

$$r - s + k = r - s + \left[ g_j + \sum_{i=j+1}^t (s_i - r_i) + \frac{1}{2}(s_j - r_j + 1) \right]$$

for some  $1 \leq j \leq t$  (by Equations 2.6 and 4.4). Note that  $k \in \mathbb{Z} + \frac{1}{2}$  and  $k \geq \frac{1}{2}$ , because  $g_j \geq \frac{1}{2}$ ,  $r_i = 0$  (hence  $s_i = 1$ ) for all  $i > j$ , and  $|s_j - r_j| \leq 1$ . Similarly, the  $(r + d + 1)^{\text{th}}$  coordinate of  $\mu$  can be written in the form

$$r - s - l,$$

for some  $l \in \mathbb{Z} + \frac{1}{2}$  and  $l \geq \frac{1}{2}$ . Hence, we can write

$$\mu = \left( \dots, \underbrace{r - s + k, \dots, r - s + k}_{p_u} \mid \underbrace{c_1, \dots, c_d}_d \mid \underbrace{r - s - l, \dots, r - s - l}_{q_u}, \dots \right)$$

with  $k, l \in \mathbb{Z} + \frac{1}{2}$  with  $k, l \geq \frac{1}{2}$ . (This formula holds true also for  $d = 0$ .)

Then, using the equation  $\mu = \mu_1 + \mu_0 + 2\rho(\mathbf{u} \cap \mathfrak{p})$  and the expression for  $2\rho(\mathbf{u} \cap \mathfrak{p})$  given by (7.14), we find that

$$\mu_1 = \left( \dots, \underbrace{-s - d - 1 + k, \dots, -s - d - 1 + k}_{p_u} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{r + d + 1 - l, \dots, r + d + 1 - l}_{q_u}, \dots \right).$$

Using (7.15), get

$$\lambda' = \mu_1 - 2\rho(\mathbf{u}' \cap \mathfrak{p})$$

$$= \left( \dots, \underbrace{-q_u - d - 1 + k, \dots, -q_u - d - 1 + k}_{p_u} \mid \underbrace{0, \dots, 0}_d \mid \underbrace{p_u + d + 1 - l, \dots, p_u + d + 1 - l}_{q_u}, \dots \right),$$



hence

$$\begin{aligned} \gamma^\Omega + \rho(\mathbf{u}_\omega) &= \lambda' + \rho(\mathbf{u}') + \rho(\mathbf{l}') + \rho(\mathbf{u}) \\ &= \left( \dots, \underbrace{p_u - 1 + k, p_u - 2 + k, \dots, k + 1, k}_{p_u} \left| d - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \right| \right. \\ &\quad \left. \underbrace{p_u - l, p_u - l - 1, \dots, p_u - q_u - l + 1, \dots}_{q_u} \right). \end{aligned}$$

If  $p_u = 0$ , then  $\gamma_{r+d+1} = -l < 0$ , which implies (8.1), so we are done. Therefore, we may assume  $p_u$  and  $q_u$  are both nonzero. Because  $\lambda'$  is constant on  $\tilde{U}(p_u, q_u)$ , we must have  $p_u + d + 1 - l = q_u + d + 1 - k$ , so that

$$l = k + p_u - q_u.$$

Then

$$\gamma^\Omega + \rho(\mathbf{u}_\omega) = \left( \dots, \underbrace{p_u - 1 + k, p_u - 2 + k, \dots, k + 1, k}_{p_u} \left| d - \frac{1}{2}, \dots, \frac{1}{2} \right| \underbrace{q_u - k, q_u - k - 1, \dots, -k + 1, \dots}_{q_u} \right).$$

By  $\omega$ -regularity and the fact that  $k > 0$ , we must have  $k \geq d + \frac{1}{2}$ , so the last entry in this  $\tilde{U}(p_u, q_u)$  factor is

$$-k + 1 \leq -d + \frac{1}{2}.$$

Because the entries  $q_u - k, q_u - k - 1, \dots, -k + 1$  form a sequence of half integers decreasing by steps of 1,  $\omega$ -regularity implies that

$$\gamma_{r+s+1} = q_u - k \leq -d - \frac{1}{2},$$

and we are done.

Notice that this argument also applies in the case  $d = 0$ . This concludes the proof of Proposition 3.6.  $\square$

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