Equivalence of atlas parameters

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The point of this note is to clarify the notion of "parameter", and make precise the relationship between some mathematical and some computational constructs.

1 Parameters

So, we're given a real group, and the output of `kgb`. By a *parameter* we mean a triple (x, λ, ν) where:

(1) `x` is a `kgb` element, specified by an integer `0,1,..., n`.

(2) $\lambda \in (\rho + X^*)/(1 - \theta_x)X^*$;

(3) $\nu \in (X^* \otimes \mathbb{Q})^{-\theta_x}$

Of course θ_x is the Cartan involution (of the Cartan subgroup) defined by x.

Recall the infinitesimal character is

$$\begin{aligned} \gamma &= \frac{1}{2}(1+\theta_x)\lambda + \nu \\ &= \frac{1}{2}(1+\theta_x)\lambda + \frac{1}{2}(1-\theta_x)\nu, \end{aligned} \tag{1.1}$$

and the integral roots are defined with respect to γ. We're allowing singular and non-integral infinitesimal character. We always work with our fixed set of positive roots.

We have the following conditions.

(1) Standard: $\langle \lambda, \alpha^\vee \rangle \geq 0 \quad (\alpha > 0 \text{ imaginary})$

(2) Final: $\langle\nu,\alpha^\vee\rangle=0\Rightarrow\langle\lambda+\rho_r,\alpha^\vee\rangle$ is even (α real)

(3) Nonzero: $\langle\lambda,\alpha^\vee\rangle=0\Rightarrow\alpha$ is noncompact (α imaginary-simple)

Here imaginary-simple means simple in the set of imaginary roots, and ρ_r is one-half the sum of the positive θ_x-real roots. If ν is weakly dominant we can write (2) as

(2′) Final: ν weakly dominant, $\langle\nu,\alpha^\vee\rangle=0\Rightarrow\langle\lambda,\alpha^\vee\rangle$ is odd (α real-simple)

We also consider conditions

(4) $\langle\gamma,\alpha^\vee\rangle\geq 0$ for all positive, integral roots.

(5) $\langle\gamma,\alpha^\vee\rangle=0$, $\alpha>0$ complex $\Rightarrow\alpha$ is not a complex descent

Recall $\alpha>0$ complex is not a descent if $\theta_x(\alpha)>0$, equivalently α is of type `C+`.

A standard parameter defines a standard module $I(x,\lambda,\nu)$, which is nonzero if and only if the nonzero condition holds. The final condition says we can't move to a more compact Cartan. If (x,λ,ν) is standard, final and nonzero then $I(x,\lambda,\nu)$ has a unique irreducible quotient $J(x,\lambda,\nu)$.

The map $(x,\lambda,\nu)\to J(x,\lambda,\nu)$, from standard, final, nonzero data to (equivalence classes of) irreducible representations is surjective.

Definition 1.2 *We say standard, final, nonzero limit data* (x,λ,ν),(x',λ',ν') *are equivalent if* $J(x,\lambda,\nu)\simeq J(x',\lambda',\nu')$.

Given x, write W_r for the Weyl group of the θ_x-real roots, and ρ_r for one-half the sum of the positive θ_x-real roots.

Lemma 1.3 *Equivalence of parameters is generated by:*
(1) $(x,\lambda,\nu)\equiv(s_\alpha\times x,s_\alpha\lambda,s_\alpha\nu)$ *for* α *simple and* θ_x*-complex;*
(2) $(x,\lambda,\nu)\equiv(x,w(\lambda+\rho_r)-\rho_r,w\nu)$ *for* $w\in W_r$.

See [2], Lemma 1.2 and Theorem 1.3. The only subtle point is the action (1). Although this doesn't appear explicitly in [2], it is implicit, in the conjugation of parameters.

Algorithm:
Given $(x,\lambda,\nu),(x',\lambda',\nu')$, standard, final, nonzero.

(1) If $\mathrm{Cartan}(x) \neq \mathrm{Cartan}(x')$ then these are not equivalent.

(2) Assume $\mathrm{Cartan}(x) = \mathrm{Cartan}(x')$. After applying complex cross actions, we may assume x and x' are in the same fiber (for example the canonical fiber). If $x \neq x'$ these are not equivalent.

(3) Assume $x = x'$. Let R_C be the complex root system defined by x [3, Proposition 3.12], [1, Section 12]. Using $W((R_C)^{\theta_x})$ we may assume (λ, ν) and (λ', ν') are weakly dominant for $(R_C)^{\theta_x}$. (Note: $w \times x = x$ for $w \in W((R_C)^{\theta_x})$.

(4) Using $w \in W_r$ and (2) of the Lemma, we may assume ν, ν' are weakly dominant for the θ_x-real roots.

(5) Given (3) and (4), $(x, \lambda, \nu) \equiv (x, \lambda', \nu')$ if and only if only if $\nu = \nu'$ and $\lambda = \lambda' \mod (1 - \theta_x)X^*$.

There are other variants of the algorithm.

2 The `nblock` command

The `nblock` command takes as input a parameter (x, λ, ν) as described at the beginning of this note. It then constructs a block of representations at this infinitesimal character γ (cf. (1.1)), as the image via translation of a block at regular infinitesimal character, say γ_{reg}.

The output of `nblock` consists of lines numbered `0,1,..., n`. Associated to each line `k` are standard and irreducible modules $I(k)_{reg}, J(k)_{reg}$ at infinitesimal character γ_{reg}. Let $P^* = \{\mathtt{k_1}, \dots, \mathtt{k_m}\}$ be the lines marked with an asterisk `*`. For $k \in P^*$ write $I(k), J(k)$ for the standard and irreducible modules at γ.

Write Ψ for the translation functor from γ_{reg} to γ. This takes irreducibles to irreducibles or 0.

Lemma 2.1 *Consider the output of* `nblock` *for a parameter* (x, λ, ν)*.*

(1) $\{J(k) \mid k \in P^*\}$ *is the set of irreducible representations in the block at infinitesimal character* γ*.*

(2) $\{I(k) \mid k \in P^*\}$ *is the set of standard modules in the block at infinitesimal character* γ*.*

(3) $\Psi(J(k)_{reg}) \neq 0$ *if and only if* $k \in P^*$*, in which case* $\Psi(J(k)_{reg}) = J(k)$*.*

(4) Ψ *is a bijection between* $\{J(k)_{reg} \mid k \in P^*\}$ *and* $\{J(k) \mid k \in P^*\}$*.*

(5) If $k \in P^*$ *then* $\Psi(I(k)_{reg}) = I(k)$*.*

(6) Ψ *is a bijection between* $\{I(k)_{reg} \mid k \in P^*\}$ *and* $\{I(k) \mid k \in P^*\}$

Remark 2.2 If $k \notin P^*$ then, although $\Psi(J(k)_{reg}) = 0$, $\Psi(I(k)_{reg})$ may be nonzero, and may be a sum of terms $I(k')$ at γ. This accounts for the "cumulated" KLV polynomials.

The Software:

The `nblock` command takes input (x, λ, ν) and does the following.

(1) It tests if the parameter satisfies the Nonzero condition above. If the module is 0 it reports this, and the offending root, and stops.

(2) It tests if the parameter satisfies the Final condition above. If not, it reports this, and the offending root, and stops.

(3) It applies cross actions by simple complex roots of type `C-`, singular on γ, until every complex simple root is of type `C+`. It then identifies the parameter with a starred line in the output of `nblock`.

(4) It then computes the KLV polynomials for this parameter.

Currently (3/8/12) the software does not do steps 1) and 2), and might give the zero module. With this change, the software will then correctly be computing standard and irreducible modules at singular infinitesimal character.

The preceding description, especially the cross action by complex roots, is explained by the following Lemma.

Lemma 2.3 *Suppose* `k` *is a parameter at regular (integral) infinitesimal character, and* α *is a simple root. Write* Ψ *for translation to the* α*-wall.*
(1) α *type* `ic` $\Leftrightarrow \Psi(I(k)_{reg}) = 0$*.*
(1) α *type* `ic,C-,r1,r2` $\Leftrightarrow \Psi(J(k)_{reg}) = 0$
(2) Suppose α *is type* `C+`*, and the cross action* s_α *take line* `k` *to line* `m`*. Then* α *is type* `C-` *for* `m`*, and* $\Psi(I(k)_{reg}) = \Psi(I(m)_{reg})$, $\Psi(J(k)_{reg}) \neq 0, \Psi(J(m)_{reg}) = 0$*.*
(3) α *is type* `r1` *or* `r2` $\Rightarrow \Psi(I(k))$ *is not final, and therefore a sum of standard modules on more compact Cartans.*

3 Examples

Example 1

```
real: nblock
choose Cartan class (one of 0,1,2,3): 2
Choose a KGB element from Cartan 2, whose canonical fiber is:
 7:  2  [C,n]    5   8     *  10  (0,0)#2 1x2^e
 8:  2  [C,n]    6   7     *  10  (0,1)#2 1x2^e
KGB number: 7
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[0,1] (>=-1)
Give lambda-rho: -1 0
denominator for nu: 1
numerator for nu: 1 0
Name an output file (return for stdout, ? to abandon):
x = 5, gamma = [0,1]/1, lambda = [0,1]/1
Subsystem on dual side is of type B2, with roots 4,5.
Given parameters define element 4 of the following block:
 0( 0,6):  0  [i1,i1]   1   2   ( 6, *)  ( 4, *)  *(x= 0, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 1( 1,6):  0  [i1,i1]   0   3   ( 6, *)  ( 5, *)  *(x= 1, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 2( 2,6):  0  [ic,i1]   2   0   ( *, *)  ( 4, *)   (x= 2, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 3( 3,6):  0  [ic,i1]   3   1   ( *, *)  ( 5, *)   (x= 3, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 4( 4,5):  1  [C+,r1]   7   4   ( *, *)  ( 0, 2)  *(x= 5, nu= [-1,1]/1;  [0,0]/1,lam=rho+ [-1,0])  1,2,1
 5( 5,5):  1  [C+,r1]   8   5   ( *, *)  ( 1, 3)  *(x= 6, nu= [-1,1]/1;  [0,0]/1,lam=rho+ [-1,0])  1,2,1
 6( 6,4):  1  [r1,C+]   6   9   ( 0, 1)  ( *, *)   (x= 4, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2
 7( 7,3):  2  [C-,i1]   4   8   ( *, *)  (10, *)   (x= 7, nu=  [1,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2
 8( 8,3):  2  [C-,i1]   5   7   ( *, *)  (10, *)   (x= 8, nu=  [1,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2
 9( 9,2):  2  [i2,C-]   9   6   (10,11)  ( *, *)  *(x= 9, nu=  [0,1]/1;  [0,0]/1,lam=rho+ [-1,0])  1
10(10,0):  3  [r2,r1]  11  10   ( 9, *)  ( 7, 8)   (x=10, nu=  [0,1]/1;  [0,0]/1,lam=rho+ [-1,0])  e
11(10,1):  3  [r2,rn]  10  11   ( 9, *)  ( *, *)   (x=10, nu=  [0,1]/1; [0,-1]/2,lam=rho+ [-1,1])  e
(cumulated) KL polynomials (-1)^{l(4)-l(x)}*P_{x,4}:
0: -1
4: 1
```

Here we entered the parameter exactly as in line 7, but this has a complex root of type `C-`. This means at infinitesimal character $\gamma = (1,1)$ (usual coordinates) the root $e_1 - e_2$ is complex and singular, and parameters 4 and 7 are equivalent by Definition 1.2.

Example 2 We get the same thing if we enter 4 directly:

```
real: nblock
choose Cartan class (one of 0,1,2,3): 2
Choose a KGB element from Cartan 2, whose canonical fiber is:
 7:  2  [C,n]    5   8     *  10  (0,0)#2 1x2^e
 8:  2  [C,n]    6   7     *  10  (0,1)#2 1x2^e
KGB number: 5
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[0,1] (>=-1)
Give lambda-rho: -1 0
denominator for nu: 1
```

```
numerator for nu: -1 1
Name an output file (return for stdout, ? to abandon):
x = 5, gamma = [0,1]/1, lambda = [0,1]/1
Subsystem on dual side is of type B2, with roots 4,5.
Given parameters define element 4 of the following block:
 0( 0,6):  0  [i1,i1]   1   2   ( 6, *)  ( 4, *)  *(x= 0, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 1( 1,6):  0  [i1,i1]   0   3   ( 6, *)  ( 5, *)  *(x= 1, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 2( 2,6):  0  [ic,i1]   2   0   ( *, *)  ( 4, *)   (x= 2, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 3( 3,6):  0  [ic,i1]   3   1   ( *, *)  ( 5, *)   (x= 3, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 4( 4,5):  1  [C+,r1]   7   4   ( *, *)  ( 0, 2)  *(x= 5, nu= [-1,1]/1;  [0,0]/1,lam=rho+ [-1,0])  1,2,1
 5( 5,5):  1  [C+,r1]   8   5   ( *, *)  ( 1, 3)  *(x= 6, nu= [-1,1]/1;  [0,0]/1,lam=rho+ [-1,0])  1,2,1
 6( 6,4):  1  [r1,C+]   6   9   ( 0, 1)  ( *, *)   (x= 4, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2
 7( 7,3):  2  [C-,i1]   4   8   ( *, *)  (10, *)   (x= 7, nu=  [1,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2
 8( 8,3):  2  [C-,i1]   5   7   ( *, *)  (10, *)   (x= 8, nu=  [1,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2
 9( 9,2):  2  [i2,C-]   9   6   (10,11)  ( *, *)  *(x= 9, nu=  [0,1]/1;  [0,0]/1,lam=rho+ [-1,0])  1
10(10,0):  3  [r2,r1]  11  10   ( 9, *)  ( 7, 8)   (x=10, nu=  [0,1]/1;  [0,0]/1,lam=rho+ [-1,0])  e
11(10,1):  3  [r2,rn]  10  11   ( 9, *)  ( *, *)   (x=10, nu=  [0,1]/1; [0,-1]/2,lam=rho+ [-1,1])  e
(cumulated) KL polynomials (-1)^{l(4)-l(x)}*P_{x,4}:
0: -1
4: 1
```

Example 3

```
real: nblock
choose Cartan class (one of 0,1,2,3): 1
Choose a KGB element from Cartan 1, whose canonical fiber is:
 9:  2  [n,C]    9   4    10   *  (0,0)#1 2x1^e
KGB number: 4
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[1,0] (>=-1)
Give lambda-rho: -1 0
denominator for nu: 1
numerator for nu: 0 0
Parameter is not final, as witnessed by coroot [1,2].
```

This says the first coroot is of type `r1`, so the parameter is not final.

I'm kidding, the software doesn't do this yet, but will soon. Here is what it does now.

```
real: nblock
choose Cartan class (one of 0,1,2,3): 1
Choose a KGB element from Cartan 1, whose canonical fiber is:
 9:  2  [n,C]    9   4    10   *  (0,0)#1 2x1^e
KGB number: 4
rho = [1,1]/1
NEED, on following imaginary coroot, at least given value:
[1,0] (>=-1)
Give lambda-rho: -1 0
denominator for nu: 1
numerator for nu: 0 0
Name an output file (return for stdout, ? to abandon):
x = 4, gamma = [0,1]/1, lambda = [0,1]/1
```

```
Subsystem on dual side is of type B2, with roots 4,5.
Given parameters define element 6 of the following block:
 0( 0,6):  0  [i1,i1]   1   2   ( 6, *)  ( 4, *)  *(x= 0, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 1( 1,6):  0  [i1,i1]   0   3   ( 6, *)  ( 5, *)  *(x= 1, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 2( 2,6):  0  [ic,i1]   2   0   ( *, *)  ( 4, *)   (x= 2, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 3( 3,6):  0  [ic,i1]   3   1   ( *, *)  ( 5, *)   (x= 3, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2,1
 4( 4,5):  1  [C+,r1]   7   4   ( *, *)  ( 0, 2)  *(x= 5, nu= [-1,1]/1;  [0,0]/1,lam=rho+ [-1,0])  1,2,1
 5( 5,5):  1  [C+,r1]   8   5   ( *, *)  ( 1, 3)  *(x= 6, nu= [-1,1]/1;  [0,0]/1,lam=rho+ [-1,0])  1,2,1
 6( 6,4):  1  [r1,C+]   6   9   ( 0, 1)  ( *, *)   (x= 4, nu=  [0,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2,1,2
 7( 7,3):  2  [C-,i1]   4   8   ( *, *)  (10, *)   (x= 7, nu=  [1,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2
 8( 8,3):  2  [C-,i1]   5   7   ( *, *)  (10, *)   (x= 8, nu=  [1,0]/1;  [0,0]/1,lam=rho+ [-1,0])  2
 9( 9,2):  2  [i2,C-]   9   6   (10,11)  ( *, *)  *(x= 9, nu=  [0,1]/1;  [0,0]/1,lam=rho+ [-1,0])  1
10(10,0):  3  [r2,r1]  11  10   ( 9, *)  ( 7, 8)   (x=10, nu=  [0,1]/1;  [0,0]/1,lam=rho+ [-1,0])  e
11(10,1):  3  [r2,rn]  10  11   ( 9, *)  ( *, *)   (x=10, nu=  [0,1]/1; [0,-1]/2,lam=rho+ [-1,1])  e
(cumulated) KL polynomials (-1)^{l(6)-l(x)}*P_{x,6}:
real:
```

References

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[3] David A. Vogan, Jr. Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality. *Duke Math. J.*, 49(4):943–1073, 1982.