Disconnected reductive groups

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1 Introduction

{sec:intro}

These notes are concerned with the general problem of understanding and classifying disconnected reductive algebraic groups. The issue of passage to covering groups is closely related, so we'll talk about that as well. I'll concentrate first on complex groups, but ultimately we're interested in real groups. {se:rootdata}

We begin with a complex connected reductive algebraic group equipped with a pinning (see for example [1, Definition 1.9]). The reason we use [1] as a reference rather than something older and closer to original sources is that this reference is concerned, as we will be, with representation theory of disconnected groups. Write {AV}

$$\begin{gathered} G \supset B \supset H \\ R(G,H) \subset X^*(H), \qquad R^\vee(G,H) \subset X_*(G,H) \\ R^+(G,H) \supset \Pi(B,H), \qquad \Pi^\vee(B,H) \subset (R^\vee)^+(G,H) \\ \{X_\alpha \mid \alpha \in \Pi(B,H)\} \end{gathered} \tag{1.1a}$$

for the fixed Borel subgroup and maximal torus, roots and coroots, positive and simple roots and coroots defined by B, and the basis vectors for simple root spaces that constitute the pinning. all these choices are determined by H and the X_α. The *root datum of* G is the quadruple

$$\mathcal{R}(G) = (X^*(H), R(G,H), X_*(H), R^\vee(G,H)) \tag{1.1b}$$ {eq:rootdata2}

and the *based root datum* is

$$\mathcal{B}(G) = (X^*(H), \Pi(B,H), X_*(H), \Pi^\vee(B,H)) \tag{1.1c}$$ {eq:basedrootdata}

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Each $\alpha \in R(G,H)$ defines a simple reflection

$$s_\alpha \in \mathrm{Aut}(X^*(H)), \quad s_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha \qquad (\lambda \in X^*(H)). \tag{1.1d}$$ {eq:salpha}

The transpose automorphism of X_* is

$$s_{\alpha^\vee} \in \mathrm{Aut}(X_*(H)), \quad s_{\alpha^\vee}(\ell) = \ell - \langle \ell, \alpha \rangle \alpha^\vee \qquad (\ell \in X_*(H)). \tag{1.1e}$$ {eq:salphavee}

The *Weyl group of H in G* is

$$\text{group generated by } \{s_\alpha \mid \alpha \in R(G,H)\} \subset \mathrm{Aut}(X_*(H)). \tag{1.1f}$$ {eq:W}

The inverse transpose isomorphism

$$\mathrm{Aut}(X^*(H)) \simeq \mathrm{Aut}(X_*(H)), \qquad T \mapsto {}^tT^{-1}$$

identifies $W(G,H)$ with the group of automorphisms of $X_*(H)$ generated by the various $s_{\alpha^\vee}$. The set of *Coxeter generators* or *simple reflections* for W is

$$S = \{s_\alpha \mid \alpha \in \Pi(B,H)\} \simeq \{s_{\alpha^\vee} \mid \alpha^\vee \in \Pi^\vee(B,H)\}. \tag{1.1g}$$ {eq:S}

{prop:pinned}

Proposition 1.2. *Suppose we are in the setting* (1.1). *If $\{H, X_\alpha\}$ and $\{H', X'_{\alpha'}\}$ are two pinnings for G, then there is a unique coset $gZ(G)$ so that*

$$\mathrm{Ad}(g)(H) = H', \qquad \mathrm{Ad}(g)(X_\alpha) = X'_{\alpha'} \qquad (\alpha \in \Pi).$$

The bijection of simple roots $\Pi \leftrightarrow \Pi'$ is uniquely fixed by the existence of g.

An automorphism of G preserving the pinning is precisely the same thing as an automorphism of the based root datum (1.1c)*:*

$$\mathrm{Aut}(\mathcal{B}(G)) = \left\{ T \in \mathrm{Aut}(X_*) \mid \begin{array}{c} T(\Pi(B,H)) = \Pi(B,H) \\ {}^tT(\Pi^\vee(B,H)) = \Pi^\vee(B,H) \end{array} \right\}.$$

{se:disc1}

Our goal is to understand *disconnected* complex reductive algebraic groups. If E is such a group, and $E_0 = G$ its identity component, then

$$\begin{gathered} \Gamma =_{\mathrm{def}} E/E_0 = E/G \\ \{1\} \longrightarrow G \longrightarrow E \xrightarrow{p_E} \Gamma \longrightarrow \{1\} \end{gathered} \tag{1.3a}$$ {eq:disc1}

with Γ a finite group. We will begin with G and Γ, and seek to understand how to describe the possibilities for E. Write

$$\begin{aligned} \mathrm{Aut}(G) &= \{\text{algebraic automorphisms of } G\} \\ \mathrm{Aut}(G, \{H, X_\alpha\}) &= \{\tau \in \mathrm{Aut}(G) \mid \tau(\{X_\alpha\}) = \{H, X_\alpha\} \\ &= \{\textit{distinguished} \text{ automorphisms of } G\} \\ \mathrm{Int}(G) &= \{\mathrm{Ad}(g) \mid g \in G\} = \{\text{inner automorphisms of } G\} \\ \mathrm{Out}(G) &= \mathrm{Aut}(G)/\mathrm{Int}(G). \end{aligned} \tag{1.3b}$$ {eq:intout0}

As for any group, there are natural short exact sequences

$$\begin{gathered} \{1\} \longrightarrow \mathrm{Int}(G) \longrightarrow \mathrm{Aut}(G) \xrightarrow{p_{\mathrm{Aut}}} \mathrm{Out}(G) \longrightarrow \{1\} \\ \{1\} \longrightarrow Z(G) \longrightarrow G \xrightarrow{p_G} \mathrm{Int}(G) \longrightarrow \{1\}. \end{gathered} \tag{1.3c}$$ {eq:intout}

What follows from Proposition 1.2 for reductive algebraic groups is this.

Corollary 1.4. *In the setting* (1.1)*, the group of algebraic automorphisms of* G *is the semidirect product of the inner automorphisms and the distinguished automorphisms:*

$$\mathrm{Aut}(G) = \mathrm{Int}(G) \rtimes \mathrm{Aut}(G, \{H, X_\alpha\}).$$

Consequently

$$\mathrm{Aut}(\mathcal{B}(G)) \simeq \mathrm{Out}(G) \simeq \mathrm{Aut}(G, \{H, X_\alpha\}).$$
$$\mathrm{Aut}(G) = \mathrm{Int}(G) \rtimes \mathrm{Aut}(\mathcal{B}(G)).$$

{se:disc2}

We can now describe the possible disconnected groups E as in (1.3a). We will take as given the connected complex reductive algebraic G as in (1.1), specified by the based root datum $\mathcal{B}(G)$. We fix also a finite group Γ, which may be specified in any convenient fashion. If we are to have a short exact sequence as in (1.3a), we will get automatically a group homomorphism

$$\mathrm{Ad}\colon \Gamma \to \mathrm{Out}(G) \simeq \mathrm{Aut}(\mathcal{B}(G)). \tag{1.5a}$$ {eq:outer}

We will therefore take the specification of such a homomorphism Ad as part of the data (along with G and Γ) which we are given.

We pause briefly to recall what sort of group is $\mathrm{Aut}(\mathcal{B}(G))$. Recall that the set of simple roots $\Pi(B, H)$ is the set of vertices of a graph $\mathrm{Dynkin}(G)$ with some directed multiple edges, the *Dynkin diagram* of G: an edge joins distinct vertices α and β if and only if $\langle \beta^\vee, \alpha \rangle \neq 0$. We will not make further use of the Dynkin diagram, so we do not recall the details. If $Z(G)$ has dimension m, then

$$\mathrm{Aut}(\mathcal{B}(G)) \subset \mathrm{Aut}(\mathrm{Dynkin}(G)) \times \mathrm{Aut}(\mathbb{Z}^m). \tag{1.5b}$$ {eq:out}

Here the first factor is a finite group ("diagram automorphisms") and the second is the discrete group of $m \times m$ integer matrices of determinant ± 1. The inclusion is a subgroup of finite index.

The automorphism group of a connected Dynkin diagram is small (order one or two except in the case of D_4, where the automorphism group is S_3). So for semisimple G the possibilities for the homomorphism Ad are generally quite limited, and can be described concretely and explicitly. Understanding possible maps to $\mathrm{Aut}(\mathbb{Z}^m)$ means understanding m-dimensional representations of Γ over $\mathbb{Z}$. This is a more subtle and complicated subject; but we will not concern ourselves with it, merely taking Ad as given somehow.

For any group G, the inner automorphisms act trivially on the center $Z(G)$, so there is a natural homomorphism

$$\mathrm{Ad}\colon \mathrm{Out}(G) \to \mathrm{Aut}(Z(G)). \tag{1.5c}$$

In our setting, (1.5a) therefore gives

$$\overline{\mathrm{Ad}}\colon \Gamma \to \mathrm{Aut}(Z(G)). \tag{1.5d}$$

With this information in hand, the quotient group $E/Z(G)$ can now be completely described. Define (using p_{Aut} from (1.3c) and Ad from (1.5a))

$$\mathrm{Aut}_\Gamma(G) = p_{\mathrm{Aut}}^{-1}(\mathrm{Ad}(\Gamma)). \tag{1.5e}$$

We immediately get from (1.3c) a short exact sequence

$$1 \longrightarrow \mathrm{Int}(G) \longrightarrow \mathrm{Aut}_\Gamma(G) \longrightarrow \mathrm{Ad}(\Gamma) \longrightarrow 1. \tag{1.5f}$$

Necessarily $E/Z(G)$ is the pushout

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Int}(G) \arrow[r] \arrow[d] & E/Z(G) \arrow[r, dashed] \arrow[d, dashed] & \Gamma \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathrm{Int}(G) \arrow[r] & \mathrm{Aut}_\Gamma(G) \arrow[r] & \mathrm{Ad}(\Gamma) \arrow[r] & 1. \end{tikzcd} \tag{1.5g}$$

Explicitly, this means

$$\begin{aligned} E/Z(G) &= \{(\tau, \gamma) \in \mathrm{Aut}_\Gamma(G) \rtimes \Gamma \mid p_{\mathrm{Aut}}(\tau) = \mathrm{Ad}(\gamma)\} \\ &\simeq [G/Z(G)] \rtimes \Gamma \end{aligned} \tag{1.5h}$$

(notation (1.3c) and (1.5a)). What is required is therefore to construct the extension E

$$1 \longrightarrow Z(G) \longrightarrow E \longrightarrow E/Z(G) \longrightarrow 1. \tag{1.5i}$$

Here is how to do that.

{thm:listE}

Theorem 1.6. *Suppose we are in the setting* (1.5)*.*

1. *Define*

$$E(\{H, X_\alpha\}) = \{e \in E \mid \mathrm{Ad}(e)(\{H, X_\alpha\}) = \{H, X_\alpha\}\},$$

the subgroup of E defining distinguished automorphisms of G (see (1.3b)*). Then*

$$1 \longrightarrow Z(G) \longrightarrow E(\{H, X_\alpha\}) \longrightarrow \Gamma \longrightarrow 1,$$

so $E(\{H, X_\alpha\})$ is an extension of Γ by the abelian group $Z(G)$.

2. *The group $E(\{H, X_\alpha\})$ meets G in $Z(G)$, and meets every coset of G in E. Consequently there is natural surjective homomorphism*

$$G \rtimes E(\{H, X_\alpha\}) \to E$$

with kernel the diagonal copy $Z(G)_\Delta$.

3. *There is a natural bijection between algebraic extensions E of Γ by G and algebraic extensions $E(\{H, X_\alpha\})$ of Γ by $Z(G)$. These latter are parametrized by the group cohomology*

$$H^2(\Gamma, Z(G))$$

(see for example [2, pages 299–303]*). The bijection is given by* {CE}

$$E = [G \rtimes E(\{H, X_\alpha\})]/Z(G)_\Delta.$$

4. Define

$$Z(G)_{\mathrm{fin}} = \textit{elements of finite order in } Z(G),$$

the torsion subgroup. Then the natural map

$$H^p(\Gamma, Z(G)_{\mathrm{fin}}) \to H^p(\Gamma, Z(G)) \qquad (p \ge 2)$$

is an isomorphism.

{se:grpcoh}

Here are some details about the description of $E(\{H, X_\alpha\})$ explained in Theorem 1.6(3). The group cohomology $H^2(\Gamma, Z(G))$ can be described as

$$Z^2(\Gamma, Z(G))/B^2(\Gamma, Z(G)) \tag{1.7a}$$

of cocycles modulo coboundaries. The set $Z^2(\Gamma, Z(G))$ of cocycles consists of maps

$$\begin{gathered} c\colon \Gamma \times \Gamma \to Z(G), \\ \gamma_1 \cdot c(\gamma_2, \gamma_3) - c(\gamma_1\gamma_2, \gamma_3) + c(\gamma_1, \gamma_2\gamma_3) - c(\gamma_1, \gamma_2) = 0 \qquad (\gamma_i \in \Gamma). \end{gathered} \tag{1.7b}$$

A coboundary begins with an arbitrary map $b\colon \Gamma \to Z(G)$. The corresponding coboundary is

$$db(\gamma_1, \gamma_2) = b(\gamma_1) + \gamma_1 \cdot b(\gamma_2) - b(\gamma_1\gamma_2); \tag{1.7c}$$

it is standard and easy to check that db is a cocycle. Given a cocycle c, the extension $E(\{H, X_\alpha\})$ is generated by $Z(G)$ and additional elements

$$\{\tilde\gamma \mid \gamma \in \Gamma\} \tag{1.7d}$$

which are subject to the relations

$$\begin{gathered} \tilde\gamma_1\tilde\gamma_2 = c(\gamma_1, \gamma_2)\widetilde{\gamma_1\gamma_2} \\ \tilde\gamma z \tilde\gamma^{-1} = \gamma \cdot z \qquad (\gamma \in \Gamma, z \in Z(G)). \end{gathered} \tag{1.7e}$$

{eq:discrel}

Another way to say this is that the extension $E(\{H, X_\alpha\})$ as a set is $Z(G) \times \Gamma$, with multiplication defined by

$$(z_1, \gamma_1)(z_2, \gamma_2) = (z_1 \cdot \overline{\mathrm{Ad}}(\gamma_1)(z_2) \cdot c(\gamma_1, \gamma_2), \gamma_1\gamma_2). \tag{1.7f}$$

{eq:discrel2}

In this picture, the distinguished coset representatives $\tilde\gamma$ of (1.7e) are the elements $(1, \gamma)$.

Conversely, if we are given an extension $E(\{H, X_\alpha\})$ as in Theorem 1.6(1), and we choose for each $\gamma \in \Gamma$ a preimage $\tilde\gamma \in E(\{H, X_\alpha\})$, then these choices must satisfy relations of the form (1.7e), with c a cocycle. Replacing the representatives $\tilde\gamma$ with alternate representatives

$$\tilde\gamma' = b(\gamma)\tilde\gamma \qquad (b(\gamma) \in Z(G))$$

(which are the only possibilities) replaces c by $c' = c + db$.

One reason that Theorem 1.6(4) is interesting is for keeping calculations accessible to a computer. We would like the cocycle defining our disconnected group to take values in $Z(G)_{\mathrm{fin}}$, because such elements are easily described in a computer. Here is a way to prove part (4).

{prop:transfer}

Proposition 1.8 (Eckmann [3, Theorem 5]). *Suppose that Γ is a finite group of order n, and that A is an abelian group on which Γ acts. Then multiplication by n acts by zero on $H^p(\Gamma, A)$. If multiplication by n is an* automorphism *of A, then $H^p(\Gamma, A) = 0$ for all $p > 0$.* {transfer}

Sketch of Proof. It is easy to calculate (with no hypotheses on A) that $H^p(\{1\}, A) = 0$ for all $p > 0$. Therefore the obvious restriction map

$$Q\colon H^p(\Gamma, A) \to H^p(\{1\}, A) \qquad (p > 0)$$

must be zero. Eckmann in [3] defines a natural *transfer* homomorphism {transfer}

$$T\colon H^p(\{1\}, A) \to H^p(\Gamma, A),$$

and proves (this is his Theorem 5) that $T \circ Q$ is multiplication by n. Since $Q = 0$, it follows that multiplication by n must be zero for all $p > 0$. The last assertion is immediate. □

To prove Theorem 1.6(4), consider the short exact sequence of Γ modules

$$1 \to Z(G)_{\mathrm{fin}} \to Z(G) \to D \to 1. \tag{1.9}$$ {eq:ses}

Because $Z(G)$ is a reductive abelian group, it is a direct sum of copies of $\mathbb{C}^\times$ and a finite abelian group; so D is a direct sum of copies of $\mathbb{C}^\times/(\text{roots of unity})$. It follows easily that multiplication by n is an automorphism of D for every positive n. By Proposition 1.8,

$$H^p(\Gamma, D) = 0 \qquad (p > 0).$$

Examining the long exact sequence in Γ-cohomology attached to (1.9), we deduce Theorem 1.6(4). (In fact we can arrange for all values of a representative cocycle to have order dividing some power of $|\Gamma|$.)

This argument works equally well over any algebraically closed field of characteristic zero (and shows that the extensions described by Theorem 1.6 are independent of the field). In finite characteristic the same is true as long as the characteristic does not divide the order of Γ.

If the characteristic *does* divide the order of Γ, then the extension E is no longer a reductive group, and matters are more complicated.

2 Center and fundamental group

{sec:zpi1}

In this section we record some standard information about the centers and coverings of a reductive algebraic group. As always in the theory of algebraic groups, it is a useful and enlightening exercise to work sometimes over an algebraically closed field

$$k = \overline{k} \tag{2.1}$$

not necessarily of characteristic zero. I don't know how to phrase the next definition properly.

{def:redabelian}

Definition 2.2. Suppose A is a reductive abelian algebraic group scheme over k, not necessarily reduced. Define the *character group of* A to be

$$X^*(A) = \mathrm{Hom}_{\mathrm{alg}}(A, \mathbb{G}_m);$$

here $\mathbb{G}_m$ is the group scheme with closed points $k^\times$, and the Hom is in the category of algebraic group schemes. Then $X^*(A)$ is a finitely generated abelian group. The functor X^* is an anti-equivalence of categories, with inverse

$$A(Y) = \mathrm{Hom}(Y, \mathbb{G}_m);$$

here Hom is in the category of abelian groups. A bit more explicitly, suppose that Y is given by p generators and q relations:

$$Y = \mathbb{Z}^p / R\mathbb{Z}^q,$$

with R a $p \times q$ matrix of integers (whose q columns are the relations). Then

$$A(Y) = \{g \in \mathbb{G}_m^p \mid gR = 1 \in \mathbb{G}_m^q\},$$

a subscheme (not necessarily closed) of $\mathbb{G}_m^p$.

{prop:ZG}

Proposition 2.3. *Suppose G is a connected reductive algebraic group scheme over the algebraically closed field k, and that*

$$\mathcal{B}(G) = (X^*(H), \Pi(B,H), X_*(H), \Pi^\vee(B,H))$$

is the based root datum of G. Then the center $Z(G)$ is a (not necessarily reduced) subscheme of H. Its character group is therefore a quotient of $X^*(H)$: precisely,

$$X^*(Z(G)) = X^*(H)/[\mathbb{Z}\Pi(B,H)],$$

the character lattice of H modulo the root lattice of G. Explicitly, it follows that

$$Z(G) \simeq \{h \in \mathrm{Hom}(X^*(H), \mathbb{G}_m) \mid \alpha(h) = 1 \ (\alpha \in \Pi(B,H))\}.$$

Still more explicitly: an element of H is specified by specifying the values in $\mathbb{G}_m$ of every character of H. The elements of $Z(G)$ are those on which all (positive simple) roots take the value 1.

I don't know a good formulation of the next result over arbitrary fields, and that is a serious gap in understanding.

{prop:pi1}

Proposition 2.4. *Suppose G is a connected reductive algebraic group over $\mathbb{C}$, and that*

$$\mathcal{B}(G) = (X^*(H), \Pi(B,H), X_*(H), \Pi^\vee(B,H))$$

is the based root datum of G. Then the inclusion of H in G defines a surjection on fundamental groups; so $\pi_1(G)$ is naturally a quotient of

$$\pi_1(H) = \pi_1(\mathbb{C}^\times \otimes_{\mathbb{Z}} X_*(H)) = X_*(H).$$

Precisely,

$$\pi_1(G) = X_*(H)/[\mathbb{Z}\Pi^\vee(B,H)],$$

the cocharacter lattice of H modulo the coroot lattice of G.

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